Integration by Stone Throwing

Problem

Imagine yourself as a water lily farmer who wishes to plant a new crop of lilies in one of your distant ponds. Being a person of action, you grab your sack of Cargill copyrighted water lily seeds and lug it all the way out to that distant pond. You arrive and then read the planting instructions (something you should have done before), only to discover that you need to know the area of the pond to determine how many seeds to scatter. As befits the impulsive type of person that you are, you did not bring any measurement instrument with you and do not have the patience to schlep all the way back to your barn to get them. So what's an impulsive, impatient farmer to do? [Think about this for a while before reading on to see if you figure out how to determine the area of the pond using just your body and what you may find lying about.]

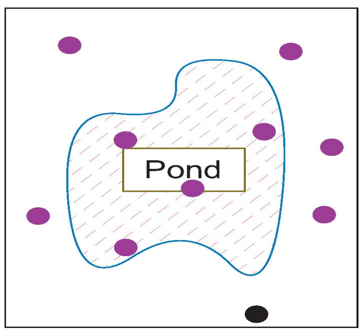

Figure 1 A pond of odd shape whose area we wish to determine. A box of known area is drawn around the pond and the ground within the box cleared of stones. Then handfuls of stones are thrown randomly and uniformly up into the air, and the number of splashes in the pond are counted, as well as the number of stones that have landed on the ground within the box.

Solution: As outlined in Figure 1, this is one way to solve the problem of the compulsive farmer:

1. Grab a stick and use it to trace out a rectangle on the ground that completely encloses the pond.
2. Clear the land area inside the box of all the stones lying on the ground, and fill your pockets with them.
3. Using your boot as an approximate one foot ruler, measure the length and the width of the box. Since the area of the box in square feet A_{box} is just its length times its width, you now know the area of the box.
4. With your eyes closed and while spinning yourself around, throw stone after stone from your pocket up into the air in a rather uniform fashion, all the time counting the number of splashes N_{splash} that you hear.
5. When you are out of stones, open your eyes, regain your equilibrium, and count the number of stones N_{ground} lying on the ground within the box.
6. Common sense, or maybe divine inspiration, tells you that if you have distributed the stones uniformly, then the ratio of the area of the pond to the area of the box is just the ratio of the number of rocks landing in the pond to the total number of rocks landing within the perimeter of the box:

   (1) $$\frac{A_{pond}}{A_{box}} = \frac{N_{splash}}{N_{splash} + N_{box}}.$$

   Now since we know the approximate area of the box in square feet (or square boots), we can determine the area of the pond by multiplying both sides of (1) by A_{box} :

(2) $$A_{pond} = A_{box} \times \frac{N_{splash}}{N_{splash} + N_{box}}.$$

Algorithm

Rather than take the chance of getting our boots all wet and muddy dealing with a real pond, let us image that our pond is a circle of radius 1. We know that the area of a circle $A = \pi r^2$, which for a unit circle (circle with radius 1) is just π. In any case, since we know that $\pi \simeq 3.1415927$, we can try out our new method of integration to calculate the area of a circular pond, and then compare it to the known value of π as a check. (This means that we will calculate π using random numbers and logic.)

1. First we generate pairs of random numbers r_i in the interval 0 to 1:

(3) $$0 \leq r_i \leq 1$$

where *i = 0, 1, …* is an integer that we use to label the numbers in our random sequence.

2. Next, as shown in Figure 2, we imagine a square of side 1 drawn around one quarter of a circle.

3. Then we generate two random numbers and use them as the *x* and *y* coordinates of our stone's landing position:

(4) $$(x_i, y_i) = (r_i, r_{i+1})$$

4. Next, for each landing position, we record a slash whenever the position lies within the circle, that is, whenever

(5) $$x_i^2 + y_i^2 \leq 1$$

5. We repeat the process for a large number of stones in order to obtain “good” statistics.

6. Since our stones (random numbers) always lie between 0 and 1, they must always land within the unit square in Figure 2. Since the area of the square is 1 while the area of the quarter circle within the square is $\pi/4$, our algorithm for integration via stone throwing can be thought of as a test of how well

(6) $$\pi \cong 4 \times \frac{N_{splash}}{N_{tot}}$$

where N_{tot} is the total number of random pairs (splashes) generated.

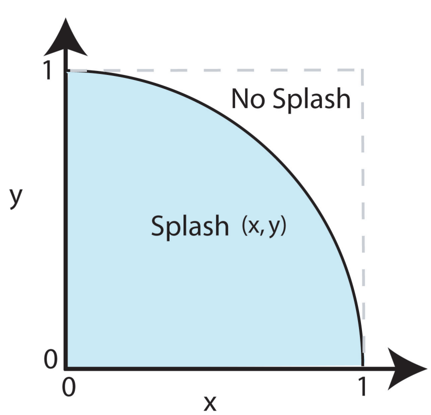

Figure 2 A quarter of a unit circle whose area is to be determined by stone throwing. If the stone lands at *(x, y)* and $x^2 + y^2 < 1$, then a splash occurs. Notice since the box touches the edges of the circle, must stones will land within the circle. As the box is made bigger, the fraction of stones that hit the circle decreases, as does the precision of the answer.

Model Assessment

Because we are throwing the stones randomly, and simulating that with random numbers, we can apply statistical analysis to this problem. We're not going to bother you with that sort of stuff here, but just state the results that such an analysis tells us that we expect that *on the average* the relative error in the computed answer to vary like

(7) $$\text{Relative Error} \simeq \frac{1}{\sqrt{N}},$$

where *N* is the number of stones thrown.

Now listen up! When we say "on the average" we mean that you need to run the simulation or experiment multiple times, and then take the average of all the results. This tends to remove much of the random fluctuations. How many simulations should you average over? Not too many. As a crude rule of thumb, if you use *N* points in a single simulation, then some number of simulations smaller than $\sqrt{N}$ should be good (even as small as $\sqrt{\sqrt{N}}$).

All of this means that to obtain two good decimal places in your answer, as we have asked you to do above, your relative error would be 1 in 100, that is 1/100 = 0.1. Accordingly, we can use Equation (7) above to determine an approximate value for the number of stones that you need to throw to obtain two good decimal places:

(8) $$\frac{1}{100} \cong \frac{1}{\sqrt{N}}$$

To solve for *N,* we square both sides of this equation to remove the square root:

(9) $$\frac{1}{10{,}000} \cong \frac{1}{N}$$

(10) $$N \cong 10{,}000$$

1. See if the number of points you needed to obtain two places of precision is within a factor of two or three of the value deduced from Equation (10).
2. The best way is to make multiple trials and take the average of the value of the value of π obtained for each. So let's say that you want to use $N \cong 10{,}000$ points. The $\sqrt{10{,}000}$ = 100, and the $\sqrt{100}$ = 10. So take the average of more than 10 and less than 100 trials.
3. This same type of analysis would indicate that you would need a million, 1,000,000 points to obtain another place of precision, that is, 100 times more points than needed for two places of precision. Compare this prediction with the increase in the number of points you needed in order to obtain an additional place of precision.
4. We have used a box of unit size that touches the sides of the semicircle whose area we are computing. Change the program so that the stones are thrown within a box of side 2. Comment on how this changes the statistics (and explain if you can).

References

[CP] Landau, R.H., M.J. Paez and C.C. Bordeianu, (2008), *A Survey of Computational Physics,* p 289-297, Princeton Univ. Press, Princeton.