

Exponential Population Growth Projects

Note: This module assumes that the reader has first worked through the *Spontaneous and Exponential Decay* module, and particularly the parts dealing with the exponential function. The present module focuses on exponential growth and on understanding the half-life (doubling time) of biological growth.

In the *Decay and Growth* module we described the average number of decaying nuclei remaining at a time t with the exponential function (see equation 13 in Student Reading of the *Decay and Growth* module):

$$N(t) = N(0) e^{-\lambda t}, \tag{1}$$

where λ is the decay rate. If we reverse the sign in the exponential in eq. (1) from negative to positive (essentially reversing the sign of λ), we obtain a model in which the population grows exponentially in time:

$$N(t) = N(0) e^{\lambda t}. \tag{2}$$

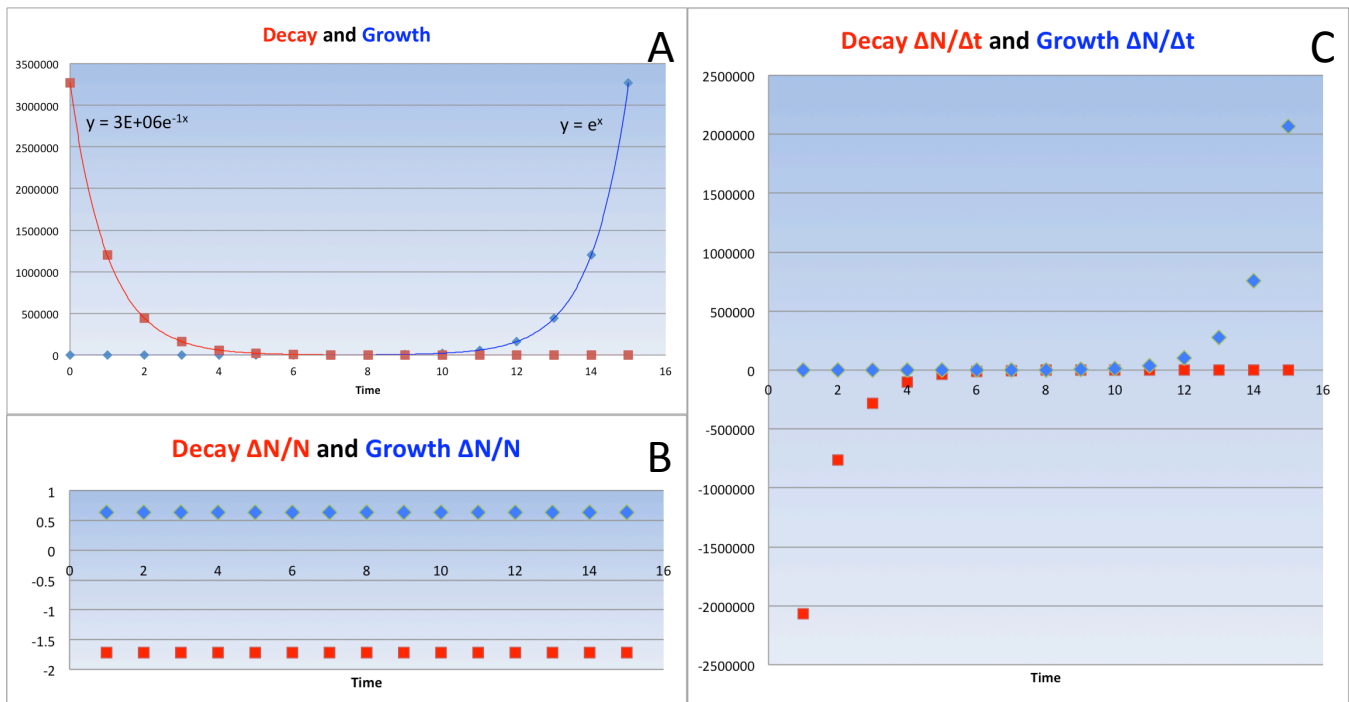


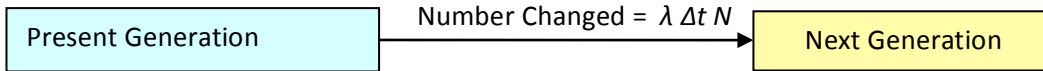
Figure 1. Decay and Growth. The plots are generated in the file [GrowthVsDecay.xlsx](#). A. Plot of Population N for Decay and Growth models. B. Plot of $\Delta N/N$. C. Plot of $\Delta N/\Delta t$.

Eq.(1) describes a population that decreases rapidly in time, while eq.(2) describes one that increases rapidly in time (Fig. 1A). The mathematics of these two equations is very similar. For example, in analogy with exponential decay, exponential growth occurs when the average fraction of growth $\Delta N/N$ (Fig. 1B) in each time interval Δt is a constant:

$$\Delta N = \lambda \Delta t N. \tag{3}$$

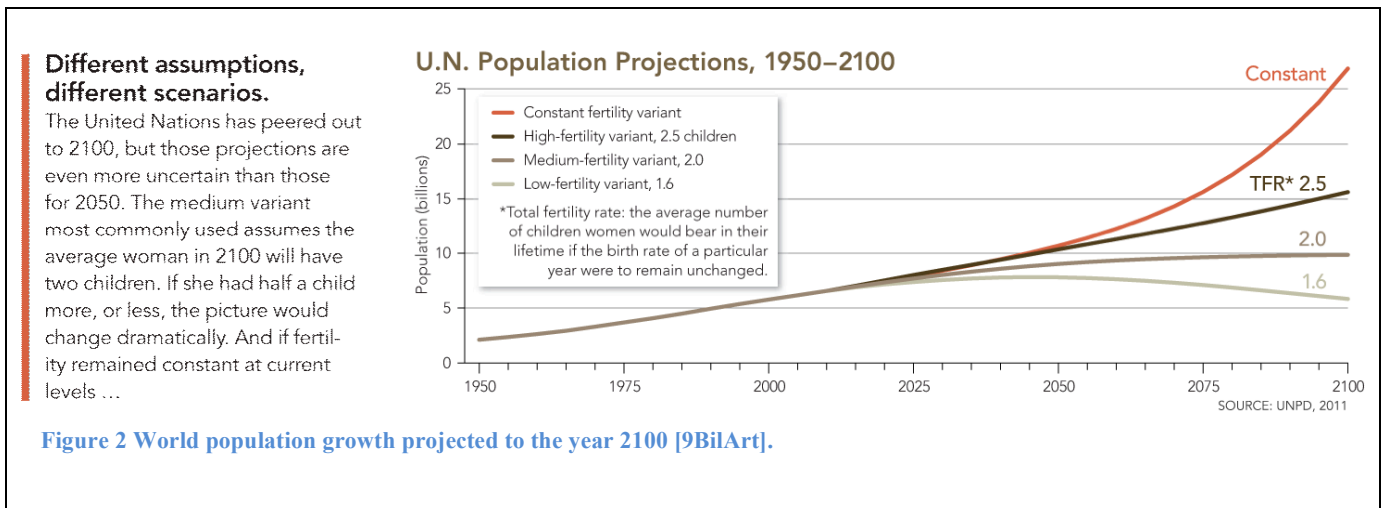
Eq. (3) describes a situation where the number of births ΔN in a population is proportional to the size N of that population. This implies that when the population grows, so does the growth rate $\frac{\Delta N}{\Delta t}$ (Fig. 1C)

A concept map describing exponential growth looks just like the one for exponential decay, except the *Number Changed* (population change from Present to the Next Generation) is now positive, indicating growth:



It is positive sign for the *Number Changed* here that leads to the positive sign in the exponential term of eq. (2).

Exponential growth is a basic model for any reproducing population, to which refinements can be added. For instance, the growth of the microorganism *E. coli* in a test tube containing a *nutrient medium* appears to be exponential. In contrast, human population growth over the last 100 years is complicated, with some aspects of it predicted to be exponential (the projections in Figure 2) [Pop, UN, 9BilArt, 9BilMov].



World Population Growth Project

On the right of Figure 2 is presented a United Nation’s plot of the past growth of the world’s population and three predictions of the future population.

1. Open an Excel file [WorldPopulation](#). This file contains UN data with world population under three different scenarios [UN]. Plot medium, low and high population predictions versus year.
2. Determine which portions of this graph, if any, correspond to exponential growth or decay, $N \propto e^{\pm \lambda t}$, where λ is a constant. Examine all three predictions of the future.
3. Determine which portions of this graph, if any, correspond to linear decay or growth, $N \propto m t$, where m is a constant. Examine all three predictions of the future.
4. Determine which portions of this graph, if any, correspond to a power-law decay or growth, $N \propto \lambda t^n$, where λ and n are constants. Examine all three predictions of the future.

5. Once you have made graphs of the data, you can curve fit an exponential best-fit trendline to the data. As you fit the trendline you will also have the option to include an equation to the graph. See if you can identify λ for each case.

$$\text{Proof: } N = A e^{\pm\lambda t} \rightarrow \log N = \pm\lambda t + \log A$$

$$N = \lambda t^n \rightarrow \log N = \log t + \log \lambda$$

Wolffia Growth Project

In this project you will model biomass growth.

Background



© W.P. Armstrong 2000

Wolffia is the world's smallest flowering plant. It is shaped like a microscopic green football, and is so small that 5,000 of them can fit into a thimble (Fig.3). An average individual plant is 0.6 mm long, 0.3 mm wide and weighs about 150 micrograms, approximately the weight of 2 grains of table salt. In addition to thimbles, Wolffia plant can be found floating at the surface of quiet streams and ponds.

Figure 3. A Photograph of Wolffia growing in a thimble [Wolf]

The growth time of *Wolffia* is measured in hours. A standard measure of the growth time is the *doubling time* τ_2 , that is, the time it takes a population to double. So if we start with an initial population of $N(0)$ then at $t = \tau_2$,

$$N(\tau_2) = 2 N(0). \quad (4)$$

We can relate the doubling time τ_2 to the growth rate λ in Eq.(2) by evaluating Eq.(2) at $t = \tau_2$:

$$N(0) e^{\lambda\tau_2} = 2 N(0). \quad (5)$$

If we take the natural logarithm of each side we obtain:

$$\lambda\tau_2 = \ln 2 \rightarrow \tau_2 = \ln 2 / \lambda. \quad (6)$$

As might be expected, this is the same relation that the half-life of a decaying nucleus has to its decay rate, Eq. (24) in the [Decay](#) module.

- Why do you think the formulas for *half-life* and *doubling time* are the same?

Objectives

- To determine the growth rate λ from the doubling time τ_2 for *Wolffia microscopica*. Although it is experimentally hard to measure the change in the number of plants over time, it is straightforward to measure the change in weight or in size of the biomass over time.
- To reproduce a previously-measured growth as a function of time using our exponential model. (Reproducing someone else's results is often done in the early stages of modeling.)
- Implement the model using two different tools (Excel and Python)

Assignments

1. The doubling time for *Wolffia* microscopic via budding is 30 hours = 1.25 days. What is its growth rate λ ?
2. Reproduce and plot sixteen days (Fig. 4) of *Wolffia* growth in Excel using a time interval of $\Delta t = 1$ day and an initial population $N(0) = 1$.
 - a. Compare your plot to the plot in Fig.4.
 - b. Use your Excel file to generate a table and a plot of *Wolffia* population size vs. time for $\Delta t = 0.5$, $N(0) = 2$, and rate λ is the same as in (a).
3. Create Python simulation of *Wolffia* plant growth
 - a. Save the Decay.py file as **Wolffia growth.py** file
 - b. Change the code to represent growth rather than decay
 - c. Use your Python model to reproduce the graphs you obtained in Excel
 - d. Is it easier (less laborious or time consuming) for you to change time step Δt and initial population $N(0)$ in Excel or Python? Post you answer into the discussion "*Wolffia Implementation*" on the Blackboard.

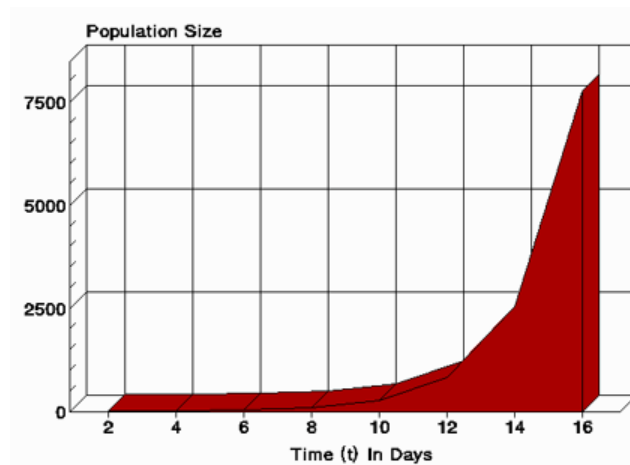


Figure 4. *Wolffia* Population growth in days

References

[Pop] *Eminent Scientist Claims Humans Will Be Extinct In 100 Years*, TheTechJournal, Technological News Portal, <http://thetechjournal.com/science/eminent-scientist-claims-humans-will-be-extinct-in-100-years.xhtml> .

[UN] *World Population to 2300*, United Nations **Department** of Economic and Social Affairs, Population Division, <http://www.un.org/esa/population/publications/longrange2/WorldPop2300final.pdf>.

[9BilArt] *Population. News. 9 Billion?* Leslie Roberts. Science 29 July 2011: Vol. 333 no. 6042 pp. 540-543 DOI: 10.1126/science.333.6042.540

[9BilMov] <http://www.sciencemag.org/site/special/population/pop-intro-movie.xhtml>

[Wolf] *Principles of Population Growth*, Wayne's Word, An On-Line Textbook of Natural History, <http://waynesword.palomar.edu/lmexer9.htm>.

Answer to Assignment 1 of Wolffia Project:

- The growth rate for **Wolffia microscopica** may be calculated from its doubling time according to the Eq.(6).
- In the population growth Eq. (6) the original starting population (N_0) will double, when
 $\Rightarrow \lambda t = \lambda \tau_2 = \ln 2 = 0.693$.
- Therefore a simple equation $\lambda \tau_2 = .693$ can be used to solve for λ and τ_2 . The growth rate (λ) can be determined by simply dividing .693
 $\Rightarrow \lambda = .693 / \tau_2$.
- Doubling time = $\tau_2 = 30$ hours = 1.25 days.
- Since the doubling time (τ_2) for **Wolffia microscopica** is 1.25 days, the growth rate λ is
 $\Rightarrow \lambda = 0.693/1.25 = 0.55$.