Shock Waves & Solitons
PDE Waves; Oft-Left-Out; CFD to Follow

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Course: **Computational Physics II**
1834, J. Scott Russell, Edinburgh-Glasgow Canal

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon. . . .”
Problem: Explain Russel’s Soliton Observation

J. Scott Russell, 1834, Edinburgh-Glasgow Canal

- We extend PDE Waves; You see String Waves 1st
- Extend: nonlinearities, dispersion, hydrodynamics
- Fluids, old but deep & challenging
- Equations: complicated, nonlinear, unstable, rare analytic
- Realistic BC ≠ intuitive (airplanes, autos)
- Solitons: computation essential, modern study
Theory: Advection = Continuity Equation

Simple Fluid Motion

- Continuity equation = conservation of mass

\[
\frac{\partial \rho(x, t)}{\partial t} + \nabla \cdot \mathbf{j} = 0
\]  

(1)

\[
\mathbf{j}(x, t) \overset{\text{def}}{=} \rho \mathbf{v} = \text{current}
\]  

(2)

- \( \rho(x, t) = \) mass density, \( \mathbf{v}(x, t) = \) fluid velocity

- \( \nabla \cdot \mathbf{j} = "\text{Divergence}" \) of current = spreading

- \( \Delta \rho: \) in + out current flow

- Advection Equation, 1-D flow, constant \( v = c: \)

\[
\frac{\partial \rho(x, t)}{\partial t} + c \frac{\partial \rho(x, t)}{\partial x} = 0
\]  

(3)
"Advection" $\overset{\text{def}}{=} \text{transport salt from thru water due to } \vec{v} \text{ field}

Solution: $u(x, t) = f(x - ct) = \text{traveling wave}$

Surfer rider on traveling wave crest

Constant shape $\Rightarrow$

$x - ct = \text{constant} \Rightarrow x = ct + C \Rightarrow \text{Surfer speed} = \frac{dx}{dt} = c$

Can leapfrog, not for shocks
Extend Theory: Burgers’ Equation

Wave Velocity $\propto$ Amplitude

$$\frac{\partial u}{\partial t} + \epsilon u \frac{\partial u}{\partial x} = 0$$  \hspace{1cm} (1)

$$\frac{\partial u}{\partial t} + \epsilon \frac{\partial (u^2/2)}{\partial x} = 0 \quad \text{(Conservative Form)} \hspace{1cm} (2)$$

- Advection: all points @ $c \Rightarrow$ constant shape
- Burgers: larger amplitudes faster $\Rightarrow$ shock wave
Lax–Wendroff Algorithm for Burgers’ Equation

\[ \frac{\partial u}{\partial t} + \epsilon \frac{\partial (u^2/2)}{\partial x} = 0 \quad \text{(Conservative Form)} \]

\[ u(x, t + \Delta t) = u(x, t - \Delta t) - \beta \left[ \frac{u^2(x + \Delta x, t) - u^2(x - \Delta x, t)}{2} \right] \]

\[ \beta = \frac{\epsilon}{\Delta x/\Delta t} = \text{measure nonlinear} < 1 \quad \text{(stable)} \]

\[ u(x, t + \Delta t) \simeq u(x, t) + \frac{\partial u}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} \Delta t^2 \]

\[ u_{i,j+1} = u_{i,j} - \frac{\beta}{4} \left( u_{i+1,j}^2 - u_{i-1,j}^2 \right) + \frac{\beta^2}{8} \left[ (u_{i+1,j} + u_{i,j}) \left( u_{i+1,j}^2 - u_{i,j}^2 \right) - (u_{i,j} + u_{i-1,j}) \left( u_{i,j}^2 - u_{i-1,j}^2 \right) \right] \]
Burger’s Assessment

1. Solve Burgers’ equation via leapfrog method
2. Study shock waves
3. Modify program to Lax–Wendroff method
4. Compare the leapfrog and Lax–Wendroff methods
5. Explore $\Delta x$ and $\Delta t$
6. Check different $\beta$ for stability
7. Separate numerical and physical instabilities
Dispersionless Propagation

Meaning of Dispersion?

- Dispersion $\Rightarrow$ $E$ loss, Dispersion $\Rightarrow$ information loss
- Physical origin: propagate spatially regular medium
- Math origin: higher-order $\partial_x$
- $u(x, t) = e^{i(kx \mp \omega t)} = R/L$ “traveling” plane wave
- Dispersion Relation: sub into advection equation

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad (1)
\]

\[
\Rightarrow \quad \omega = \pm ck \quad \text{(dispersionless propagation)} \quad (2)
\]

\[
v_g = \frac{\partial \omega}{\partial k} = \text{group velocity} = \pm c \quad \text{(linear)} \quad (3)
\]
Including Dispersion (Wave Broadening)

**Small-Dispersion Relation \( w(k) \)**

- \( \omega = ck = \) dispersionless
  \[
  \omega \simeq ck - \beta k^3
  \]  \hspace{1cm} (1)

- \( v_g = \frac{d\omega}{dk} \simeq c - 3\beta k^2 \)
  \hspace{1cm} (2)

- Even powers \( \rightarrow \) R-L asymmetry in \( v_g \)

- Work back to wave equation, \( k^3 \Rightarrow \partial_x^3 \):
  \[
  \frac{\partial u(x, t)}{\partial t} + c \frac{\partial u(x, t)}{\partial x} + \beta \frac{\partial^3 u(x, t)}{\partial x^3} = 0
  \]  \hspace{1cm} (3)
Korteweg & deVries (KdeV) Equation, 1895

\[
\frac{\partial u(x, t)}{\partial t} + \varepsilon u(x, t) \frac{\partial u(x, t)}{\partial x} + \mu \frac{\partial^3 u(x, t)}{\partial x^3} = 0
\]

- Nonlinear $\varepsilon u \partial u / \partial t \rightarrow$ sharpening $\rightarrow$ shock
- $\partial^3 u / \partial x^3 \rightarrow$ dispersion
- Stable: dispersion $\simeq$ shock; (parameters, IC)
- Rediscovered numerically Zabusky & Kruskal, 1965
- 8 Solitons, larger = faster, pass through each other!
Analytic Soliton Solution

Convert Nonlinear PDE to Linear ODE

- Guess traveling wave $\rightarrow$ solvable ODE

\[
0 = \frac{\partial u(x, t)}{\partial t} + \epsilon u(x, t) \frac{\partial u(x, t)}{\partial x} + \mu \frac{\partial^3 u(x, t)}{\partial x^3} \tag{1}
\]

\[
u(x, t) = u(\xi = x - ct) \tag{2}
\]

$\Rightarrow$ \[
0 = \frac{\partial u}{\partial \xi} + \epsilon u \frac{\partial u}{\partial \xi} + \mu \frac{d^3 u}{d\xi^3} \tag{3}
\]

$\Rightarrow u(x, t) = \frac{-c}{2} \sech^2 \left[ \frac{1}{2} \sqrt{c} (x - ct - \xi_0) \right] \tag{4}$

- $\sech^2 \Rightarrow$ solitary lump
Algorithm for KdeV Solitons

**CD for $\partial_t$, $\partial_x$; 4 points $\partial_x^3$**

\[
u_{i,j+1} \approx \nu_{i,j-1} - \frac{\epsilon}{3} \frac{\Delta t}{\Delta x} [u_{i+1,j} + u_{i,j} + u_{i-1,j}] [u_{i+1,j} - u_{i-1,j}] \\
- \mu \frac{\Delta t}{(\Delta x)^3} [u_{i+2,j} + 2u_{i-1,j} - 2u_{i+1,j} - u_{i-2,j}]
\]

- **IC + FD to start** (see text)
- **Truncation error & stability:**

\[
\mathcal{E}(u) = \mathcal{O}[(\Delta t)^3] + \mathcal{O}[\Delta t(\Delta x)^2]
\]

\[
\frac{1}{(\Delta x/\Delta t)} \left[ \epsilon |u| + 4 \frac{\mu}{(\Delta x)^2} \right] \leq 1
\]
Implementation: KdeV Solitons

- Bore $\rightarrow$ solitons
- Solitons crossing
- Stability check
- Solitons in a box