The Chaotic Pendulum I

Continuous Nonlinear Dynamics

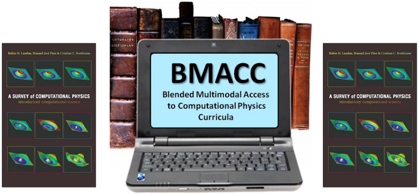
Rubin H Landau

Sally Haerer, Producer-Director

Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

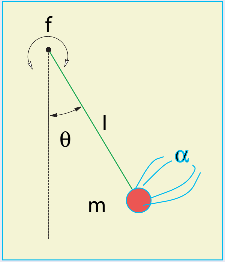
with Support from the National Science Foundation

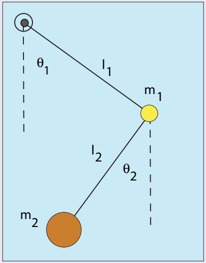
Course: **Computational Physics I**



Problem: Realistic Single or Double Pendulum

Simulate Nonlinear, Chaotic System





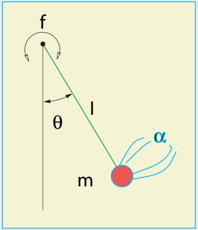
loading TwoPend

- Driven single pendulum
- Free, double pendulum
- Large oscillations, even over-the-top

Chaotic Pendulum ODE

Newton's Laws for Rotational Motion $\sum \tau = I \frac{d^2\theta}{dt^2}$

- Gravitation τ: $-mgl \sin\theta$
- Friction τ: $-\beta\dot{\theta}$
- External τ: $\tau_0 \cos \omega t$



$$I \frac{d^2\theta}{dt^2} = -mgl \sin\theta - \beta \frac{d\theta}{dt} + \tau_0 \cos\omega t \quad (1)$$

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin\theta - \alpha \frac{d\theta}{dt} + f \cos\omega t \quad (2)$$

$$\omega_0 = \frac{mgl}{I}, \quad \alpha = \frac{\beta}{I}, \quad f = \frac{\tau_0}{I}$$

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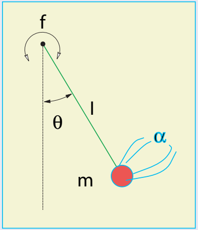

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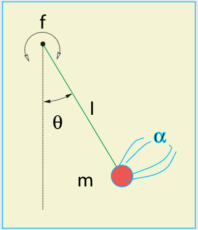

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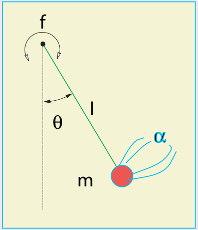

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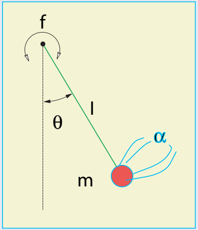

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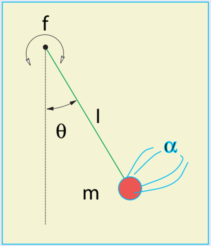
Standard ODE Form (rk4): $\dot{\vec{y}} = \vec{f}(\vec{y}, t)$



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- 2^{nd} O t-dependent nonlinear ODE
- Nonlinearity: $\sin\theta \simeq \theta - \theta^3/3! \cdots$
- $y^{(0)} = \theta(t), \quad y^{(1)} = \frac{d\theta(t)}{dt}$

$$\frac{dy^{(0)}}{dt} = y^{(1)} \quad (2)$$

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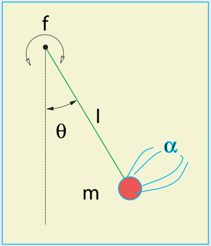

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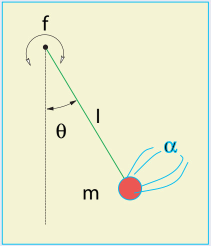

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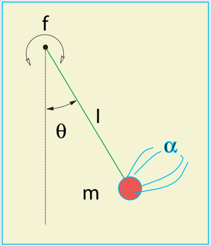

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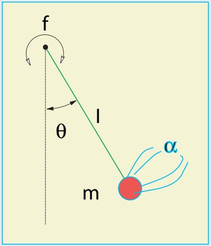

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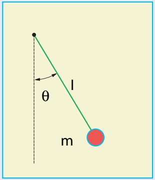
Start Simply: Free Oscillations (Test Algorithm & Physics)

Ignore Friction & External Torques $(f = \alpha = 0)$



$$\ddot{\theta} = -\,\omega_0^2 \sin\theta \qquad (1)$$

$$\ddot{\theta} \simeq -\,\omega_0^2\theta \qquad \text{(linear, } \theta \simeq 0\text{)}$$

$$\Rightarrow \theta(t) = \theta_0 \sin(\omega_0 t + \phi) \qquad (2)$$

(1): "Analytic solution"; sort of:

$$T \propto \int_0^{\theta_m} \frac{d\theta}{\left[\sin^2(\theta_m/2) - \sin^2(\theta/2)\right]^{1/2}} \qquad (3)$$

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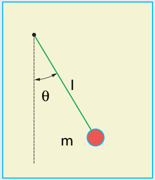

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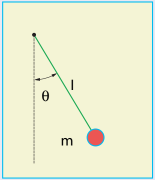

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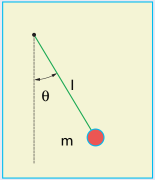

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Solve ODE with rk4

1. Initial conditions: $\{\theta = 0, \dot{\theta}(0) \neq 0\}$; increase $\dot{\theta}(0)$
2. Verify SHM $\ddot{\theta} = -\omega_0^2\theta \Rightarrow \omega = \omega_0 = 2\pi/T$ = constant
3. Devise algorithm to determine period T ($3 \times \theta = 0$)
4. Determine $T(\theta)$ for realistic pendulum, compare
5. Verify as $KE(0) \leq 2mgl$: non harmonic oscillations
6. Verify $\Rightarrow$ separatrix ($KE(0) \rightarrow 2mgl$), $T \rightarrow \infty$
7. Listen harmonic & anharmonic motion (Hear now)
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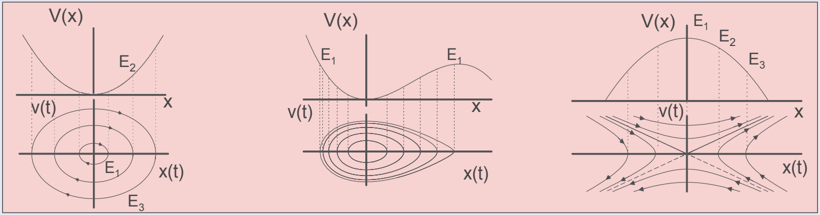
Visualization: Phase Space Orbits

Abstract Space: $v(t)$ vs $x(t)$ (x vs t, v vs t= Complicated)



- Geometry easy to "see"
- SHM: Ellipse, $E \rightarrow$ size
- Anharmonic: + corners
- Ossc $\Rightarrow$ CW Closed
- Non Ossc, repulse = open

$$x(t) = A\sin(\omega t), \quad v(t) = \omega A\cos(\omega t) \quad (SHM) \tag{1}$$

$$E = \mathrm{KE} + \mathrm{PE} = mv^2/2 + \omega^2 m^2 x^2/2 = \text{ellipse} \tag{2}$$

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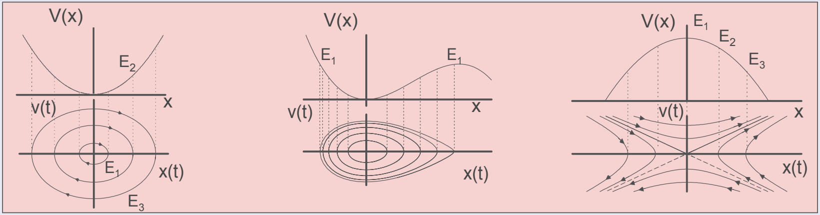

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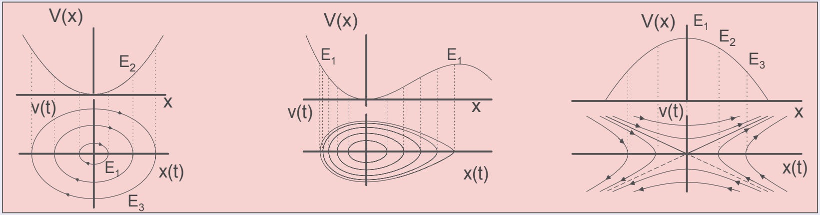

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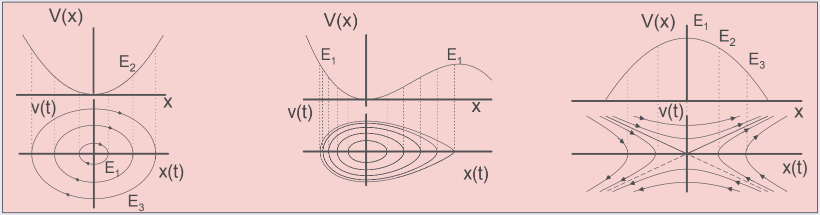

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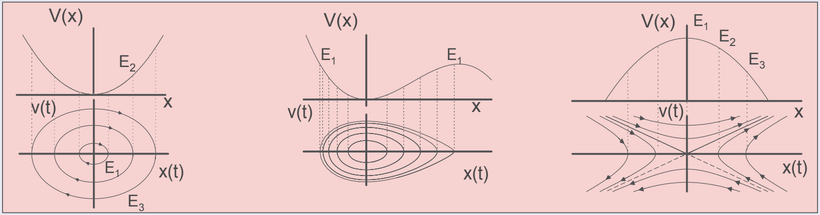

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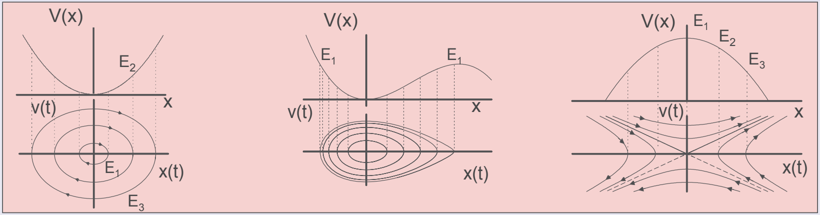
- Non Ossc, repulse = open

$$x(t) = A\sin(\omega t), \quad v(t) = \omega A\cos(\omega t) \quad (SHM) \tag{1}$$

$$E = \mathrm{KE} + \mathrm{PE} = mv^2/2 + \omega^2 m^2 x^2/2 = \text{ellipse} \tag{2}$$

Visualization: Phase Space Orbits

Abstract Space: $v(t)$ vs $x(t)$ (x vs t, v vs t= Complicated)



- Geometry easy to "see"
- SHM: Ellipse, $E \rightarrow$ size
- Anharmonic: + corners
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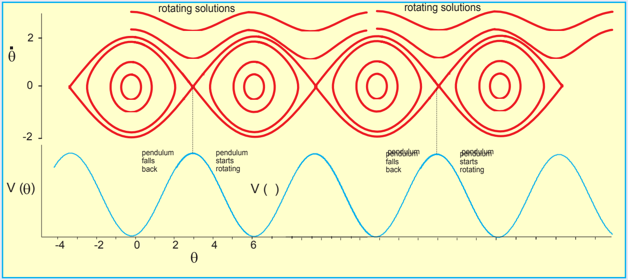
Undriven, Frictionless Pendulum in Phase Space

Separatrix Separates Open & Closed Orbits



- Closed: oscillation
- Open: rotation
- Both: periodic
- Orbits do not cross

- Open orbits touch
- Hyperbolic points
- Unstable equilibrium

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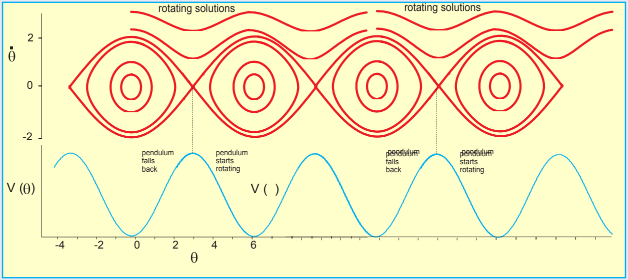

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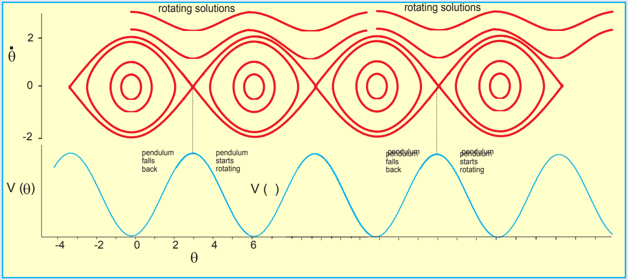

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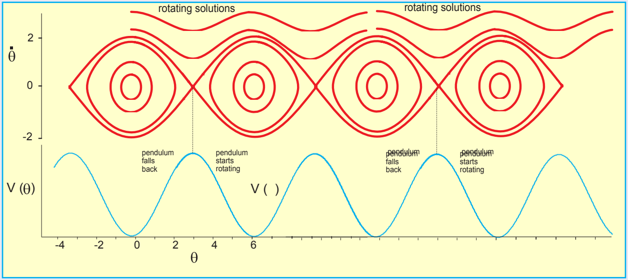

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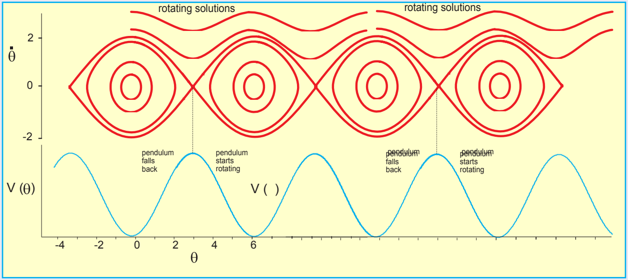

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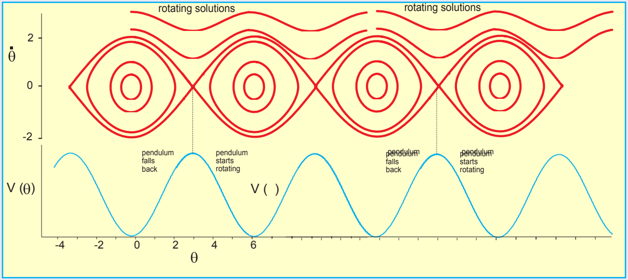

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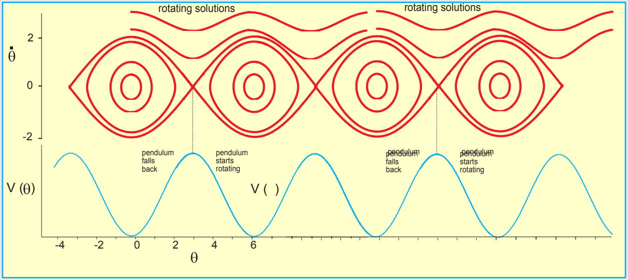

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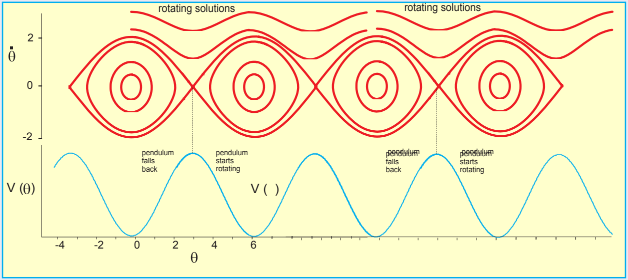

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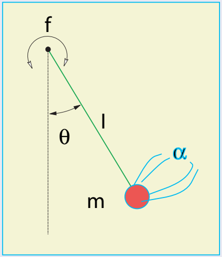
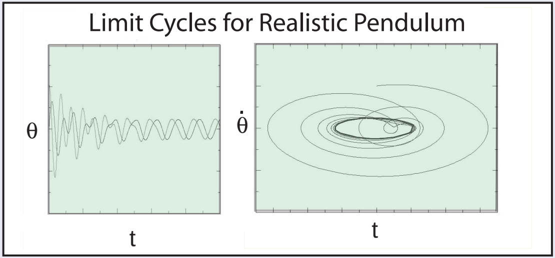
Include Friction, Driving Torque (small t steps)

Geometry Tends to Remain





- Friction $\Rightarrow \downarrow E$
- Inward Spiral
- τ_{ext} can put E back
- Limit cycle = Balance
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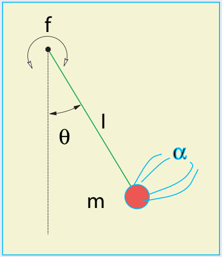

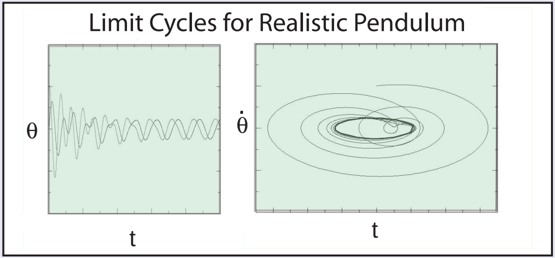

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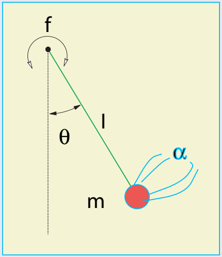

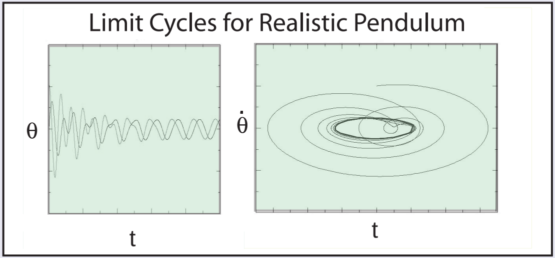

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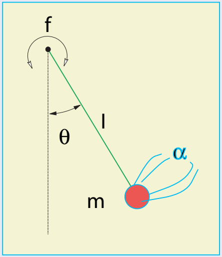

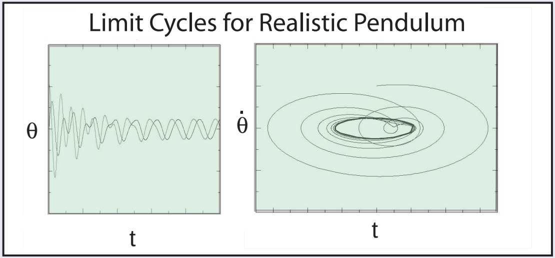

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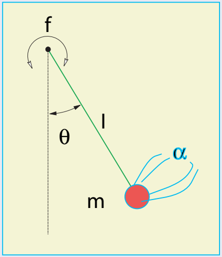

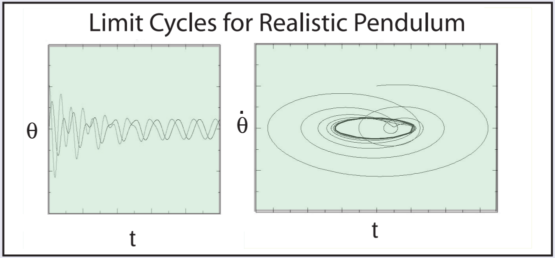

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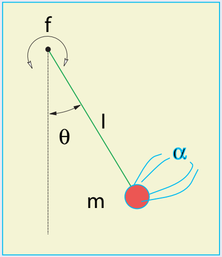

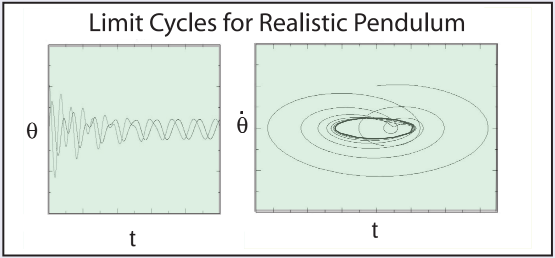

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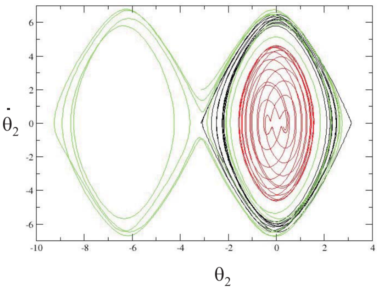
Chaos As Viewed in Phase Space (Full Solution)

Look for in Your Simulations



- Complex $\leq$ Chaos $\leq$ Rand
- Fixed Params, all x_0, ts: flows

- Chaos complex $\neq$ mess
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- Simplicity in chaos [PS, $\neq \theta(t)$]
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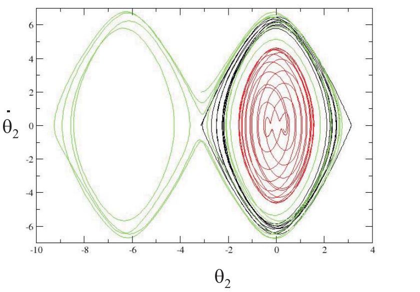

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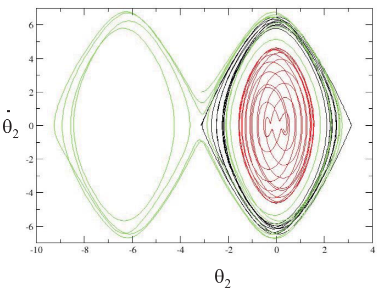

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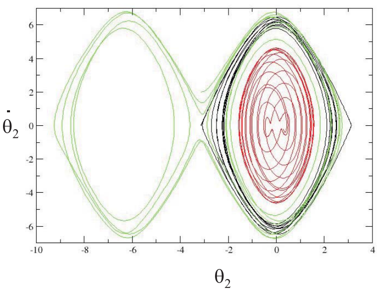

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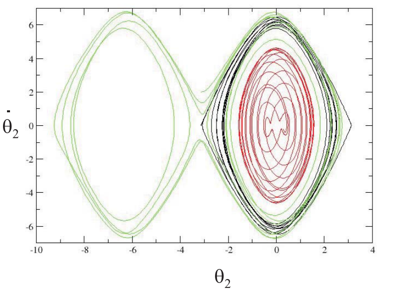

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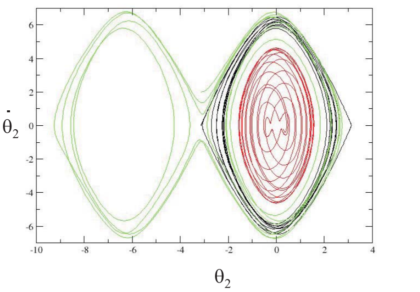

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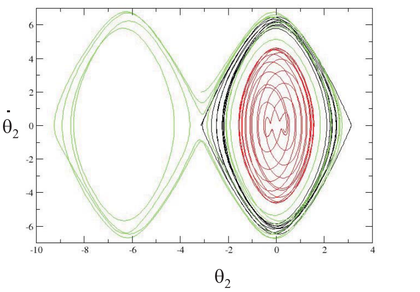

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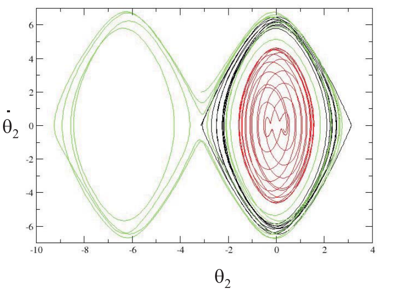

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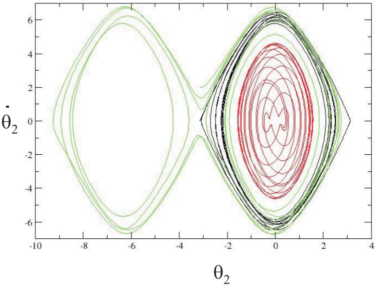

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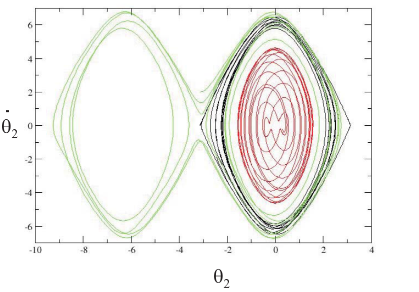

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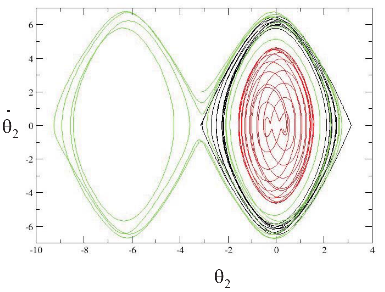

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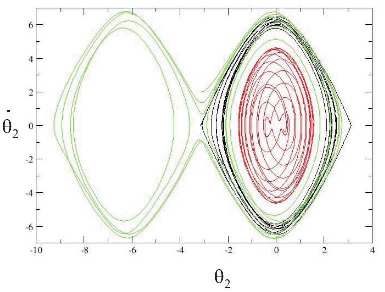

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Implementation: Let’s Get Down to Work

Good Time for a Break!