Numerical Solution of Differential Equations

- Big topic; we’ll cover in parts
- Not hard, very useful
- Context of practical problem
- Can browse beginning, if already known

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Problem: Forced Nonlinear Oscillator

- Mass $m$, spring, 2 forces:
- Spring: any $x$ dependence
- Force: external $(t)$, friction ...
- **Problem: position $x(t)$ (1-D)**

- Computational: arbitrary forces = easy
- Traditional treatments: small $x$, linear $F(x)$
Physics Theory: Newton's Laws

- Newton's 2nd law ⇒ equation of motion (Solve This!)

\[
\sum_i F_i = ma \quad (1)
\]

\[
F_k(x) + F_{\text{ext}}(x, t) = m \frac{d^2x}{dt^2} \quad (2)
\]

- Solve (2) = Ordinary Differential Eqn = (ODE)
- ODE: many physical laws; nature, concrete?
- Integral equations (easy now): maybe more?
Model: Nonlinear Oscillator

Potential: arbitrary power $p$

$$V(x) = \frac{1}{p} k x^p$$  \hspace{1cm} (3)

- Even $p \Rightarrow$ restoring
- $p > 6$: particle in a box

Force on mass

$$F_k(x) = -\frac{dV(x)}{dx} = -k x^{p-1}$$  \hspace{1cm} (4)

- Newton’s law: 2\textsuperscript{nd} - ODE

$$F_{\text{ext}}(x, t) - k x^{p-1} = m \frac{d^2 x}{dt^2}$$  \hspace{1cm} (5)
Math: Types of Differential Equations

- **Landau's 1st Rule of education**
- **Order** = degree of derivative

- **First Order ODE**
  \[ \frac{dy}{dt} = f(t, y) \]  

- **RHS**: arbitrary derivative, "force" function \( f(t, y) \)
  - e.g. \( f(t, y) \) nonlinear in \( y \)
  \[ \frac{dy}{dt} = -3t^2y + t^9 + y^7 \]  

- **Second Order ODE** (e.g. Newton's law)
  \[ m \frac{d^2y}{dt^2} = -3t^2 \left( \frac{dy}{dt} \right)^4 + t^9 y(t), \]  

- **Independent variable**: \( t = \) time, **Dependent** \( y(t) = \) position; \( t(x) \)
More Words: *Ordinary, Partial, Initial, Boundary*

- **Ordinary Differential Equation** = ODE
  - only 1 independent variable:
    \[ V(x)\psi(x) = -\frac{d\psi(x)}{dx} \]  \((9)\)

- **Partial Differential Equation** = PDE (later)
  - > 1 independent variable:
    \[ i\frac{\partial\psi(x,y,t)}{\partial t} = -\frac{\partial^2\psi(x,y,t)}{\partial x^2} + \frac{\partial^2\psi(x,y,t)}{\partial y^2} \]  \((10)\)

- **Initial Conditions**: solve 1st-order ODE: 1 constant, \(\psi(0)\)
- 2nd-order: 2 constants, \(\psi(0), \psi'(0)\)

- **Boundary Conditions**
  - solution: fixed value(s) in space
  - needed for PDE (more degrees freedom)
  - ODE: extra restriction \(\Rightarrow\) eigenvalue problem
Words: Linear and Nonlinear ODEs

• Nonlinear DE (dependent): \( y(t) \) or \( \frac{dy}{dt} \):

\[
\frac{dy}{dt} = g^3(t)y(t), \quad \text{linear}
\]

\[
\frac{dy}{dt} = \lambda y(t) - \lambda^2 y^2(t), \quad \text{nonlinear}
\]  \hspace{1cm} (11) \hspace{1cm} (12)

• Nonlinear: hard analytically; all same numerically

• Law linear superposition:

Also a solution \( y(t) = \alpha A(t) + \beta B(t) \) \hspace{1cm} (13)

\[
\frac{dy}{dt} = \lambda y(t) - \lambda^2 y^2(t)
\]  \hspace{1cm} (Nonlinear ODE) \hspace{1cm} (14)

\[
y(t) = \frac{a}{1 + be^{-\lambda t}}
\]  \hspace{1cm} (Soltn) \hspace{1cm} (15)

\[
y_1(t) = \frac{a}{1 + be^{-\lambda t}} + \frac{a'}{1 + b'e^{-\lambda t}}
\]  \hspace{1cm} (Not soltn) \hspace{1cm} (16)
You deserve a break now!
Applied Math & Classical Dynamics: Standard Form of ODEs

• All order ODEs

\[
\frac{dy(t)}{dt} = f(t,y).
\]  
(no \(dy(i)/dt\))

N-D vectors

\[
y = \begin{bmatrix}
    y^{(0)}(t) \\
    y^{(1)}(t) \\
    \vdots \\
    y^{(N-1)}(t)
\end{bmatrix},
\quad f = \begin{bmatrix}
    f^{(0)}[t,y] \\
    f^{(1)}[t,y] \\
    \vdots \\
    f^{(N-1)}[t,y]
\end{bmatrix},
\]

\(N\) simultaneous 1st-order ODEs:

\[
\frac{dy^{(0)}(t)}{dt} = f^{(0)}[t,y]
\]

\[
\frac{dy^{(1)}(t)}{dt} = f^{(1)}[t,y]
\]

\(\vdots\)
E.G.: Dynamical Form for 2nd-Order ODE (Problem!)

\[ \frac{d^2 x}{dt^2} = \frac{1}{m} F \left( t, \frac{dx}{dt}, x \right) \]

\[ \Rightarrow \quad \frac{dy(t)}{dt} = f(t, y) \quad \text{(21)} \]

- Rules: RHS no explicit derivatives, LHS only 1st derivatives
- Start: define position \( x = \) dependent variable
- Trick: define velocity = dependent variable

\[ y^{(0)}(t) \overset{\text{def}}{=} x(t) \quad \text{(22)} \]

\[ y^{(1)}(t) \overset{\text{def}}{=} \frac{dx}{dt} \overset{\text{def}}{=} \frac{dy^{(0)}}{dt} \quad \text{(23)} \]

\[ \frac{dy^{(0)}(t)}{dt} = y^{(1)}(t) \]

\[ \frac{dy^{(1)}(t)}{dt} = \frac{1}{m} F(t, y^{(0)}, y^{(1)}) \]

\[ f^{(0)} = y^{(1)}(t) \quad \text{(24)} \]

\[ f^{(1)} = F(t, y^{(0)}, y^{(1)}) \quad \text{(25)} \]