Filters and Noise

Optional Assessment of Practical Importance

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Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

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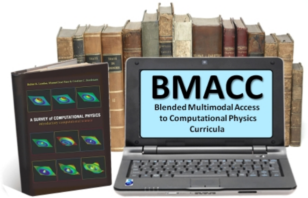
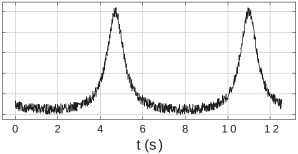
Course: **Computational Physics II**



Problem: Cleaning Up Noisy a Signal



Problem: What is pure signal?

- Two Simple Approaches
  1. Autocorrelation functions
  2. Digital Filters
- Both wide applications
- More filters in Wavelet Analysis & Data Compression

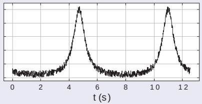
Noise Reduction via Autocorrelation (Theory)

Assumption: Noise just adds to signal

Measure $=$ Signal + Noise

$$y(t) = s(t) + n(t)$$

$$s(t) = ?$$



- Science: assume simplest (+)
- Science: noise $\simeq$ random $(\infty\ ^{o}\ \mathrm{F})$
- Recall "random" sequence; $r_i \quad \not\Rightarrow \quad r_{i+1}$
- $\Rightarrow n(t)$ not correlated with $s(t)$, $n(t)$

Correlation Function $c(t)$

How measure correlation?

$$y(t) = \sin\omega, \quad x(t) = \sin(n\omega t + \phi) \quad \textit{correlated}$$

$$c(\tau) = \int_{-\infty}^{+\infty} dt\; y^*(t)\, x(t+\tau) \quad \text{(Correlation Function)}$$

- Correlated (τ = lag time = variable):
  - Integrand > 0 for some τ
  - $\Rightarrow$ Constructive interference $\Rightarrow c(\tau) \to \infty$
- Not correlated:
  - 2 functions oscillate independently
  - +, - equally likely
  - $\Rightarrow$ Destructive interference $\Rightarrow c(\tau) \simeq 0$

More Correlation Function

Properties

- Express c, y^*, x via FT & substitute:

$$\text{(FT)} \quad c(\tau) = \int_{-\infty}^{+\infty} d\omega' \; C(\omega') \frac{e^{i\omega'\tau}}{\sqrt{2\pi}} \tag{1}$$

$$\text{(Def)} \quad c(\tau) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} dt \; y^*(t)\, x(t+\tau) \tag{2}$$

$$\Rightarrow \quad C(\omega) = \sqrt{2\pi}\; Y^*(\omega) X(\omega) \tag{3}$$

- Requires convergence to rearrange
- Related to convolution theorem (soon)

Special Correlation Function: Autocorrelation

Measure Correlation with Itself: $a(\tau)$

$$a(\tau) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} dt\, y^*(t)\, y(t+\tau)$$

- To compute: fold or convolute with self:
- $y(t)$ = measured signal
- Average over time for "all" τ values
- $a(0)$ = "large"

Averaging Removes Random Noise from $a(t)$

Proof by substitution

$$y(t) = s(t) + n(t) \quad \text{(Noisy Signal)} \tag{4}$$

$$a_y(\tau) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} dt\, y^*(t)\, y(t+\tau) \quad \text{(Def } a(t)\text{)} \tag{5}$$

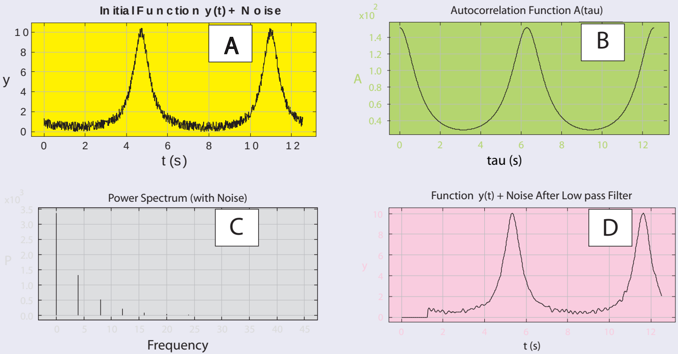
$$= \int_{-\infty}^{+\infty} dt\, [s(t)s(t+\tau) + s(t)n(t+\tau) + n(t)n(t+\tau)]$$

$$\Rightarrow \quad a_y(\tau) \simeq \int_{-\infty}^{+\infty} dt\, s(t)\, s(t+\tau) = a_s(\tau) \quad \text{QED Magic} \tag{6}$$

$$\text{So} \quad A_s(\omega) \simeq \sqrt{2\pi}\, |S(\omega)|^2 \propto \text{Power Spectrum} \tag{7}$$

How Apply to Data?

Start with Noisy signal



1. Compute autocorrelation function
2. DFT: $a(t) \Rightarrow A(\omega)$
3. $\Rightarrow$ power spectrum w/o random noise

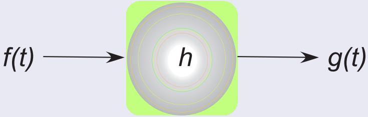
Autocorrelation Function Exercises (`Noise.java`)

Example

1. Signal: $s(t) = \frac{1}{1-0.9\sin t} \simeq 1 + 0.9\sin t + (0.9\sin t)^2 \cdots$
2. DFT $\Rightarrow S(\omega)$, Plot $|S(\omega)|^2$.
3. Autocorrelation function $a(t)$ of $s(t)$?
4. Power spectrum $a(t)$ *vs* $|S(\omega)|^2$*?*
5. *Add noise* $y(t_i) = s(t_i) + \alpha(2r_i - 1), \quad 0 fuss \leq \alpha \leq hide$
6. *Plot* $y(t)$, $Y(\omega)$, *Power spectrum.*
7. $a(t) \quad \rightarrow \quad A(\omega)$.
8. *Compare* $A(\omega)$ *to power spectrum.*

Filtering with Transforms (Theory)

Action of Filter



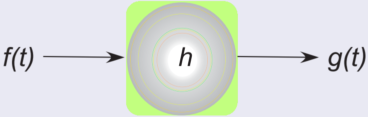
$$g(t) = \int_{-\infty}^{+\infty} d\tau\, f(\tau)\, h(t-\tau) \tag{8}$$

$$\stackrel{\text{def}}{=} f(t) * h(t) \quad \text{(analog filter)} \tag{9}$$

- $h(t) \stackrel{\text{def}}{=}$ unit impulse response
- $h(t) = \int_{-\infty}^{+\infty} d\tau\, \delta(\tau)\, h(t-\tau)$
- Greens function
- $h(0)$ = max, $h(<0) = 0$
- $*$ = Convolution

Convolution Theorem

Filter as Convolution



$$g(t) = \int_{-\infty}^{+\infty} d\tau \, f(\tau) \, h(t-\tau) \quad (10)$$

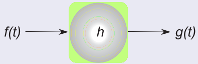
$$G(\omega) = \sqrt{2\pi} \, F(\omega) \, H(\omega) \quad (11)$$

- Proof: FT,δ, $\int$
- Simpler in ω than t
- Digital: response (ω_n)

- Lowpass: $\downarrow$ high ω
- Highpass: $\downarrow$ low ω

Digital Filters

Filter Coefficients c_n = Complete Description



Filter Def: $$g(t) = \int_{-\infty}^{+\infty} d\tau\, f(\tau)\, h(t-\tau) \qquad (12)$$

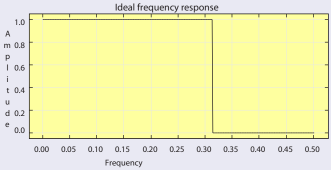
Digital Transfer: $$h(t) = \sum_{n=0}^{N} c_n\, \delta(t - n\tau) \qquad (13)$$

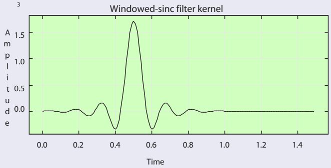
$$\Rightarrow \quad g(t) = \sum_{n=0}^{N} c_n\, f(t - n\tau) \qquad (14)$$

c_n: integration wts N point DFT + response

Exploration: Windowed Sinc Filters (Filter.java)⊙

Lowpass Filter to Reduce Noise, Aliasing





- Ideal lowpass
- $Y(\omega)$ = rectangular pulse $\Rightarrow y(t)$ = sinc function

$$\int_{-\infty}^{+\infty} d\omega \, e^{-i\omega t} \mathrm{rect}(\omega) = \mathrm{sinc}\left(\frac{t}{2}\right) \stackrel{\text{def}}{=} \frac{\sin(\pi t/2)}{\pi t/2}$$

- $\Rightarrow$ filter out high ω: convolute with $\sin(\omega_c t)/(\omega_c t)$
- **Exercise:** Repeat random noise addition using sinc filter