Electromagnetic Waves
The Finite-Difference Time Domain (FDTD) Algorithm

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Based on A Survey of Computational Physics by Landau, Páez, & Bordeianu
with Support from the National Science Foundation

Course: Computational Physics II
Problem: Determine E & H Fields for All Times

Given: Space $0 \leq z \leq 200$

$$E_x(z, t = 0) = 0.1 \sin \frac{2\pi z}{100}, \quad H_y(z, t = 0) = 0.1 \sin \frac{2\pi z}{100}$$

- E&M practical import
- FDTD $\Delta z, \Delta t = \text{step}$
- New: coupled fields
- New: vector fields, 3-D
**Title: Maxwell’s Equations in Free Space**

\[ \mathbf{E} = E_x, \quad \mathbf{H} = H_y, \quad \mathbf{S} = \mathbf{E} \times \mathbf{H} = S_z \Rightarrow 3-D \]

\[ \nabla \cdot \mathbf{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x(z, t)}{\partial x} = 0 \quad \text{(Transverse)} \quad (1) \]

\[ \nabla \cdot \mathbf{H} = 0 \quad \Rightarrow \quad \frac{\partial H_y(z, t)}{\partial y} = 0 \quad \text{(Transverse)} \quad (2) \]

\[ \frac{\partial \mathbf{E}}{\partial t} = + \frac{1}{\epsilon_0} \nabla \times \mathbf{H} \quad \Rightarrow \quad \frac{\partial E_x}{\partial t} = - \frac{1}{\epsilon_0} \frac{\partial H_y(z, t)}{\partial z} \quad (3) \]

\[ \frac{\partial \mathbf{H}}{\partial t} = - \frac{1}{\mu_0} \nabla \times \mathbf{E} \quad \Rightarrow \quad \frac{\partial H_y}{\partial t} = - \frac{1}{\mu_0} \frac{\partial E_x(z, t)}{\partial z} \quad (4) \]
Central-Difference Derivatives \(\Rightarrow\)

\[ E_{x}^{z,t} = E_{x}^{k,n+1/2}, \quad H_{y}^{z,t} = H_{y}^{k+1/2,n} \]

\[
\frac{\partial E(z, t)}{\partial t} \simeq \frac{E(z, t + \frac{\Delta t}{2}) - E(z, t - \frac{\Delta t}{2})}{\Delta t}, \quad (1)
\]

\[
\frac{\partial E(z, t)}{\partial z} \simeq \frac{E(z + \frac{\Delta z}{2}, t) - E(z - \frac{\Delta z}{2}, t)}{\Delta z}, \quad (2)
\]

Substitute into Maxwell, rearrange for \(t\) stepping

\[
E_{x}^{k,n+1/2} = E_{x}^{k,n-1/2} - \frac{\Delta t}{\epsilon_0 \Delta z} (H_{y}^{k+1/2,n} - H_{y}^{k-1/2,n}), \quad (3)
\]

\[
H_{y}^{k+1/2,n+1} = H_{y}^{k+1/2,n} - \frac{\Delta t}{\mu_0 \Delta z} (E_{x}^{k+1,n+1/2} - E_{x}^{k,n+1/2}) \quad (4)
\]
Displaced $E_x, H_y$ Space-Time Lattices

$$E_{x}^{z,t} = E_{x}^{k,n+1/2}, \quad H_{y}^{z,t} = H_{y}^{k+1/2,n}$$

- Space variation $H_y \Rightarrow$ time variation $E_x$
- Space variation $E_x \Rightarrow$ time variation $H_y$
**Alternate Formulation: Even & Odd Times**

Double Index Values

\[ E_{x}^{k,n} = E_{x}^{k,n-2} - \frac{\Delta t}{\varepsilon_0 \Delta z} \left( H_{y}^{k+1,n-1} - H_{y}^{k-1,n-1} \right), \quad k \text{ even, odd}, \quad (1) \]

\[ H_{y}^{k,n} = H_{y}^{k,n-2} - \frac{\Delta t}{\mu_0 \Delta z} \left( E_{x}^{k+1,n-1} - E_{x}^{k-1,n-1} \right), \quad k \text{ odd, even}. \quad (2) \]

- **E**: even \( z \), odd \( t \)
- **H**: odd \( z \), even \( t \)
Normalized Algorithm; Stability Analysis

\[ \tilde{E} \text{ With Same Dimension as } H, \quad \tilde{E} = \sqrt{\varepsilon_0/\mu_0} E \]

\[ \tilde{E}_x^{k,n+1/2} = \tilde{E}_x^{k,n-1/2} + \beta \left( H_y^{k-1/2,n} - H_y^{k+1/2,n} \right) \]  

(1)

\[ H_y^{k+1/2,n+1} = H_y^{k+1/2,n} + \beta \left( \tilde{E}_x^{k,n+1/2} - \tilde{E}_x^{k+1,n+1/2} \right) \]  

(2)

\[ \beta = \frac{c}{\Delta z/\Delta t}, \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \] (light)  

(3)

- \( \beta = \) light/grid speed
- \( \omega_{\text{wave}} \Rightarrow t \) scale
- \( \lambda_{\text{wave}} \Rightarrow z \) scale
- \( > 10 \text{ points/} \lambda \)
- Courant Stability: \( \beta \leq 1/2 \)
- Smaller \( \Delta t \) \( \uparrow \) precision
- Smaller \( \Delta t \) \( \uparrow \) stability
- Smaller \( \Delta z \) \( \Rightarrow \) smaller \( \Delta t \)
**Implementation** **FDTD.py**

- Initial conditions ($0 \leq z(k) \leq 200$):

  $$E_x(z, t = 0) = 0.1 \sin \frac{2\pi z}{100}, \quad H_y(z, t = 0) = 0.1 \sin \frac{2\pi z}{100}$$

- Discrete Maxwell equations:

  $$Ex[k, 1] = Ex[k, 0] + beta \times (Hy[k - 1, 0] - Hy[k + 1, 0])$$
  $$Hy[k, 1] = Hy[k, 0] + beta \times (Ex[k - 1, 0] - Ex[k + 1, 0])$$

- $0 = \text{old time}, \quad 1 = \text{new time}$

- Spatial endpoints via periodic boundary conditions:
Assessment

1. Impose BC such that fields vanish on boundaries
2. Show effect of these BCs
3. Test Courant stability condition
4. Solve with inserted dielectric slab
5. Note transmission, reflection at slab boundaries
6. Verify that $\mathbf{H}(t = 0) = 0 \implies$ right & left pulses
7. Investigate resonator modes for plane waves with nodes at boundaries
Extension: Circularly Polarized Waves

CircPolartzn.py