Computational Fourier Analysis

Mathematics, Computing and Nonlinear Oscillations

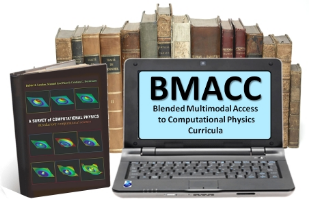
Rubin H Landau

Sally Haerer, Producer-Director

Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

with Support from the National Science Foundation

Course: **Computational Physics II**



Outline

Applied Math: Approximate Fourier Integral

Numerical Integration

$$\text{Transform} \quad Y(\omega) = \int_{-\infty}^{+\infty} dt \;\; y(t)\, \frac{e^{-i\omega t}}{\sqrt{2\pi}} \tag{1}$$

$$\simeq \sum_{i=0}^{N} h\, y(t_i)\, \frac{e^{-i\omega t_i}}{\sqrt{2\pi}} \tag{2}$$

- Approximate Fourier integral → finite Fourier series
- Consequences to follow

Experimental Constraints Too!

Transform, Spectral: $$Y(\omega) = \int_{-\infty}^{+\infty} dt \; y(t) \, \frac{e^{-i\omega t}}{\sqrt{2\pi}} \tag{3}$$

Inverse, Synthesis: $$y(t) = \int_{-\infty}^{+\infty} d\omega \; Y(\omega) \, \frac{e^{+i\omega t}}{\sqrt{2\pi}} \tag{4}$$

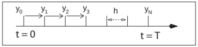
Real World: Data Restrict Us

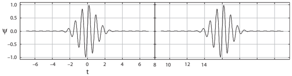
- Measured $y(t)$ only @ N times (t_i's)
- Discrete not continuous & not $-\infty \le t \le +\infty$
- Can't measure enough data to determine $Y(\omega)$
- The inverse problem with incomplete data
- DFT: one possible solution

Algorithm with Discrete & Finite Times

Measure: N signal values

- Uniform time steps $\Delta t = h, \quad t_k = kh$
- $y_k = y(t_k), \quad k = 0, 1, \ldots, N$
- Finite $T \Rightarrow$ ambiguity
- Integrate over all t; $y(t < 0)$, $y(t > T)$ = ?
- Assume periodicity $y(t + T) = y(t)$ (removes ambiguity)
- $\Rightarrow Y(\omega)$ at N discrete ω_i's
- $\Rightarrow y_0 \equiv y_N$ repeats!
- $\Rightarrow N + 1$ values, N independent

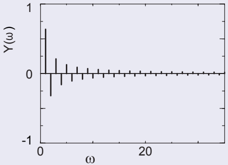




“A” Solution to Indeterminant Problem

Discrete $y(t_i)s \Rightarrow$ Discrete ω_i

- N independent $y(t_i)$ measured
- $\Rightarrow$ N independent $Y(\omega_i)$
- $\omega_i = ?$
- Total time $T = Nh \Rightarrow$ min ω_1:



$$\omega_1 = \frac{2\pi}{T} = \frac{2\pi}{Nh} \tag{5}$$

- Only N ω_is (+1)

$$\omega_k = k\omega_1, \qquad k = 0, 1, \ldots, N \tag{6}$$

$$\omega_0 = 0 \times \omega_1 = 0 \quad \text{(DC)} \tag{7}$$

Algorithm: Discrete Fourier Transform (DFT)

Trapezoid Rule: $\int f(t)dt \simeq \sum h\, f(t_i)$

$$Y(\omega_n) = \int_{-\infty}^{+\infty} dt\, \frac{e^{-i\omega_n t}}{\sqrt{2\pi}} y(t) \simeq \int_0^T dt\, \frac{e^{-i\omega_n t}}{\sqrt{2\pi}} y(t) \tag{8}$$

$$\simeq \sum_{k=1}^{N} h\, y(t_k) \frac{e^{-i\omega_n t_k}}{\sqrt{2\pi}} \tag{9}$$

Symmetrize notation, substitute ω_n:

$$Y_n = \frac{1}{h} Y(\omega_n) = \sum_{k=1}^{N} y_k \frac{e^{-2\pi i k n/N}}{\sqrt{2\pi}} \tag{10}$$

Discrete Inverse Transform (Function Synthesis)

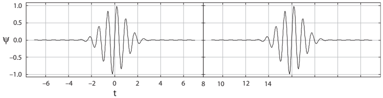
Trapezoid Rule for Inverse

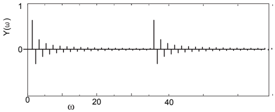
- Frequency Step $\Delta\omega = \omega_1 = \frac{2\pi}{T} = \frac{2\pi}{Nh}$

$$y(t) = \int_{-\infty}^{+\infty} d\omega \, \frac{e^{i\omega t}}{\sqrt{2\pi}} Y(\omega) \simeq \sum_{n=1}^{N} \frac{2\pi}{Nh} \frac{e^{i\omega_n t}}{\sqrt{2\pi}} Y(\omega_n) \quad (11)$$

- Trig functions $\Rightarrow$ periodicity for y and Y:

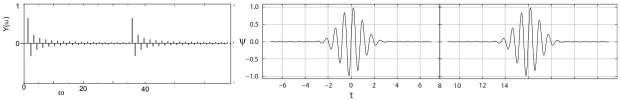
$$y(t_{k+N}) = y(t_k), \quad Y(\omega_{n+N}) = Y(\omega_n) \quad (12)$$





Consequences of Discreteness

$$Y(\omega_n) \simeq \sum_{k=1}^{N} h\, y(t_k) \frac{e^{-i\omega_n t_k}}{\sqrt{2\pi}} \qquad\qquad y(t) \simeq \sum_{n=1}^{N} \frac{2\pi}{Nh} \frac{e^{i\omega_n t}}{\sqrt{2\pi}} Y(\omega_n)$$



- $\Delta\omega = \omega_1 = \frac{2\pi}{T} = \frac{2\pi}{Nh}$
- Finer $\omega \Leftrightarrow$ larger $T = Nh$
- Finer $\omega \rightarrow$ smoother $Y(\omega)$
- "Pad" $y(t)$ $\Rightarrow$ smoother $Y(\omega)$

- Ethical question
- Synthetic $y(t)$ = bad $t \rightarrow T$
- Periodicity $\downarrow$ as $T \rightarrow \infty$
- Aliases and ghosts: special lecture

Concise & Efficient DFT Computation

Compute 1 Complex number

$$Z = e^{-2\pi i/N} = \cos\frac{2\pi}{N} - i\sin\frac{2\pi}{N} \tag{13}$$

$$Y_n = \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{N} Z^{nk} y_k \qquad \text{(Transform)} \tag{14}$$

$$y_k = \frac{\sqrt{2\pi}}{N}\sum_{n=1}^{N} Z^{-nk} Y_n \qquad \text{(Synthesis, TF}^{-1}\text{)} \tag{15}$$

- Z rotates y into complex Y and *visa versa*
- Compute only powers of Z (basis of FFT)

$$Z^{nk} \equiv (Z^n)^k = \cos\frac{2\pi nk}{N} - i\sin\frac{2\pi nk}{wN} \tag{16}$$

DFT Fourier Series: $y(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) \simeq ?$

Discrete: Sample $y(t)$ @ N times

$$y(t = t_k) \equiv y_k, \quad k = 0, 1, \ldots, N \tag{17}$$

- Repeat period $T = Nh \ \Rightarrow \ y_0 = y_N$
- $\Rightarrow \quad N$ independent a_ns
- Use trapezoid rule, $\omega_1 = 2\pi/Nh$:

$$a_n = \frac{2}{T}\int_0^T dt \cos(k\omega t) y(t) \simeq \frac{2}{N}\sum_{k=1}^{N} \cos(nk\omega_1)\, y_k \tag{18}$$

- Truncate sum:

$$y(t) \simeq \sum_{n=1}^{N} a_n \cos(n\omega_1 t) \tag{19}$$

Code Implementation: DFTcomplex.py

Example

```
N = 1000;
twopi  = 2.*math.pi;
sq2pi = 1./math.sqrt(twopi);

def fourier(dftz):
    for n in range(0, N):
      zsum = complex(0.0, 0.0)
      for  k in range(0, N):
          zexpo = complex(0, twopi*k*n/N)
          zsum += signal[k]*exp(-zexpo)
      dftz[n] = zsum * sq2pi
```

Assessment: Test Where know Answers

Simple Analytic Cases

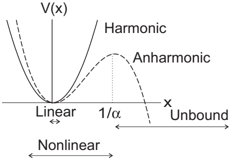
$$\text{e.g.} \quad y(t) = 3\cos(\omega t) + 2\cos(3\omega t) + \cos(5\omega t) \tag{20}$$

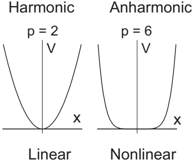
1. Known: $Y_1 : Y_2 : Y_3 = 3 : 2 : 1$ (9 :4 :1 power spectrum)
2. Resum to input signal? (> graph, idea of error)
3. Effect of: time step h, period $T = Nh$
4. Sample mixed signal:

$$y(t) = 4 + 5\sin(\omega t + 7) + 2\cos(3\omega t) + \sin(5\omega t) \tag{21}$$

5. Effects of varying 4 & 7?

Physics Assessment





Determine Spectrum and Check Inversion

1. **Nonlinearly perturbed oscillator:**

$$V(x) = \frac{1}{2}kx^2\left(1 - \frac{2}{3}\alpha x\right) \qquad (1)$$

2. Determine when $>$ 10% higher harmonics ($b_{n>1} \geq 10\%$)
3. **Highly nonlinear oscillator:**

$$V(x) = kx^{12} \qquad (2)$$

4. Compare to sawtooth.

Summary

- Represent periodic or nonperiodic functions with DFT.
- Finiteness of measurements $\rightarrow$ ambiguities (T)
- Infinite series or integral not practical algorithm or in experiment.
- Approximate integration $\rightarrow$ simplicity & approximations
- Better high frequency components: smaller h, same T.
- Smoother transform: larger T, same h (padding).
- Less periodicity: more measurements.
- DFT is simple, elegant and powerful.
- Rotation between signal and transform space.
- $(e^{i\phi})^n \rightarrow$ Fast Fourier Transform.