Continuous Wavelet Transforms

Part I (Discrete to Follow)

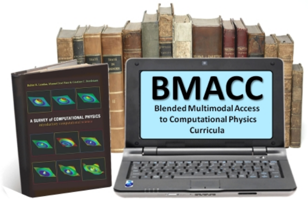
Rubin H Landau

Sally Haerer, Producer-Director

Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

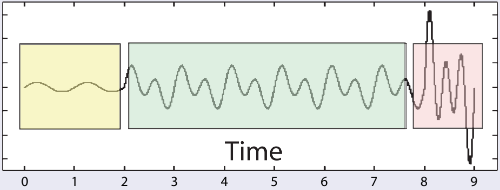
with Support from the National Science Foundation

Course: **Computational Physics II**



Problem: Multiple Frequencies in Time

Non Stationary Signals



- Amount ω_i at each t?
- Δ number of ω's in t
- Numerical signal OK
- Here analytic:

$$y(t) = \begin{cases} \sin 2\pi t, & \text{for } 0 \le t \le 2, \\ 5\sin 2\pi t + 10\sin 4\pi t, & \text{for } 2 \le t \le 8, \\ 2.5\sin 2\pi t + 6\sin 4\pi t + 10\sin 6\pi t, & \text{for } 8 \le t \le 12. \end{cases}$$

Why Not Fourier Analysis?

Fourier Limitation: amount of $\sin(n\omega t)$

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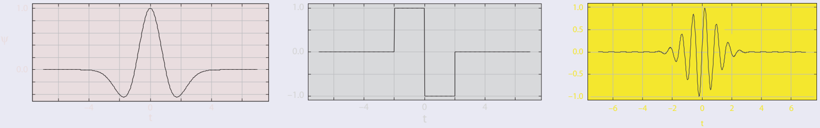
- OK for stationary signals
- Not OK for **Problem**
- Fourier: all ω_i all time
- No time resolution
- Fourier: correlated ω_i's
- Poor data compression; recompute c_i

Wavelets in a Nutshell

Three Wavelet Examples

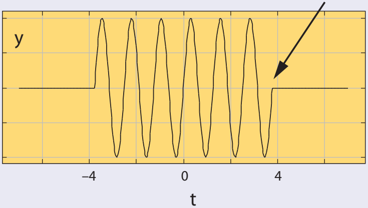


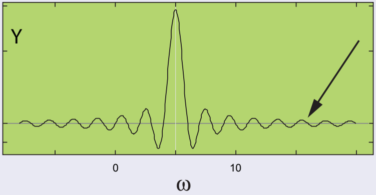
- Extend Fourier
- Nonstationary signals
- Fairly recent
- Extensive applications
- E.g.: all oscillate

- Varied functional forms
- Wavelet basis expansion
- "let": small wave (pack)
- Each: finite & $\Delta\ T$
- Each: center different t

Wave Packets $= \sum$ Waves

Wave Packet e.g. N Cycle Sine





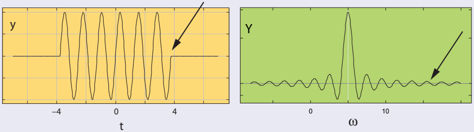
- Packet $\Rightarrow y(t) = \text{pulse } \Delta t$
- $\Rightarrow Y(\omega) = \text{pulse } \Delta\omega$

$$y(t) = \begin{cases} \sin\omega_0 t, & \text{for} \quad |t| < N\frac{T}{2}, \\ 0, & \text{for} \quad |t| > N\frac{T}{2}, \end{cases}$$

$$\Rightarrow \Delta t = NT = N\frac{2\pi}{\omega_0}, \qquad \Delta\omega \simeq \frac{\omega_0}{N}$$

Uncertainty Principle (Theory)

Fundamental Relation: $\Delta t \leftrightarrow \Delta\omega$



- N cycle example $\Rightarrow$ general truth
- $\Delta\omega \simeq$ first 0's of $Y(\omega)$:

$$\frac{\omega - \omega_0}{\omega_0} = \pm\frac{1}{N} \quad \Rightarrow \quad \Delta\omega \simeq \omega - \omega_0 = \frac{\omega_0}{N}$$

$$\text{N cycle} \Rightarrow \Delta t \simeq NT = N\frac{2\pi}{\omega_0}$$

$$\Rightarrow \quad \Delta t\, \Delta\omega \geq 2\pi$$

- QM: "Heisenberg Uncertainty Principle"

Wave Packet Assessment (before break)

Example

Given three wave packets:

$$y_1(t) = e^{-t^2/2}, \quad y_2(t) = \sin(8t)e^{-t^2/2}, \quad y_3(t) = (1 - t^2)\, e^{-t^2/2}$$

For each wave packet:

1. Estimate the width Δt. A good measure might be the *full width at half-maxima* (FWHM) of $|y(t)|$.
2. Evaluate and plot the Fourier transform $Y(\omega)$.
3. Estimate the width $\Delta\omega$ of the transform. A good measure might be the *full width at half-maxima* of $|Y(\omega)|$.
4. Determine the constant C for the uncertainty principle

$$\Delta t\, \Delta\omega \geq 2\pi C.$$

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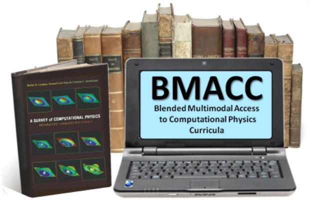
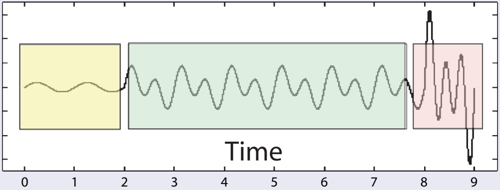


Aside: Wavelet Precursor Sets Stage

Colored Boxes $\longrightarrow$ Windows $w(t)$

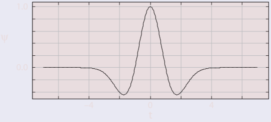


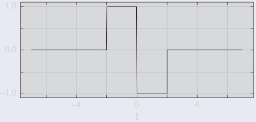
- Seen: $\sin n\omega t\ \exists$ all t's
- Overlap $\Rightarrow$ correlated
- Dependent components
- $\Rightarrow$ FT short time interval
- Boxes = windows $= w(t)$
- $\Rightarrow\ Y_{\tau_1}(\omega), Y_{\tau_2}(\omega), \ldots Y_{\tau_N}(\omega)$

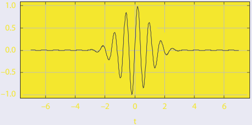
$$Y^{(\mathrm{ST})}(\omega,\tau) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\, w(t-\tau)\, y(t)$$

The Wavelet Transform

$Y(\omega): \ \exp(i\omega t) \ \rightarrow \ Y(s,\tau): \ \ \psi_{s,\tau}(t)$







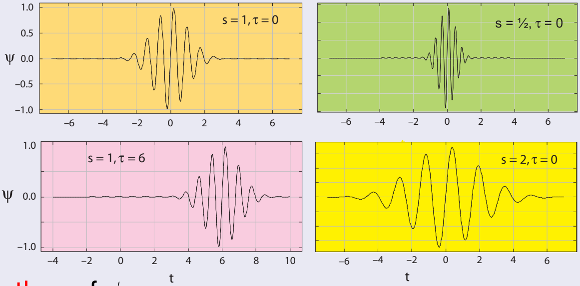
$$Y(s,\tau) = \int_{-\infty}^{+\infty} dt \ \psi^*_{s,\tau}(t) \ y(t) \qquad \text{(wavelet transform)}$$

- $\sim$ Short-time FT
- Wavelet localized in t
- $\Rightarrow$ Own window
- Oscillations $\Rightarrow \Delta\omega$
- Y = amt $\psi_{s,\tau}(t)$ in $y(t)$

- τ: time interval analyzed
- s = scale = $2\pi/\omega$
- t details $\Rightarrow$ small s
- Small scale $\Rightarrow$ high ω

Generating Wavelet Basis Functions

Scale by s, Translate by τ: $\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right)$

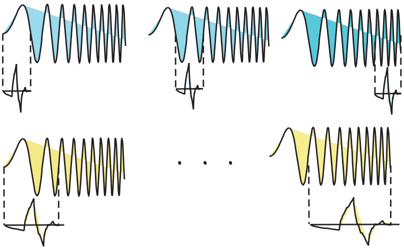


- Ψ = mother of ψ
- Fixed # oscills; vary T, 0
- $s <, > 1 \;\rightarrow$ high, low ω
- Large s: smooth envelope
- Need fewer large s
- Small s: details
- Need for hi resolution

Visualization: Transform of Chirp $\sin(60t^2)$

$$Y(s,\tau) = \frac{1}{\sqrt{s}} \int_{-\infty}^{+\infty} dt\, \Psi^*\left(\frac{t-\tau}{s}\right) y(t) \qquad \text{(Transform)}$$

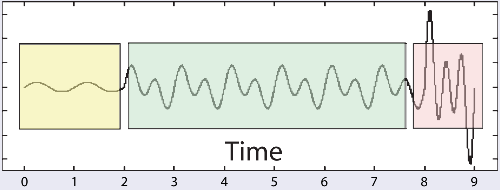
$$y(t) = \frac{1}{C} \int_{-\infty}^{+\infty} d\tau \int_{0}^{+\infty} \frac{ds}{s^{3/2}}\, \psi^*_{s,\tau}(t)\, Y(s,\tau) \qquad \text{(Inverse)}$$

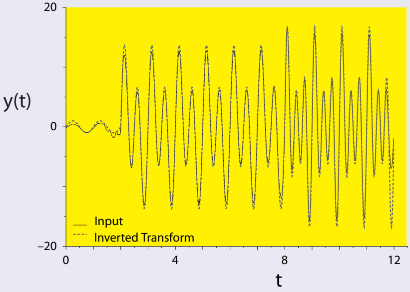


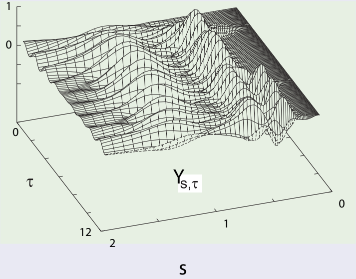
- Convolute low scale
- Cover all
- $\Rightarrow$ High res
- Expand
- $\Rightarrow$ Shape

Solution to Problem

Recall Nonstationary Signal







Required of Mother Wavelet Ψ

For Math to Work

1. $\Psi(t)$ is real
2. $\Psi(t)$ oscillates around 0 such that the average

$$\int_{-\infty}^{+\infty} \Psi(t)\, dt = 0$$

3. $\Psi(t)$ is local (wave packet) & square integrable

$$\int_{-\infty}^{+\infty} |\Psi(t)|^2\, dt < \infty$$

4. The first p moments vanish (for details):

$$\int_{-\infty}^{+\infty} t^0\, \Psi(t)\, dt = \int_{-\infty}^{+\infty} t^1\, \Psi(t)\, dt = \cdots = \int_{-\infty}^{+\infty} t^{p-1}\, \Psi(t)\, dt = 0$$

Implementation: Visualizing Wavelet Transforms

Example

1. Convert your DFT program to a CWT one.
2. Examine different mother wavelets. Write methods for
    1. a Morlet wavelet
    2. a Mexican hat wavelet
    3. a Haar wavelet
3. Test your transform on input:
    1. $y(t) = \sin 2\pi t$,
    2. $y(t) = 2.5 \sin 2\pi t + 6 \sin 4\pi t + 10 \sin 6\pi t$,
    3. The nonstationary signal for our problem:

$$y(t) = \begin{cases} \sin 2\pi t, & \text{for } 0 \le t \le 2, \\ 5 \sin 2\pi t + 10 \sin 4\pi t, & \text{for } 2 \le t \le 8, \\ 2.5 \sin 2\pi t + 6 \sin 4\pi t + 10 \sin 6\pi t, & \text{for } 8 \le t \le 12. \end{cases}$$

4. Invert your CWT & compare to input.