Discrete Nonlinear Dynamics; Bugs

A Success Story of Computational Science
(solitons, chaos, fractals)

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Based on *A Survey of Computational Physics* by Landau, Páez, & Bordeianu

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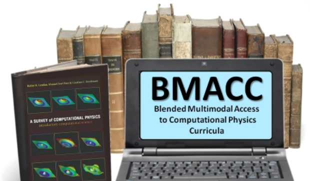
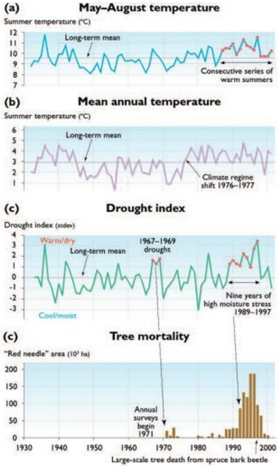
Course: **Computational Physics II**



Problem: Why Is Nature So Complicated?

- Insect populations, weather patterns
- Complex behavior
- Stable, periodic, chaotic, stable, ...
- **Problem:** can a simple, discrete law produce such complicated behavior?



Model Realistic Problem: Bug Cycles

Bugs Reproduce Generation after Generation = i

- $N_0 \;\rightarrow\; N_1, N_2, \ldots N_\infty$
- $N_i = f(i)$?
- Seen discrete law,

$$\frac{\Delta N}{\Delta t} = -\lambda N$$

$$\Rightarrow \simeq e^{-\lambda t}$$

- $-\lambda \rightarrow +\lambda \;\Rightarrow$ growth

Refine Model: Maximum Population N_*

Incorporate Carrying Capacity into Rate

- Assume breeding rate proportional to number of bugs:

$$\frac{\Delta N_i}{\Delta t} = \lambda \, N_i$$

- Want growth rate $\downarrow$ as $N_i \to N_*$
- Assume $\lambda = \lambda'(N_* - N_i)$

$$\Rightarrow \quad \frac{\Delta N_i}{\Delta t} = \lambda'(N_* - N_i)N_i \qquad \text{(Logistic Map)}$$

- Small N_i/N_* $\Rightarrow$ exponential growth
- $N_i \to N_*$ $\Rightarrow$ slow growth, stable, decay

Logistic as Map in Dimensionless Variables

As Population, Change Variables

$$N_{i+1} = N_i + \lambda' \Delta t (N_* - N_i) N_i \quad (1)$$

$$x_{i+1} = \mu x_i (1 - x_i) \qquad \text{(Logistic Map)} \quad (2)$$

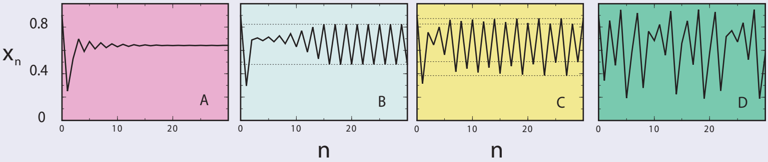
$$\mu \stackrel{\text{def}}{=} 1 + \lambda' \Delta t N_*, \qquad x_i \stackrel{\text{def}}{=} \frac{\lambda' \Delta t}{\mu} N_i \simeq \frac{N_i}{N_*} \quad (3)$$

$$x_i \simeq \frac{N_i}{N_*} = \text{fraction of max} \quad (4)$$

- $0 \leq x_i \leq 1$
- Map: $x_{i+1} = f(x_i)$
- Quadratic, 1-D map
- $f(x) = \mu x (1 - x)$

Properties of Nonlinear Maps (Theory)

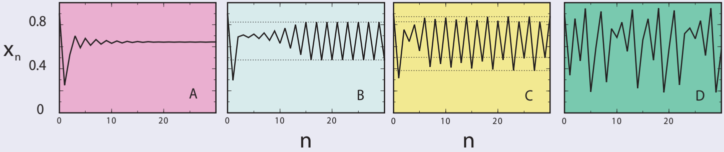
Empirical Study: Plot x_i vs i



- A: $\mu = 2.8$, equilibration into single population
- B: $\mu = 3.3$, oscillation between 2 population levels
- C: $\mu = 3.5$ oscillation among 4 levels
- D: chaos

Fixed Points

x_i Stays at x_* or Returns



$$x_{i+1} = \mu x_i(1 - x_i) \tag{5}$$

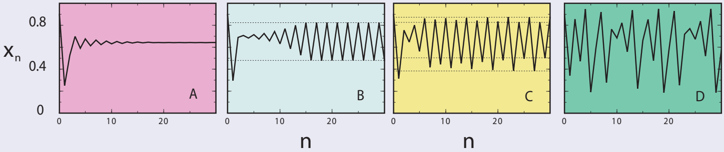
- One-cycle: $x_{i+1} = x_i = x_*$

$$\mu x_*(1 - x_*) = x_* \tag{6}$$

$$\Rightarrow \quad x_* = 0, \quad x_* = \frac{\mu - 1}{\mu} \tag{7}$$

Period Doubling, Attractors

Unstable via Bifurcation into 2-Cycle



- Attractors, cycle points
- Predict: same population generation i, $i+2$

$$x_i = x_{i+2} = \mu x_{i+1}(1 - x_{i+1}) \quad \Rightarrow \quad x_* = \frac{1 + \mu \pm \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$$

- $\mu > 3$: real solutions
- Continues $1 \rightarrow 2$ populations

Exercise 1

Produce sequence x_i

1. Confirm behavior patterns A, B, C, D
2. Identify the following:

Transients

Asymptotes

Extinction

Stable states

Multiple cycles

Four-cycle

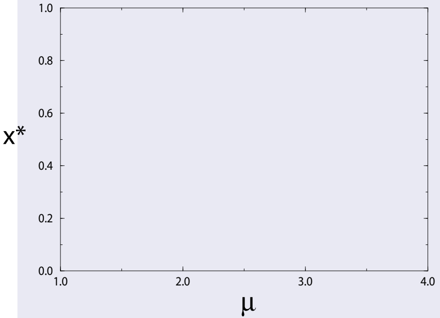
Intermittency $3.8264 < \mu < 3.8304$

Chaos deterministic irregularity; hypersensitivity

$\Rightarrow$ nonpredictable, $\mu = 4,\ 4(1 - \epsilon)$

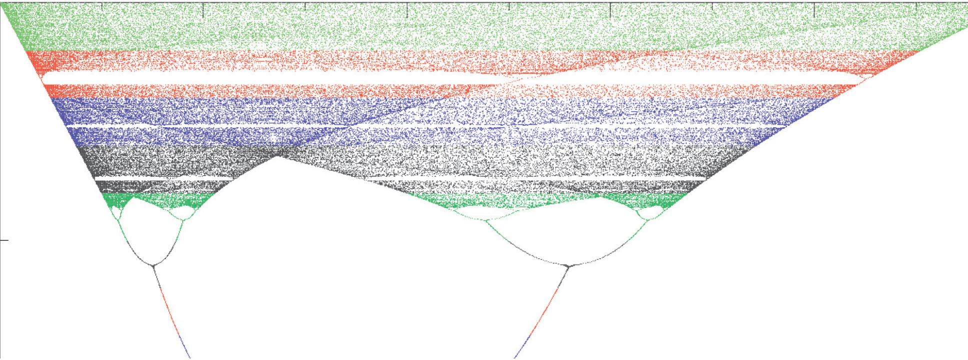
Bifurcation Diagram (Assessment)

Concentrate on Attractors



- Simplicity in chaos
- Attractors as $f(\mu)$
- Scan x_0, μ
- Let transients die
- Output $(\mu, x_*)s$
- n cycle = n values
- See enlargements

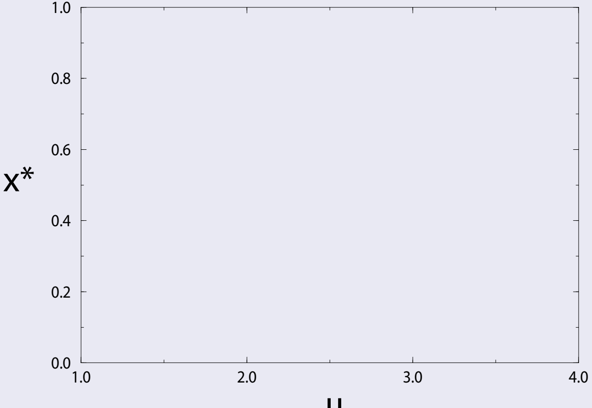
Detailed Bifurcation Diagram



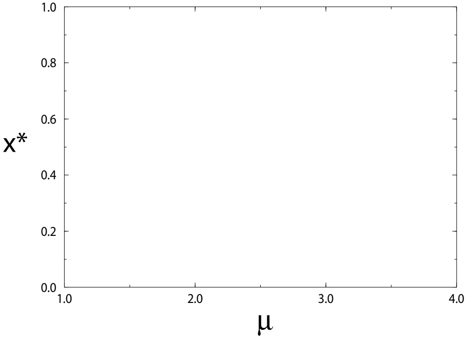
Bifurcation Diagram Sonification

Play Bifurcation Diagram

- Hear each bifurcation
- Each branch = one ω
- $\omega \propto x^*$
- Bifurcation = new ω, cord



Exercise 2: Bifurcation Diagram



- Can't vary intensity
- Vary point density
- Resolution $\sim$ 300 DPI
- $3000 \times 3000 \simeq 10^7$ pts
- Big, more = waste
- Create 1000 bins
- $1 \leq \mu \leq 4$
- Print x_* 3-4 decimal places
- Remove duplicates
- Enlarge: self-similarity
- Observe windows

Summary & Conclusion

Simplicity & Beauty within Chaos

- Yes, simple discrete maps can lead to complexity
- Models of real world complexity
- Complexity related to nonlinearity (x^2)
- Computation crucial for nonlinear systems
- Signals of simplicity, chaos

  Bifurcation Diagram

  Feigenbaum Constants

  Lyapunov Coefficients

  Shannon Entropy

  Fractal Dimension