Critical Tokunaga Branching Processes

Yevgeniy Kovchegov
Oregon State University

Joint work with
Ilya Zaliapin
University of Nevada, Reno
**Introduction.**

Let $T$ be a random rooted tree that is also reduced, i.e., has no vertices of degree two.

**Horton pruning:** removes the leaves and their parental edges from $T$, followed by series reduction (removing each degree-two non-root vertex by merging its adjacent edges into one).

**Horton pruning** induces the **Horton-Strahler orders**.

The **Horton-Strahler order** of $T$ is defined as the minimal number of Horton prunings necessary to eliminate the tree $T$.

We let $N_k[T]$ denote the number of branches of order $k$ in $T$. 
tree $T$

ROOT

ROOT
$N_1 = 15$ branches of order one (leaves).

$N_2 = 3$ branches of order two

$N_3 = 1$ branches of order three

Horton Law: $\frac{N_k}{N_1} = R^{1-k}$

Horton exponent: $R > 2$

Horton-Strahler order of the tree = 3
Critical Tokunaga Branching Processes

Critical Tokunaga process.

For parameters $\gamma > 0$ and $c > 1$, a continuous-time multi-type branching process $S(t)$ is a critical Tokunaga process if

- It starts ($t = 0$) with a single progenitor, whose Horton-Strahler order is $K \geq 1$ with probability $2^{-K}$.

- A branch of order $j \leq K$ produces offspring (side branches) of every order $i < j$ with rate $\gamma (c - 1) c^{-i}$.

- A branch of order $j$ terminates with rate $\gamma c^{1-j}$.

- At its termination time, a branch of order $j \geq 2$ splits into two independent branches of order $j - 1$.

- A branch of order $j = 1$ terminates without leaving offspring.

We write $S(t) \overset{d}{\sim} S_{\text{Tok}}(t; c, \gamma)$.

**Theorem.** For $c = 2$, $S(t) \overset{d}{\sim} S_{\text{Tok}}(t; 2, \gamma)$ is a continuous-time critical binary Galton-Watson process with intensity $\gamma$. 
• It starts \((t = 0)\) with a single progenitor, whose Horton-Strahler order is \(K \geq 1\) with probability \(2^{-K}\).

• A branch of order \(j \leq K\) produces offspring (side branches) of every order \(i < j\) with rate \(\gamma (c - 1) c^{-i}\).

• A branch of order \(j\) terminates with rate \(\gamma c^{1-j}\).

• At its termination time, a branch of order \(j \geq 2\) splits into two independent branches of order \(j - 1\).

• A branch of order \(j = 1\) terminates without leaving offspring.
Critical Tokunaga process.

Critical Tokunaga processes satisfy a number of self-similarity and invariance properties as observed in the following publications:


Let $\mu_K$ denote the tree measure induced by the critical Tokunaga process conditioned on having order $K$. 
A Markov tree process.

Next, we construct a discrete time Markov tree process \( \left\{ \gamma_K \right\}_{K \in \mathbb{N}} \) such that each \( \gamma_K \) is distributed as a tree induced by the critical Tokunaga process conditioned on having order \( K \), i.e. \( \gamma_K \overset{d}{\sim} \mu_K \).

Let \( X_K = N_1[\gamma_K] \) (number of leaves) and \( Y_K = \text{length}(\gamma_K) \).

- \( \gamma_1 \) is I-shaped tree of order one, with \( X_1 = 1 \) and \( Y_1 \overset{d}{\sim} \text{Exp}(\gamma) \).
- Conditioned on \( \gamma_K \), tree \( \gamma_{K+1} \) is obtained as follows:

  1. Obtain \( \gamma'_K \) by multiplying the edge lengths in \( \gamma_K \) by \( c \), while preserving the combinatorial shape.
  2. Attach new leaf edges to \( \gamma'_K \) at the points sampled with a homogeneous Poisson point process with intensity \( \gamma(c - 1)c^{-1} \) along the carrier space \( \gamma'_K \).
  3. Attach a pair of new leaf edges to each of the leaves in \( \gamma'_K \).

The lengths of all the newly attached leaf edges are i.i.d. exponential random variables with parameter \( \gamma \).
- $\mathcal{T}_1$ is I-shaped tree of order one, with $X_1 = 1$ and $Y_1 \sim \text{Exp}(\gamma)$.

- Conditioned on $\mathcal{T}_K$, tree $\mathcal{T}_{K+1}$ is obtained as follows:

  1. Obtain $\mathcal{T}'_K$ by multiplying the edge lengths in $\mathcal{T}_K$ by $c$, while preserving the combinatorial shape.
  2. Attach new leaf edges to $\mathcal{T}'_K$ at the points sampled with a homogeneous Poisson point process with intensity $\gamma(c - 1)c^{-1}$ along the carrier space $\mathcal{T}'_K$.
  3. Attach a pair of new leaf edges to each of the leaves in $\mathcal{T}'_K$.

  The lengths of all the newly attached leaf edges are i.i.d. exponential random variables with parameter $\gamma$. 

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\[ \mathcal{T}_1 \] \[ \mathcal{T}'_1 \] \[ \mathcal{T}_2 \] \[ \mathcal{T}'_2 \] \[ \ldots \]
**Proving the Strong Horton Law via Martingales.**

We prove the **Strong Horton Law** with Horton exponent $R = 2c$.

**Lemma.** The sequence

$$M_K = R^{1-K} \left( X_K + \gamma(c - 1)Y_K \right) \text{ with } K \in \mathbb{N}$$

is a martingale with respect to the Markov tree process $\{\gamma_K\}_{K \in \mathbb{N}}$.

**Theorem.** Suppose $S^{Tok}(t; c, \gamma)$ is the distribution of a critical Tokunaga process and $\{\gamma_K\}_{K \in \mathbb{N}}$ is the corresponding Markov tree process. Then,

$$\frac{N_k[\gamma_K]}{N_1[\gamma_K]} \xrightarrow{a.s.} R^{1-k} \quad \text{as } K \to \infty.$$ 

Recall that $\mu_K$ denotes the tree measure induced by the critical Tokunaga process conditioned on having order $K$.

**Strong Horton law for branch numbers.** For any $\epsilon > 0$,

$$\mu_K \left( \left| \frac{N_k[T]}{N_1[T]} - R^{1-k} \right| > \epsilon \right) \to 0 \quad \text{as } K \to \infty.$$
Proving the Strong Horton Law via Martingales.

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