Critical Tokunaga Branching Processes

Yevgeniy Kovchegov
Oregon State University

Joint work with
Ilya Zaliapin
University of Nevada, Reno

Introduction.

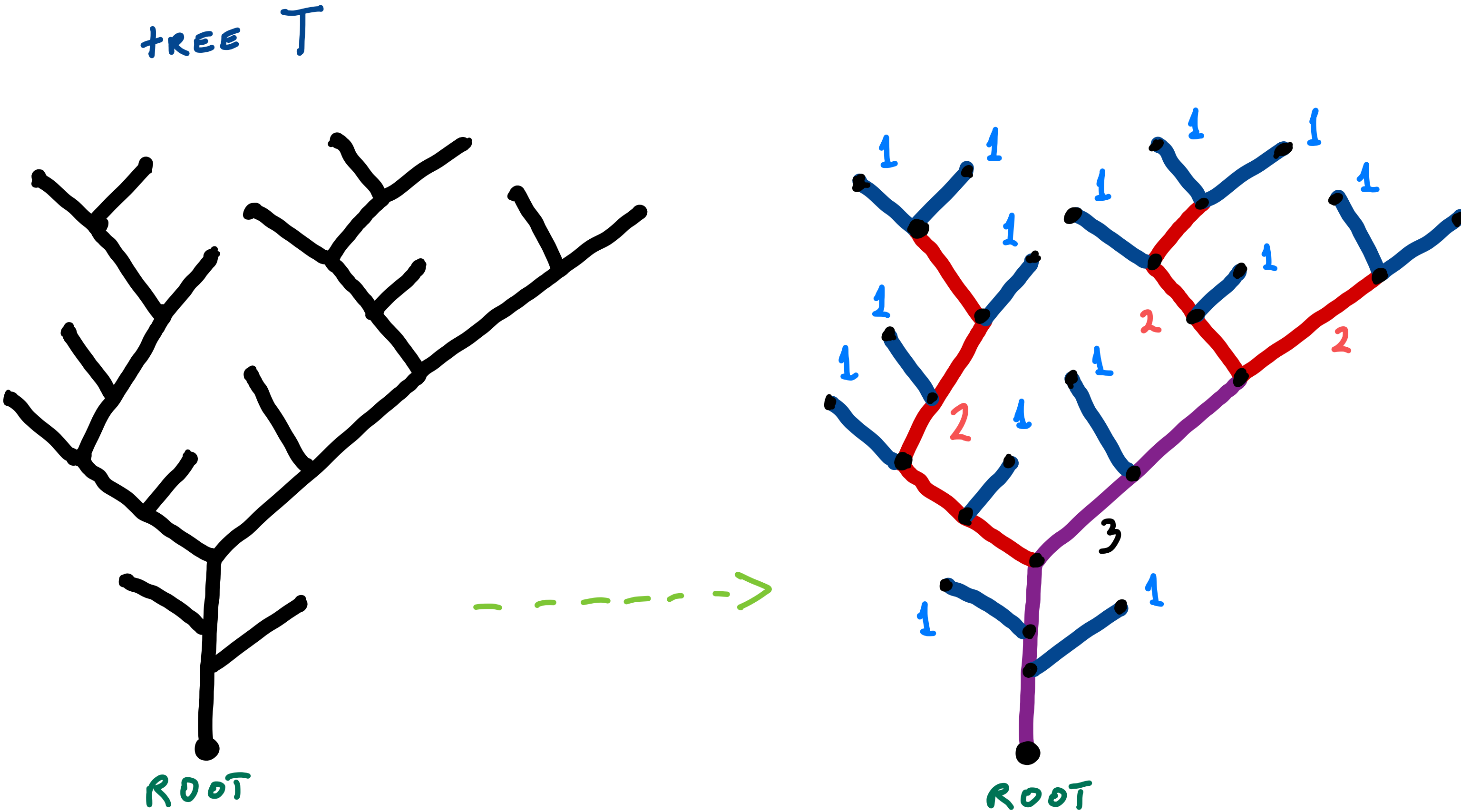
Let T be a random rooted tree that is also reduced, i.e., has no vertices of degree two.

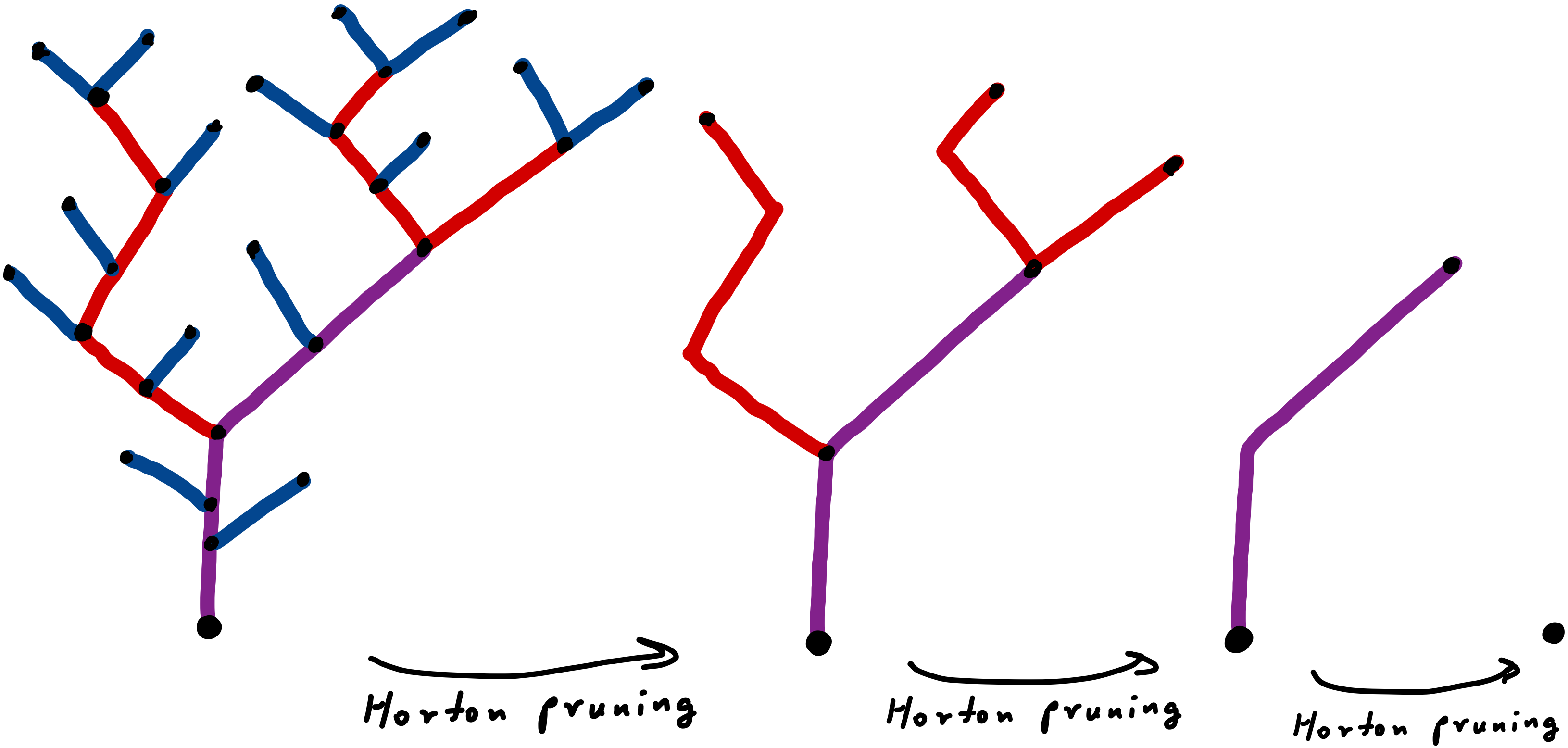
Horton pruning: removes the leaves and their parental edges from T, followed by series reduction (removing each degree-two non-root vertex by merging its adjacent edges into one).

Horton pruning induces the Horton-Strahler orders.

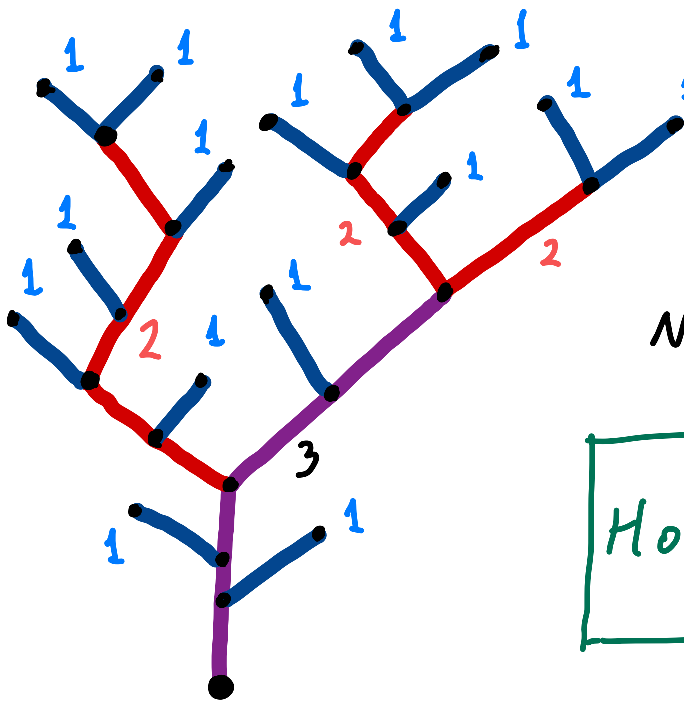
The Horton-Strahler order of T is defined as the minimal number of Horton prunings necessary to eliminate the tree T.

We let $N_k[T]$ denote the number of branches of order k in T.


tree T
ROOT
1
2
3
ROOT



Horton pruning
Horton pruning
Horton pruning



$N_1 = 15$ BRANCHES OF ORDER ONE (LEAVES).

$N_2 = 3$ BRANCHES OF ORDER TWO

$N_3 = 1$ BRANCHES OF ORDER THREE

Horton Law: $\frac{N_k}{N_1} \simeq R^{1-k}$

Horton exponent: $R \geq 2$

Horton-Strahler ORDER OF THE TREE = 3

Critical Tokunaga process.

For parameters $\gamma > 0$ and $c > 1$, a continuous-time multi-type branching process $S(t)$ is a critical Tokunaga process if

• It starts ($t = 0$) with a single progenitor, whose Horton-Strahler order is $K \geq 1$ with probability 2^{-K}.

• A branch of order $j \leq K$ produces offspring (side branches) of every order $i < j$ with rate $\gamma\,(c-1)\,c^{-i}$.

• A branch of order j terminates with rate $\gamma\,c^{1-j}$.

• At its termination time, a branch of order $j \geq 2$ splits into two independent branches of order $j-1$.

• A branch of order $j = 1$ terminates without leaving offspring.

We write $S(t) \overset{d}{\sim} S^{\mathsf{Tok}}(t; c, \gamma)$.

Theorem. For $c = 2$, $S(t) \overset{d}{\sim} S^{\mathsf{Tok}}(t; 2, \gamma)$ is a continuous-time critical binary Galton-Watson process with intensity γ.

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- A branch of order $j \leq K$ produces offspring (side branches) of every order $i < j$ with rate $\gamma\,(c-1)\,c^{-i}$.
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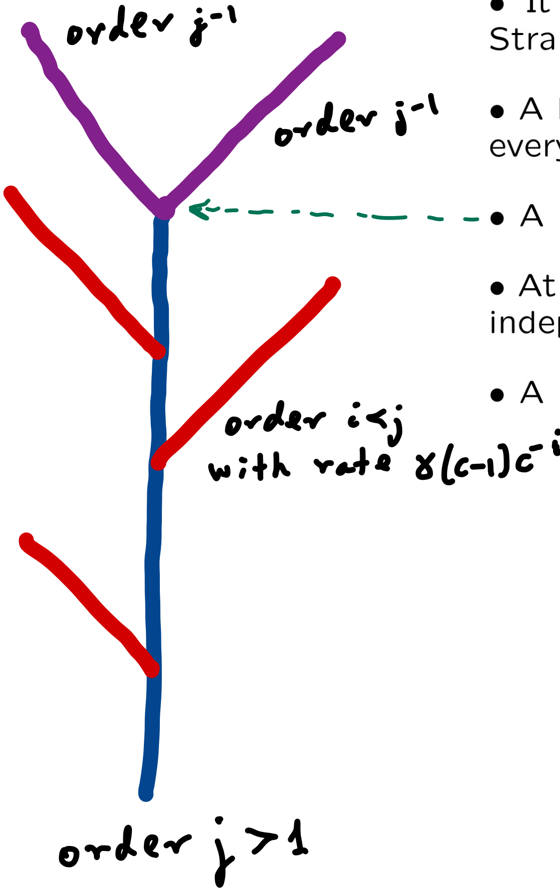

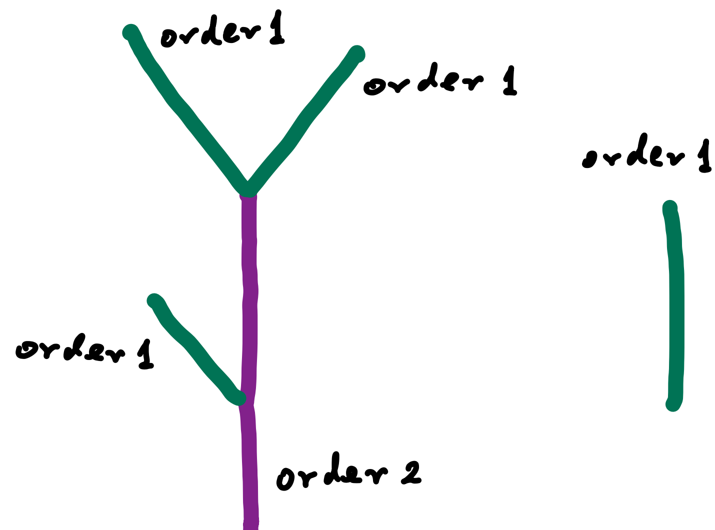

Critical Tokunaga process.

Critical Tokunaga processes satisfy a number of self-similarity and invariance properties as observed in the following publications:

- Y. K. and Ilya Zaliapin, "Random Self-Similar Trees: A mathematical theory of Horton laws" Probability Surveys Vol. 17 (2020), 1–213

- Y. K. and Ilya Zaliapin, "Random self-similar trees and a hierarchical branching process" Stochastic Processes and their Applications Vol. 129, Issue 7 (2019), 2528–2560

- Y. K. and Ilya Zaliapin, "Tokunaga self-similarity arises naturally from time invariance" Chaos Vol. 28, 041102 (2018)

Let μ_K denote the tree measure induced by the critical Tokunaga process conditioned on having order K.

A Markov tree process.

Next, we construct a discrete time Markov tree process $\left\{\Upsilon_K\right\}_{K\in\mathbb{N}}$ such that each Υ_K is distributed as a tree induced by the critical Tokunaga process conditioned on having order K, i.e. $\Upsilon_K \stackrel{d}{\sim} \mu_K$.

Let $X_K = N_1[\Upsilon_K]$ (number of leaves) and $Y_K =$ length(Υ_K).

- Υ_1 is I-shaped tree of order one, with $X_1 = 1$ and $Y_1 \stackrel{d}{\sim}$ Exp(γ).

- Conditioned on Υ_K, tree Υ_{K+1} is obtained as follows:

(1) Obtain Υ'_K by multiplying the edge lengths in Υ_K by c, while preserving the combinatorial shape.
(2) Attach new leaf edges to Υ'_K at the points sampled with a homogeneous Poisson point process with intensity $\gamma(c-1)c^{-1}$ along the carrier space Υ'_K.
(3) Attach a pair of new leaf edges to each of the leaves in Υ'_K.

The lengths of all the newly attached leaf edges are i.i.d. exponential random variables with parameter γ.

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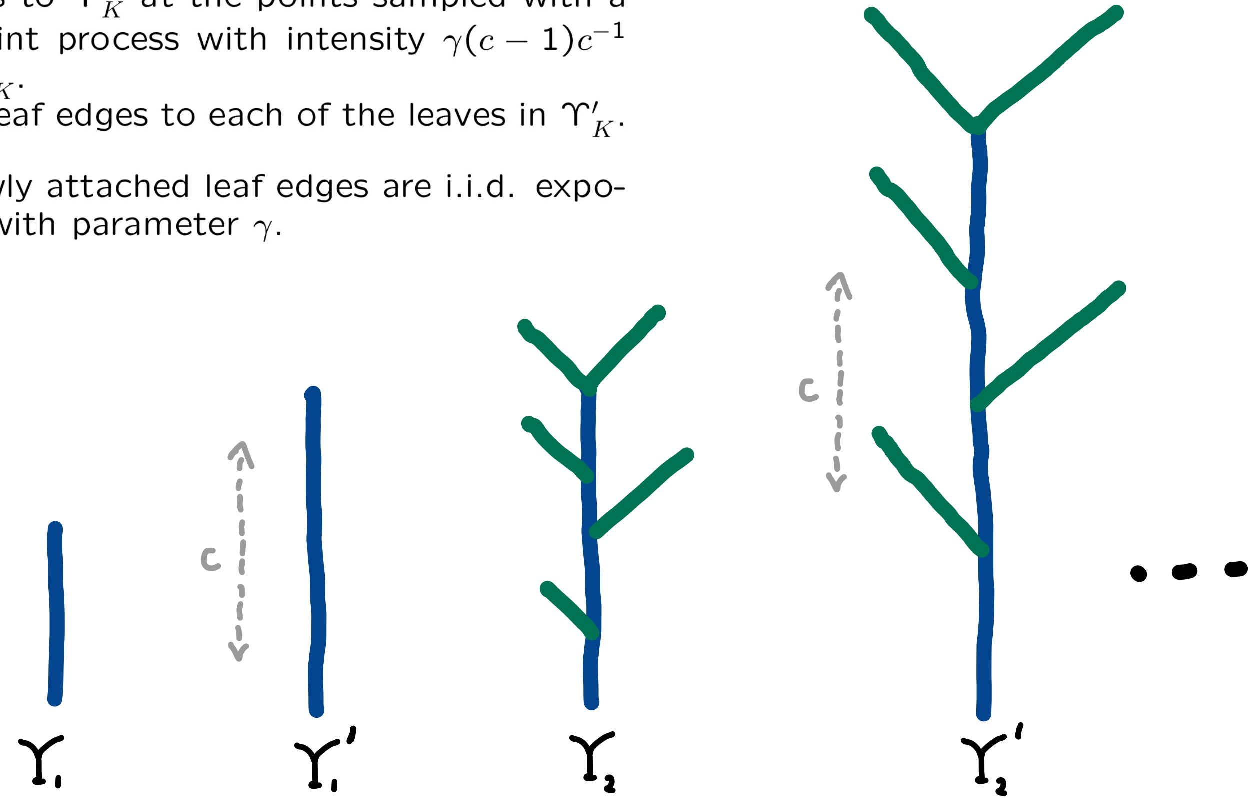

Proving the Strong Horton Law via Martingales.

We prove the Strong Horton Law with Horton exponent $R = 2c$.

Lemma. The sequence

$$M_K = R^{1-K}\left(X_K + \gamma(c-1)Y_K\right) \text{ with } K \in \mathbb{N}$$

is a martingale with respect to the Markov tree process $\left\{\Upsilon_K\right\}_{K\in\mathbb{N}}$.

Theorem. Suppose $S^{\mathsf{Tok}}(t;c,\gamma)$ is the distribution of a critical Tokunaga process and $\left\{\Upsilon_K\right\}_{K\in\mathbb{N}}$ is the corresponding Markov tree process. Then,

$$\frac{N_k[\Upsilon_K]}{N_1[\Upsilon_K]} \overset{a.s.}{\to} R^{1-k} \quad \text{as } K \to \infty.$$

Recall that μ_K denotes the tree measure induced by the critical Tokunaga process conditioned on having order K.

Strong Horton law for branch numbers. For any $\epsilon > 0$,

$$\mu_K\left(\left|\frac{N_k[T]}{N_1[T]} - R^{1-k}\right| > \epsilon\right) \to 0 \quad \text{as} \quad K \to \infty.$$

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