BROWNIAN BRIDGE, PERCOLATION AND RELATED PROCESSES

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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Abstract

We introduce and solve a complex of problems for various statistical mechanics processes conditioned on arriving to a faraway point from the origin. We derive a Brownian bridge asymptotics for percolation, self-avoiding walks and some types of random walk models. We outline a number of possible future directions for the development in the field.

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Contents

Chapter 1

Introduction

The material presented in this thesis can be viewed as yet another step in the development of the techniques used to study rare events in a wide variety of statistical mechanical systems. Here we concentrate on the subcritical bond percolation model, self-avoiding walks and some forms of a random walk. We want to mention some possible approaches used to study rare events in percolation, self-avoiding walks, Ising and related models:

• For the bond percolation (and $q \geq 1$ Fortuin-Kasteleyn random-cluster models with implications in Potts and Ising models), the finite clusters of large size ("volume") were studied for dimensions $d \geq 3$ via renormalization techniques (the so called 'slab technology') and optimization of the cluster "surface area". As a result some neat large deviation estimates were produced for near-critical percolation probabilities in [26] as a refinement of the techniques used in [19] and [5].

• In higher dimensions some very precise assymptotics were produced using lace expansion techniques for the self-avoiding walks (dimension $d \geq 5$) and percolation (dimension $d \geq 19$) models. For more details we refer to [13], [14], [24] and references therein.

• At very high temperatures the perturbation techniques were used in Ising and related models. See, for example, [25].

• An elaborate methodology, crucial for this thesis, was produced in the process of derivation of Ornstei-Zernike limiting behavior for various statistical mechanical systems in [8], [2], [4], [16], [6], [7] and other related research manuscripts (for more on the above techniques, see chapter 5).

We think it is important to mention the result of [15] studying some of the statistical properties of the phase separation line in the 2D low-temperature Ising model, producing a Brownian bridge asymptotics (after scaling). This result, though in nature similar to the results of this thesis was produced using a significantly different approach.

In the chapters to follow, we first briefly go over basic definitions and results concerning the notions of Brownian bridge, percolation and self-avoiding walks (chapters 2, 3 and 4). In chapter 5 we introduce the reader to the methods used in obtaining the results of this thesis as well as the history of some of them. We formulate the main results of the manuscript and outline the strategy of some of the proofs from the perspective of the latest developments in the theory of regeneration structures in statistical mechanical models. There, we state the technical theorem of chapter 8 together with its "directed" random walk interpretation. The technical theorem is to be used in chapters 6 and 7 as part of the derivation of the main results, producing the Brownian bridge asymptotics for the subcritical percolation and self-avoiding walks models. We prove the technical theorem in chapter 8. We conclude the thesis with chapter 9 outlining the latest developments in the field explored inhere and stating some possible directions for the future research on the subject.

Chapter 2

Introduction to Brownian Bridge

A Brownian bridge (or tied-down Brownian Motion) is defined as a sample-continuous Gaussian process B^0 on $[0, 1]$ with mean 0 and $\mathbf{E}B_s^0 B_t^0 = s(1-t)$ for $0 \le s \le t \le 1$. So, $B_0^0 = B_1^0 = 0$ a.s. Also, if B is a Brownian motion, then the process $B_t - tB_1$ $(0 \le t \le 1)$ is a Brownian bridge. For more details see [3], [9] and [10]. In a more general setting, we call the process $B_t^{0,\vec{\mathbf{a}}} \equiv B_t^0 + t\vec{\mathbf{a}}$ "a Brownian bridge connecting points zero and $\vec{\mathbf{a}}$".

Of all the classical theorems concerning the notion of Brownian bridge the following is the closest in spirit to the theorems proved in this thesis. Let again B_t denote a Brownian Motion, and B_t^0 denote the Brownian bridge.

Theorem 1. *The probability distribution $P_\varepsilon(A) \equiv P[B \in A \,|0 \le B_1 \le \varepsilon]$ (where A is in $\mathcal{C}$, the σ-algebra on $C[0, 1]$ corresponding to the $\|\cdot\|_\infty$ norm) converges weakly to B_0, as $\varepsilon \downarrow 0$, e.g. $P_\varepsilon \Rightarrow B^0$.*

The proof of the theorem can be found in [3]. We refer the reader to appendix B for the notion of weak convergence, and corresponding theorems. We notice that the theorem practically implies that the Brownian bridge B^0 behaves like a Winer path B conditioned by the requirement that $B_1 = 0$.

Chapter 3

Percolation

Here we summarize the basic notions defining the Bernoulli bond percolation model as they were elaborately presented in [18] and [12]:

For each edge of the d-dimensional square lattice $\mathbb{Z}^d$ in turn, we declare the edge *open* with probability p and *closed* with probability $1-p$, independently of all other edges. If we delete the closed edges, we are left with a random subgraph of $\mathbb{Z}^d$. A connected component of the subgraph is called a "cluster", and the number of edges in a cluster is the "size" of the cluster. The probability $\theta(p)$ that the point $(0,0)$ belongs to a cluster of an infinite size is zero if $p = 0$, and one if $p = 1$. However, there exists a critical probability $0 < p_c < 1$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$. In the first case, we say that we are dealing with a *subcritical* percolation model, and in the second case, we say that we are dealing with a *supercritical* percolation model. The critical probability for dimension $d = 2$ was proved to be equal to one half in [17] using dual lattice techniques.

We say that two vertices of a square lattice are "connected" to each other if they both belong to the same open cluster.

The *exponential decay* of clusters in subcritical bond percolation was established in [23], and later on enhanced in [1]:

Theorem 2. *If $p < p_c$ then $\exists\ \psi(p) > 0$ such that*

$$P_p[0 \leftrightarrow \partial S(n)] < e^{-n\psi(p)} \textit{ for all } n.$$

Here $S(n)$ is a unit ball of radius n:

$$S(n) = \{\vec{x} \in \mathbb{Z}^d : \, |x_1| + ... + |x_d| = n\}.$$

Chapter 4

Self-Avoiding Walks

In this chapter we introduce the notion of self-avoiding walks. We refer the reader to [24] for the detailed presentation of the model.

An N-step self-avoiding walk (path) ω on $\mathbb{Z}^d$, beginning at 0 is a sequence of sites: $\omega(0) = 0, \omega(1), ..., \omega(N)$ with $|\omega(j+1) - \omega(j)| = 1$ and $\omega(i) \neq \omega(j)$ for all $i \neq j$. We let c_N denote the number of N-step self-avoiding walks beginning at zero. It had been established that the limit representing the connective constant $\mu = \lim_{N\to\infty} c_N{}^{\frac{1}{N}}$ exists due to a subadditivity property of $\log c_N$ (see [24]). We also let $c_N(x,y)$ to be the number of N-step self-avoiding walks ω with $\omega(0) = x$ and $\omega(N) = y$. The two-point function $g_\beta(x,y)$ (as defined below) is an important tool in the theory of self-avoiding walks:

$$g_\beta(x,y) \equiv \sum_{N=0}^{\infty} c_N(x,y)e^{-\beta N} = \sum_{\omega:x\to y} e^{-\beta|\omega|},$$

where the second sum is taken only over all self-avoiding walks $\omega : x \to y$ on the lattice. For the simplicity of notation (due to the shift-invariance property of $c_N(x,y)$) we denote $g_\beta(x) \equiv g_\beta(0,x)$. The supercritical $\beta > \beta_c(d)$ is the one for which the equivalent sums above are finite. It was shown (see [24]) that for the supercritical β, the "bubble diagram"

$$B_d(\beta) \equiv \sum_x g_\beta(x)^2$$

is finite. The significance of the bubble diagram is discussed in Section 1.5 of [24].

Since the radius of convergence $e^{-\beta_c(d)} = \frac{1}{\mu}$, it is apparent that the two-point function decays exponentially:

$$g_\beta(0,x) \leq C_\beta e^{-c_\beta \|x\|} \tag{4.1}$$

for all $\beta > \beta_c(d)$ and some corresponding $C_\beta, c_\beta > 0$.

The notion of a "mass" of a two-point function applies here as well. The mass $m(\beta)$ is the rate of exponential decay of $g_\beta(x,y)$ in the direction of the first coordinate vector:

$$m(\beta) = \liminf_{n\to\infty} \frac{-\log g_\beta(0,(n,0,...,0))}{n}.$$

It can be shown that the lim inf above can be replaced by the limit.

Chapter 5

History of the Problem, the Results and the Methods Used

The technique used in the proofs originates from the methodology developed in the process of establishing a precise Ornstein-Zernike decay for a variety of spin systems and lattice field theories and the development of the renewal theory. It turned out that the technique developed by Ornstein and Zernike in 1914 for the case of the classical fluid can be implemented in many classical models of statistical mechanics (self-avoiding walks, percolation, 2D Ising model and many other spin systems) for all noncritical temperatures. For this, for the given two-point function, one needs to construct a "direct correlation function" with a strictly greater rate of decay. This approach was implemented in in the case of the d-dimensional self-avoiding walks [8] giving the precise Ornstein-Zernike behavior of the two-point function $g_\beta(0, n\tilde{\mathbf{a}})$ connecting the origin to a point on an axis (the case $\tilde{\mathbf{a}} = (\|\tilde{\mathbf{a}}\|, 0, ..., 0)$) for all non-critical β. There the so called "mass gap" condition (or separation of mass)is proved. In that case, the two-point function with a different rate of decay is the generating function corresponding to the self-avoiding walks with all non-trivial (more than one) intersections with the hyper spaces $\{x_1 = c\}$ situated in between the origin and the destination point. The work of proving the Ornstein-Zernike behavior (the coefficient of order $\|x\|^{\frac{d-1}{2}}$ near the decay exponent of the two-point function) was completed in [16] for any supercritical value of the parameter $\beta > \beta_c(d)$. There the complete precise asymptotics (4) of the decay was derived in any direction $\vec{\mathbf{a}}$ as the result of an extensive study of the geometric properties of corresponding equi-decay level sets, broadening the methodology of [8].

The corresponding developments in subcritical bond percolation model followed the above advances in the theory of self-avoiding walks. In [2], [4] and [6] some similar equi-decay level sets are studied, and corresponding Ornstein-Zernike asymptotics is produced. This technique will be used in chapter 6 (see also [20]) together with the technical result of chapter 8 (and also cited below) to produce a Brownian bridge asymptotics of a scaled percolation cluster conditioned on reaching a faraway point, and also proving the shrinking of such clusters. There we consider the d-dimensional model subcritical bond percolation model ($p < p_c$) and a point $\tilde{\mathbf{a}}$ in $\mathbb{Z}^d$, conditioned on the event of zero being connected to $n\tilde{\mathbf{a}}$. We first show that a specifically chosen path connecting points zero and $n\tilde{\mathbf{a}}$ and going through some appropriately defined points on the cluster (regeneration points), if scaled $\frac{1}{n\|\tilde{\mathbf{a}}\|}$ times along $\tilde{\mathbf{a}}$ and $\frac{1}{\sqrt{n}}$ times in the direction orthogonal to $\tilde{\mathbf{a}}$, converges to Time $\times$ $(d-1)$-dimensional Brownian bridge as $n \to +\infty$, where the scaled interval connecting points zero and $n\tilde{\mathbf{a}}$ serves as a $[0, 1]$ time interval. In other words, we prove that a scaled "skeleton" going through the regeneration points of the cluster converges to Time $\times$ $(d-1)$-dimensional Brownian bridge. In a subsequent step, we show that if scaled, then the hitting area of the orthogonal hyper-planes shrinks, implying that for n large enough, all the points of the scaled cluster are within an ε-neighborhood of the points in the "skeleton". Chronologically, we would like to mention the result preceding our study (in subcritical percolation), establishing that for $\tilde{\mathbf{a}} = (1, 0, ..., 0)$, the hitting distribution of the cluster in the intermediate planes, $x_1 = tn\tilde{\mathbf{a}}$, $0 < t < 1$ obeys a multidimensional local limit theorem (see [4]). Dealing with all other $\tilde{\mathbf{a}} \neq (k, 0, ..., 0)$ became possible only after the corresponding techniques mastering the regeneration structures and geometry of equi-decay profiles was developed in [6] (and in [16] for self-avoiding walks). These techniques played a central role in obtaining these research results.

Here, we also establish a similar result for the self-avoiding walks (see also [21]). In chapter 7, we prove the weak convergence of a scaled interpolation "skeleton" going through the regeneration points (see definition 7) of a self-avoiding walk, and terminating at a faraway point $n\vec{\mathbf{a}}$ to Time$\times(d-1)$-dimensional Brownian bridge as $n \to \infty$. Later, the shrinking of the self-avoiding walk to the above interpolation skeleton is proved (see section 7.5). We prove the result for $\tilde{\mathbf{a}} = (\|\vec{\mathbf{a}}\|, 0, ..., 0)$ given an appropriate measure on such self-avoiding walks (see (7.4)). We outline the proof of the result for all other $\vec{\mathbf{a}}$ in $\mathbb{Z}^d$.

The result of chapter 8 (see section 8.2) that we cite below is to play an important role in proving the Brownian bridge asymptotics for percolation and self-avoiding walks in chapters 6 and 7. It can be also interpreted on its own as a similar result establishing a Brownian bridge asymptotics for a scaled "directed" random walk, conditioned on arriving to a faraway point $n\vec{\mathbf{a}}$, where by a directed random walk we mean a random walk in which the steps $\{\zeta_i\}$ are i.i.d. and the probability $P[\zeta_i \cdot \vec{\mathbf{a}} > 0] = 1$.

Let $X_1, X_2, ...$ be i.i.d. random variables on $\mathbb{Z}^d$ with the span of the lattice distribution equal to one (see [10], section 2.5), and let there be a $\bar{\lambda} > 0$ such that the moment-generating function

$$\mathbf{E}[e^{\theta \cdot X_1}] < \infty$$

for all $\theta \in B_{\bar{\lambda}}$.

Now, for a given vector $\tilde{\mathbf{a}} \in \mathbb{Z}^d$, let $X_1 + ... + X_i = [t_i, Y_i]_f \in \mathbb{Z}^d$ when written in the new orthonormal basis such that $\tilde{\mathbf{a}} = [\|\tilde{\mathbf{a}}\|, 0]_f$ (in the new basis $[\cdot, \cdot]_f \in \mathbb{R} \times \mathbb{R}^{d-1}$). Also let $P[\tilde{\mathbf{a}} \cdot X_i] > 0] = 1$. We define the process $[t, Y^*_{n,k}(t)]_f$ to be the interpolation of 0 and $[\frac{1}{n\|\tilde{\mathbf{a}}\|} t_i, \frac{1}{\sqrt{n}} Y_i]_f^{i=0,1,...,k}$, in Section 2.2 we will show that

Technical Theorem. *The process*

$$\{Y^*_{n,k} \text{ for some } k \text{ such that } [t_k, Y_k]_f = n\tilde{\mathbf{a}}\} \tag{5.1}$$

conditioned on the existence of such k converges weakly to the Brownian bridge (of variance that depends only on the law of X_1).

Chapter 6

The Result in Subcritical Percolation

In this chapter we work only with subcritical percolation probabilities $p < p_c$.

6.1 Preliminaries

Here we briefly go over the definitions that one can find in Section 4 of [6].
We start with the inverse correlation length $\xi_p(\vec{x})$:

$$\xi_p(\vec{x}) \equiv - \lim_{n\to\infty} \frac{1}{n} \log P_p(0 \leftrightarrow [n\vec{x}]),$$

where the limit is always defined due to the FKG property of the Bernoulli bond percolation (see [12]). Now, $\xi_p(\vec{x})$ is the support function of the compact convex set

$$\mathbf{K}^p \equiv \bigcap_{\vec{n}\in\mathbb{S}^{d-1}} \{\vec{r} \in \mathbb{R}^d : \vec{r}\cdot\vec{n} \le \xi_p(\vec{n})\},$$

with non-empty interior $\mathrm{int}\{\mathbf{K}^p\}$ containing point zero.
Let $\tilde{\mathbf{r}} \in \partial\mathbf{K}^p$, and let $\vec{e}$ be a basis vector such that $\vec{e}\cdot\tilde{\mathbf{r}}$ is maximal. For $\vec{x}, \vec{y} \in \mathbb{Z}^d$ define

$$S^r_{\vec{x},\vec{y}} \equiv \{\vec{z} \in \mathbb{R}^d | \tilde{\mathbf{r}}\cdot\vec{x} \le \tilde{\mathbf{r}}\cdot\vec{z} \le \tilde{\mathbf{r}}\cdot\vec{y}\}.$$

Note that $S^r_{\vec{x},\vec{y}} = \emptyset$ if $\tilde{\mathbf{r}} \cdot \vec{y} < \tilde{\mathbf{r}} \cdot \vec{x}$.
Let $\mathbf{C}^r_{\vec{x},\vec{y}}$ denote the corresponding common open cluster of x and y when we run the percolation process on $S^r_{\vec{x},\vec{y}} \bigcap \mathbb{Z}^d$. Let also Δ_r be the set of all basis vectors orthogonal to $\vec{\mathbf{r}}$, and their negatives. For the simplicity of notations (avoiding writing $(1-p)^{|\Delta_r|}$ coefficient) in the future, we restrict ourself to the case when vector $\vec{\mathbf{r}}$ has all non-zero coefficients (e.g. $|\Delta_r| = 0$).

Definition 1. For $\vec{x}, \vec{y} \in \mathbb{Z}^d$ lets define h_r-connectivity $\{\vec{x} \xleftarrow{h_r}\!\!\rightarrow \vec{y}\}$ of $\vec{x}$ and $\vec{y}$ to be the event that
1. $\vec{x}$ and $\vec{y}$ are connected in the restriction of the percolation configuration to the slab $S^r_{\vec{x},\vec{y}}$.
2. If $\vec{x} \neq \vec{y}$, then $\mathbf{C}^r_{\vec{x},\vec{y}} \bigcap S^r_{\vec{x},\vec{x}+\vec{e}} = \{\vec{x}, \vec{x}+\vec{e}\}$ and $\mathbf{C}^r_{\vec{x},\vec{y}} \bigcap S^r_{\vec{y}-\vec{e},\vec{y}} = \{\vec{y}-\vec{e}, \vec{y}\}$.

Set

$$h_r(\vec{x}) \equiv P_p[0 \xleftarrow{h_r}\!\!\rightarrow \vec{x}]$$

and $h_r(0) = 1$.

Definition 2. For $\vec{x}, \vec{y} \in \mathbb{Z}^d$ lets define f_r-connectivity $\{\vec{x} \xleftarrow{f_r}\!\!\rightarrow \vec{y}\}$ of $\vec{x}$ and $\vec{y}$ to be the event that
1. $\vec{x} \neq \vec{y}$
2. $\vec{x} \xleftarrow{h_r}\!\!\rightarrow \vec{y}$.
3. For no $\vec{z} \in \mathbb{Z}^d \setminus \{\vec{x}, \vec{y}\}$ both $\{\vec{x} \xleftarrow{h_r}\!\!\rightarrow \vec{z}\}$ and $\{\vec{z} \xleftarrow{h_r}\!\!\rightarrow \vec{y}\}$ take place.

Set

$$f_r(\vec{x}) \equiv P_p[0 \xleftarrow{f_r}\!\!\rightarrow \vec{x}]$$

and $f_r(0) = 0$.

Definition 3. Suppose $0 \xleftarrow{h_r}\!\!\rightarrow \vec{x}$, we say that $\vec{z} \in \mathbb{Z}^d$ is **a regeneration point** of $\mathbf{C}^r_{0,\vec{x}}$ if
1. $\tilde{\mathbf{r}} \cdot \vec{e} \leq \tilde{\mathbf{r}} \cdot \vec{z} \leq \tilde{\mathbf{r}} \cdot (\vec{x} - \vec{e})$
2. $S^r_{\vec{z}-\vec{e},\vec{z}+\vec{e}} \bigcap \mathbf{C}^r_{0,\vec{x}}$ contains exactly three points: $\vec{z}-\vec{e}$, $\vec{z}$ and $\vec{z}+\vec{e}$, where $\vec{e}$ is defined as before.

The following Ornstein-Zernike equality is due to be used soon:

Theorem. $\exists\ A(\cdot,\cdot)$ *on* $(0,p_c)\times \mathbf{S}^{d-1}$ *s. t.*

$$P_p[0\leftrightarrow \vec{x}] = \frac{A(p,n(\vec{x}))}{\|\vec{x}\|^{\frac{d-1}{2}}}e^{-\xi_p(\vec{x})}(1+o(1)) \tag{6.1}$$

uniformly in $\vec{x}\in\mathbb{Z}^d$, *where* $n(\vec{x})\equiv \frac{\vec{x}}{\|\vec{x}\|}$.

We refer to [6] for the proof of the theorem.

6.2 Measure $Q^r_{r_0}(x)$

It had been proved in section 4 of [6] that for a given $\tilde{\mathbf{r}}_0\in\partial\mathbf{K}^p$ there exists $\bar{\lambda}>0$ such that

$$F_{r_0}(\tilde{\mathbf{r}}) = \sum_{x\in\mathbb{Z}^d} f_{\tilde{\mathbf{r}}_0}(x)e^{\tilde{\mathbf{r}}\cdot\vec{x}} = 1 \text{ whenever } \tilde{\mathbf{r}}\in B_{\bar{\lambda}}(\tilde{\mathbf{r}}_0)\bigcap\partial\mathbf{K}^p$$

and therefore

$$Q^r_{r_0}(\vec{x})\equiv f_{r_0}(\vec{x})e^{\tilde{\mathbf{r}}\cdot\vec{x}} \text{ is a measure on } \mathbb{Z}^d.$$

Also, it was shown that

$$\mu=\mu_{r_0}(\tilde{\mathbf{r}})\equiv \mathbf{E}^r_{r_0}X = \sum_{\vec{x}\in\mathbb{Z}^d}\vec{x}Q^r_{r_0}(\vec{x}) = \nabla_r log F_{r_0}(\tilde{\mathbf{r}})\neq 0$$

and

$$F_{r_0}(\tilde{\mathbf{r}})<\infty \text{ for all } \tilde{\mathbf{r}} \text{ in } B_{\bar{\lambda}}(\tilde{\mathbf{r}}_0).$$

The later implies

$$F_{r_0}(\tilde{\mathbf{r}}) = \sum_{\vec{x}\in\mathbb{Z}^d} f_{r_0}(\vec{x})e^{\tilde{\mathbf{r}}\cdot\vec{x}} = \sum_{\vec{x}\in\mathbb{Z}^d} Q^{r_0}_{r_0}(\vec{x})e^{\theta\cdot\vec{x}}<\infty$$

for $\theta=\tilde{\mathbf{r}}-\tilde{\mathbf{r}}_0\in B_{\bar{\lambda}}(0)$,
i.e. the moment generating function $\mathbf{E}^{r_0}_{r_0}(e^{\theta\cdot X_1})$ of the law $Q^{r_0}_{r_0}$ is finite for all $\theta\in B_{\bar{\lambda}}(0)$.

Now, there is a renewal relation (see section 1 and section 4 of [6]),

$$h_{r_0}(\vec{x}) = \sum_{\vec{z}\in\mathbb{Z}^d} f_{r_0}(\vec{z})h_{r_0}(\vec{x}-\vec{z}) \text{ with } h_{r_0}(0) = 1$$

and therefore

$$h_{r_0}([N\mu]) = e^{-r\cdot[N\mu]}\sum_k \bigotimes_1^k Q^r_{r_0}(X_1 + ... + X_k = [N\mu]) \text{ for } N > 0,$$

where $X_1, X_2, ...$ is a sequence of i.i.d. random variables distributed according to $Q^r_{r_0}$, as h_{r_0}-connection is a chain of f_{r_0}-connections with junctions at the regeneration points of $\mathbf{C}^{r_0}_{0,x}$.

6.3 Important Observation

We would like the reader to notice a certain relationship between the notions of the regeneration points and that of f_{r_0}-connectivity as they were defined in section 6.1. That is for a given vector $\vec{x} \in \mathbb{Z}^d$, the event of

- $\{0 \overset{h_{r_0}}{\longleftrightarrow} \vec{x}$ with exactly one regeneration point $\vec{x}_1$ $\}$

is equivalent to the two independent events:

- $\{\ 0 \overset{f_{r_0}}{\longleftrightarrow} \vec{x}_1\ \}$,
- $\{\ \vec{x}_1 \overset{f_{r_0}}{\longleftrightarrow} \vec{x}\ \}$.

Thus the probability of the event is equal to

$$f_{r_0}(\vec{x}_1)f_{r_0}(\vec{x}-\vec{x}_1).$$

More generally, the probability P_X that $0 \overset{h_{r_0}}{\longleftrightarrow} \vec{x}$ with exactly k-1 regeneration points $\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^{k-1}\vec{x}_i$ (where $\sum_{i=1}^{k}\vec{x}_i = \vec{x}$) can be factored as following

$$
\begin{aligned}
P_X &\equiv P[0 \longleftarrow^{h_{r_0}}\longrightarrow \vec{x} \text{ ; regeneration points: } \vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^{k-1}\vec{x}_i] \\
&= P[0 \longleftarrow^{f_{r_0}}\longrightarrow \vec{x}_1]P[\vec{x}_1 \longleftarrow^{f_{r_0}}\longrightarrow \vec{x}_1+\vec{x}_2]...P[\sum_{i=1}^{k-1}\vec{x}_i \longleftarrow^{f_{r_0}}\longrightarrow \sum_{i=1}^{k}\vec{x}_i = \vec{x}] \\
&= f_{r_0}(\vec{x}_1)f_{r_0}(\vec{x}_2)...f_{r_0}(\vec{x}_k). \qquad (6.2)
\end{aligned}
$$

6.4 The Result

In this section we fix $\tilde{\mathbf{a}} \in \mathbb{Z}^d$, and let $\mathbf{r} = \mathbf{r}_0 = \tilde{\mathbf{a}}\mathbb{R}^+ \bigcap \partial\mathbf{K}^p$. Then we recall that

$$\mathbf{E}^r_{r_0}[e^{\theta \cdot X_1}] < \infty$$

for all $\theta \in B_{\bar{\lambda}}(0)$. We also denote $h(x) \equiv h_{r_0}(x)$ and $f(x) \equiv f_{r_0}(x)$.
First, we introduce a new basis $\{\vec{f}_1, \vec{f}_2, ..., \vec{f}_d\}$, where $\vec{f}_1 = \frac{\tilde{\mathbf{a}}}{\|\tilde{\mathbf{a}}\|}$. We use $[\cdot,\cdot]_f \in \mathbb{R}\times\mathbb{R}^{d-1}$ to denote the coordinates of a vector with respect to the new basis. Obviously $\tilde{\mathbf{a}} = [\|\tilde{\mathbf{a}}\|, 0]_f$. We want to prove that the process corresponding to the last $d-1$ coordinates in the new basis of the scaled ($\frac{1}{n\|\tilde{\mathbf{a}}\|}$ times along $\tilde{\mathbf{a}}$ and $\frac{1}{\sqrt{n}}$ times in the orthogonal d-1 dimensions) interpolation of regeneration points of $\mathbf{C}^{r_0}_{0,n\tilde{\mathbf{a}}}$ conditioned on $0 \longleftarrow^{h}\longrightarrow n\tilde{\mathbf{a}}$ converges weakly to the Brownian bridge $B^o(t)$ (with variance that depends only on measure $Q^r_{r_0}$) where t represents the scaled first coordinate in the new basis.

Let $X_1, X_2, ...$ be i.i.d. random variables distributed according to $Q^r_{r_0}$ law. We interpolate $0, X_1, (X_1+X_2), ..., (X_1+...+X_k)$ and scale by $\frac{1}{n\|\tilde{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}}$ along $< \tilde{\mathbf{a}} > \times < \tilde{\mathbf{a}} >^{\perp}$ to get the process $[t, Y^*_{n,k}(t)]_f$. The technical theorem (see chapter 5 or section 8.2) implies the following

Theorem 3. *The process*

$$\{Y^*_{n,k} \textit{ for some } k \textit{ such that } X_1 + ... + X_k = n\tilde{\mathbf{a}}\}$$

conditioned on the existence of such k converges weakly to the Brownian bridge (with variance that depends only on measure $Q^r_{r_0}$).

Now, let for $\vec{y}_1, ..., \vec{y}_k \in \mathbb{Z}^d$ with positive increasing first coordinates $\gamma(\vec{y}_1, ..., \vec{y}_k)$ be the last $(d-1)$ coordinates in the new basis of the scaled $(\frac{1}{n\|\tilde{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}})$ interpolation of points $0, \vec{y}_1, ..., \vec{y}_k$ (where the first coordinate is time). Notice that $\gamma(\vec{y}_1, ..., \vec{y}_k) \in C_o[0,1]^{d-1}$ as a function of scaled first coordinate whenever $\vec{y}_k = n\vec{\mathbf{a}}$. By the important observation (7.5) that we have made before, for any function $F(\cdot)$ on $C[0,1]^{d-1}$,

$$\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} F(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \textstyle\sum_{i=1}^k \vec{x}_i))$$

$$\times P[0 \overset{h_{r_0}}{\longleftrightarrow} n\vec{\mathbf{a}} \text{ ; regeneration points: } \vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^{k-1} \vec{x}_i]$$

$$= \sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} F(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) f(\vec{x}_1)...f(\vec{x}_k)$$

$$= e^{-r\cdot n\vec{\mathbf{a}}} \sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} F(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) Q^r_{r_0}(\vec{x}_1)...Q^r_{r_0}(\vec{x}_k).$$

Therefore, for any $A \subset C[0,1]^{d-1}$

$$P_p[\gamma(\text{regeneration points}, n\vec{\mathbf{a}}) \in A \mid 0 \overset{h}{\longleftrightarrow} n\vec{\mathbf{a}}]$$

$$= \frac{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} I_A(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) f(\vec{x}_1)...f(\vec{x}_k)}{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} f(\vec{x}_1)...f(\vec{x}_k)}$$

$$= \frac{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} I_A(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) Q^r_{r_0}(\vec{x}_1)...Q^r_{r_0}(\vec{x}_k)}{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} Q^r_{r_0}(\vec{x}_1)...Q^r_{r_0}(\vec{x}_k)}$$

$$= P[Y^*_{n,k} \in A \text{ for the } k \text{ such that } X_1+...+X_k = n\tilde{\mathbf{a}} \mid \exists k \text{ such that } X_1+...+X_k = n\tilde{\mathbf{a}}].$$

Hence, we have proved the following

Corollary. *The process corresponding to the last $d-1$ coordinates (in the new basis*

$\{\vec{f}_1, \vec{f}_2, ..., \vec{f}_d\}$) *of the scaled* ($\frac{1}{n\|\tilde{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}}$) *interpolation of regeneration points of* $\mathbf{C}^{r_0}_{0,n\tilde{\mathbf{a}}}$ *(where the first coordinate is time) conditioned on* $0 \longleftarrow^{h} \longrightarrow n\tilde{\mathbf{a}}$ *converges weakly to the Brownian bridge (with variance that depends only on measure* $Q^r_{r_0}$*).*

6.5 Shrinking of the Cluster and Main Theorem

Here for $\tilde{\mathbf{a}} \in \mathbb{Z}^d$ we let $\mathbf{r}_0 = \tilde{\mathbf{a}}\mathbb{R}^+ \bigcap \partial\mathbf{K}^p$ again. Before we proceed with the proof that the scaled percolation cluster $\mathbf{C}^{r_0}_{0,n\tilde{\mathbf{a}}}$ shrinks to the scaled interpolation skeleton of regeneration points, we need to prove the following

Proposition. *If* $\tilde{\mathbf{r}} = \nabla\xi_p(\tilde{\mathbf{r}}_0)$ *then* $Q^r_{r_0}$ *is a probability measure.*

Proof. First we notice that $\tilde{\mathbf{r}}_0 \cdot \tilde{\mathbf{r}} = \tilde{\mathbf{r}}_0 \cdot \nabla\xi_p(\tilde{\mathbf{r}}_0) = D_{\tilde{\mathbf{r}}_0}(\xi_p(\tilde{\mathbf{r}}_0)) = \xi_p(\tilde{\mathbf{r}}_0)$, and thus

$$H_{r_0}(\tilde{\mathbf{r}}) \equiv \sum_{\vec{x}\in\mathbb{Z}^d} h_{r_0}(x)e^{\tilde{\mathbf{r}}\cdot\vec{x}} \geq \sum_{\vec{x}\in<\tilde{\mathbf{a}}>\cap\mathbb{Z}^d} h_{r_0}(x)e^{\tilde{\mathbf{r}}\cdot\vec{x}} = \sum_{\vec{x}\in<\tilde{\mathbf{a}}>\cap\mathbb{Z}^d} h_{r_0}(x)e^{\xi_p(\vec{x})} = +\infty$$

for $d \leq 3$ by Ornstein-Zernike equation (6.1). For all other d we sum over all $\vec{x}$ inside a small enough cone around $\tilde{\mathbf{a}}$ to get $H_{r_0}(\tilde{\mathbf{r}}) = +\infty$.

Now, for all $\vec{n} \in \mathbb{S}^{d-1}$, $\vec{n} \cdot \nabla\xi_p(\tilde{\mathbf{r}}_0) = D_{\vec{n}}\xi_p(\tilde{\mathbf{r}}_0) \leq \xi_p(\vec{n})$ by convexity of ξ_p, and therefore $\tilde{\mathbf{r}} = \nabla\xi_p(\tilde{\mathbf{r}}_0) \in \partial\mathbf{K}^p$. Notice that due to the strict convexity of ξ_p and the way $\mathbf{K}^p$ was defined, $\tilde{\mathbf{r}} = \nabla\xi_p(\tilde{\mathbf{r}}_0)$ is the only point on $\partial\mathbf{K}^p$ such that $\tilde{\mathbf{r}}_0 \cdot \tilde{\mathbf{r}} = \xi_p(\tilde{\mathbf{r}}_0)$.

Now, Ornstein-Zernike equation (6.1) also implies that the sums $H_{r_0}(\tilde{r})$ and $F_{r_0}(\tilde{r})$ are finite whenever $\tilde{r} \in \alpha\mathbf{K}^p = \bigcap_{\vec{n}\in\mathbb{S}^{d-1}}\{\vec{r} \in \mathbb{R}^d : \vec{r}\cdot\vec{n} \leq \alpha\xi_p(\vec{n})\}$ with $\alpha \in (0,1)$, and due to the recurrence relation of f_{r_0} and h_{r_0} connectivity functions, $H_{r_0}(\tilde{r}) = \frac{1}{1-F_{r_0}(\tilde{r})}$ (see [6]). Therefore $F_{r_0}(\tilde{\mathbf{r}}) \equiv \sum_{\vec{x}\in\mathbb{Z}^d} f_{r_0}(x)e^{\tilde{\mathbf{r}}\cdot\vec{x}} = 1$, where the probability measure $Q^r_{r_0}$ has an exponentially decaying tail due to the same reasoning as in chapter 4 of [6] ("mass-gap" property).

□

With the help of the proposition above we shell show that the consequent regeneration points are situated relatively close to each other:

Lemma.

$$P_p[\max_i |x_i - x_{i-1}| > n^{1/3},\ x_i\text{- } reg.\ points \mid 0 \overset{h}{\longleftrightarrow} n\tilde{\mathbf{a}}] < \frac{1}{n}$$

for n large enough.

Proof. Let $\tilde{\mathbf{r}} \equiv \nabla\xi_p(\tilde{\mathbf{r}}_0) = \nabla\xi_p(\tilde{\mathbf{a}})$. Since $\xi_p(x)$ is strictly convex (see section 4 in [6]),

$$\frac{\xi_p(\tilde{\mathbf{a}}) - \xi_p(\tilde{\mathbf{a}} - \frac{\vec{x}}{n})}{(\frac{\|\vec{x}\|}{n})} < \frac{\vec{x}}{\|\vec{x}\|} \cdot \nabla\xi_p(\tilde{\mathbf{a}})$$

for $\vec{x} \in \mathbb{Z}^d$ ($\vec{x} \neq 0$), and therefore

$$\xi_p(n\tilde{\mathbf{a}}) - \xi_p(n\tilde{\mathbf{a}} - \vec{x}) = \|\vec{x}\| \frac{\xi_p(\tilde{\mathbf{a}}) - \xi_p(\tilde{\mathbf{a}} - \frac{\vec{x}}{n})}{(\frac{\|\vec{x}\|}{n})} < \vec{x} \cdot \nabla\xi_p(\tilde{\mathbf{a}}) = \tilde{\mathbf{r}} \cdot \vec{x}.$$

Thus, since $Q^r_{r_0}(x)$ decays exponentially and therefore

$$e^{\xi_p(n\tilde{\mathbf{a}}) - \xi_p(n\tilde{\mathbf{a}} - x)} < Q^r_{r_0}(x)$$

and also decays exponentially. Hence by Ornstein-Zernike result (6.1),

$$P_p[n^{1/3} < |x|,\ x\text{-first reg. point } |0 \overset{h}{\longleftrightarrow} n\tilde{\mathbf{a}}] = \sum_{n^{1/3} < |x|} f(x) \frac{h(n\tilde{\mathbf{a}} - x)}{h(n\tilde{\mathbf{a}})} < \frac{1}{n^2}$$

for n large enough. So, since the number of the regeneration points is no greater than n,

$$P_p[\max_i |x_i - x_{i-1}| > n^{1/3},\ x_i\text{- reg. points} \mid 0 \overset{h}{\longleftrightarrow} n\tilde{\mathbf{a}}] < \frac{1}{n}$$

for n large enough.

□

Now, it is really easy to check that there is a constant $\lambda_f > 0$ such that

$$f(\vec{x}) > e^{-\lambda_f \|\vec{x}\|}$$

for all $\vec{x}$ such that $f(\vec{x}) \neq 0$ (here we only need to connect points $\vec{e}$ and $\vec{x} - \vec{e}$ with two non-intersecting open paths surrounded by the closed edges), and there exists a constant $\lambda_u > 0$ such that

$$P_p[\text{ percolation cluster } \mathbf{C}(0) \not\subset [\mathbb{R}; B_R^{d-1}(0)]_f] < e^{-\lambda_u R}$$

for R large enough due to the exponential decay of the radius distribution for sub-critical probabilities (see [12]). Hence, for a given $\epsilon > 0$

$$P_p[\text{ cluster } \mathbf{C}_{0,\vec{x}}^{r_0} \not\subset [\mathbb{R}, B_{\epsilon\sqrt{n}}^{d-1}(0)]_f \mid 0 \longleftarrow^{f} \longrightarrow x] < e^{\lambda_f \|\vec{x}\| - \lambda_u \epsilon \sqrt{n}},$$

and therefore, summing over the regeneration points, we get

$$P_p[\text{ scaled cluster } \mathbf{C}_{0,n\tilde{\mathbf{a}}}^{r_0} \not\subset \epsilon\text{-neighbd. of } [0,1] \times \gamma(\text{ reg. points }) \mid 0 \longleftarrow^{g} \longrightarrow n\tilde{\mathbf{a}}]$$

$$< \frac{1}{n} + n e^{\lambda_f n^{1/3} - \lambda_u \epsilon \sqrt{n}}$$

for n large enough.

We can now state the main result of this paper:

Main Theorem (Percolation). *The process corresponding to the last $d-1$ coordinates (in the new basis $\{\vec{f}_1, \vec{f}_2, ..., \vec{f}_d\}$) of the scaled ($\frac{1}{n\|\tilde{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}}$) interpolation of regeneration points of $\mathbf{C}_{0,n\tilde{\mathbf{a}}}^{r_0}$ (where the first coordinate is time) conditioned on $0 \longleftarrow^{h} \longrightarrow n\tilde{\mathbf{a}}$ converges weakly to the Brownian bridge (with variance that depends only on measure $Q_{r_0}^r$).*

Also for a given $\epsilon > 0$

$$P_p[\text{ scaled cluster } \mathbf{C}^{r_0}_{0,n\tilde{\mathbf{a}}} \not\subset \epsilon\text{-neighbd. of } [0,1] \times \gamma(\text{ reg. points }) \mid 0 \overset{h}{\longleftrightarrow} n\tilde{\mathbf{a}}] \to 0$$

as $n \to \infty$.

Chapter 7

The Result in Self-Avoiding Walks

In this chapter we work only with supercritical SARW ($\beta > \beta_c(d)$).

7.1 Preliminaries.

Here we briefly go over the definitions that one can find in [24]. We start with the decay rate $\tau_\beta(\vec{x})$:

$$\tau_\beta(\vec{x}) \equiv -\lim_{n\to\infty} \frac{1}{n} \log g_\beta([n\vec{x}]),$$

where the limit is always defined since

$$\frac{g_\beta(\vec{x}+\vec{y})}{B_d(\beta)} \geq \frac{g_\beta(\vec{x})}{B_d(\beta)} \frac{g_\beta(\vec{y})}{B_d(\beta)}.$$

Now, $\tau_\beta(\vec{x})$ is the support function of the compact convex set

$$\mathbf{K}^\beta \equiv \bigcap_{\vec{n}\in\mathbb{S}^{d-1}} \{\vec{r} \in \mathbb{R}^d : \vec{r}\cdot\vec{n} \leq \tau_\beta(\vec{n})\},$$

with non-empty interior $\text{int}\{\mathbf{K}^\beta\}$ containing point zero.

Let $\omega(j) = (\omega_1(j), ..., \omega_d(j))$ be a self-avoiding path defined for $j \in [a,b] \bigcap \mathbb{N}$, $a \leq b \in \mathbb{Z}^+$.

Definition 4. We call ω **a bridge** if

$$\omega_1(a) < \omega_1(j) \leq \omega_1(b)$$

for all $a < j \leq b$. If $x = \omega(a)$ is the initial point and $y = \omega(b)$ is the final point, we write $\omega : x -^{b} \to y$.

For $\vec{x} \in \mathbb{Z}^d$, we define the *cylindrical* two-point function

$$h(\vec{x}) \equiv \sum_{\omega : 0 -^{b} \to \vec{x}} e^{-\beta|\omega|},$$

where $h(\vec{x}) = \delta_0(\vec{x})$ for all $\vec{x} \in \{0\} \times \mathbb{Z}^{d-1}$.

Definition 5. We say that $k \in \mathbb{N}$ $(\omega_1(a) < k < \omega_1(b))$ is **a break point** of ω if there exists $r \in [a, b]$ such that $\omega_1(j) \leq k$ whenever $j \leq r$ and $\omega_1(j) > k$ whenever $j > r$.

Definition 6. A bridge $\omega : x -^{b} \to y$ (where, as before, $x = \omega(a)$ and $y = \omega(b)$) is called **irreducible** if it has no break points. In that case we write $\omega : x -^{ib} \to y$.

Now, for $\vec{x} \in \mathbb{Z}^d$, we define the *irreducible* two-point function

$$f(\vec{x}) \equiv \sum_{\omega : 0 -^{ib} \to \vec{x}} e^{-\beta|\omega|},$$

with $f(\vec{x}) = \delta_0(\vec{x})$ for all $\vec{x} \in \{0\} \times \mathbb{Z}^{d-1}$.

7.2 SARW and Regeneration Structures.

It turned out that if counting the bridges between the origin and a point $\vec{k} = (k_x, k_y) \in \mathbb{N} \times \mathbb{Z}^{d-1}$, that f and h satisfy the recurrence equation (see [24]):

$$h(\vec{k}) = \sum_{i=1}^{k_x} \sum_{l \in \mathbb{Z}^{d-1}} f(i, l) h(k_x - i, k_y - l), \tag{7.1}$$

which together with $h(0,\tilde{k}) = \delta_0(\tilde{k})$ (for $\tilde{k} \in \mathbb{Z}^{d-1}$) are called the Ornstein-Zernike equations.
Now, for any $\tilde{r} \in \mathbb{Z}^{d-1}$, we define

$$H_n(\tilde{r}) \equiv \sum_{\tilde{k}\in\mathbb{Z}^{d-1}} e^{\tilde{r}\cdot\tilde{k}} h(n,\tilde{k}) \text{ and } F_n(\tilde{r}) \equiv \sum_{\tilde{k}\in\mathbb{Z}^{d-1}} e^{\tilde{r}\cdot\tilde{k}} f(n,\tilde{k}), \tag{7.2}$$

as well as the corresponding mass

$$m_H(\tilde{r}) \equiv \lim_{n\to+\infty} \frac{1}{n} \log H_n(\tilde{r}) \text{ and } m_F(\tilde{r}) \equiv \lim_{n\to+\infty} \frac{1}{n} \log F_n(\tilde{r}).$$

The Ornstein-Zernike asymptotics has been proved for the cylindrical two-point function $h(\cdot)$ (see [8] and [16]), using the "mass gap" condition, e.g. existence of a point $\tilde{r}_o \in \mathbb{Z}^{d-1}$, inside a neighborhood of points with finite mass m_H, such that $m_H(\tilde{r}_o) > m_F(\tilde{r}_o)$. It was also shown (see [16], Section 2) that the mass gap condition with the renewal theorem ([24], Appendix B) imply that $\exp\{-nm_H(\tilde{r}_o)\}F_n(\tilde{r}_o)$ is a probability distribution (where $\tilde{r}_o$ is as above):

$$\sum_{n\in\mathbb{N}} \exp\{-nm_H(\tilde{r}_o)\}F_n(\tilde{r}_o) = 1. \tag{7.3}$$

As it was mentioned in the introduction, the mass gap condition was crucial in obtaining the Ornstein-Zernike decay (see [16]):

Theorem 4. *For all $d \geq 2$ and $\beta > \beta_c(d)$,*

$$g_\beta(x) = \psi_\beta(\frac{x}{\|x\|})\frac{e^{-\tau_\beta(x)}}{\|x\|^{\frac{d-1}{2}}}(1+o(1))$$

uniformly in $\|x\|$, where $\psi_\beta(\cdot)$ is analytic on the unit circle.

7.3 Measure $Q_{r_0}(x)$.

We notice that substituting the sum $F_n(\tilde{r}_o)$, as defined in (7.2), into (7.3) we obtain (after some simple manipulations) an enhanced version of (7.3):

$$\sum_{\vec{x}\in\mathbb{N}\times\mathbb{Z}^{d-1}} f(\vec{x})e^{\vec{x}\cdot(-m_H(\tilde{r}_o),\tilde{r}_o)} = 1,$$

where $\vec{r}_o \equiv (-m_H(\tilde{r}_o), \tilde{r}_o) \in \partial\mathbf{K}^\beta$ as it was shown in [16], Section 3.

Now, let for $\vec{x} \in \mathbb{N} \times \mathbb{Z}^{d-1}$,

$$Q_{r_0}(\vec{x}) \equiv f(\vec{x})e^{\vec{x}\cdot\vec{r}_0}.$$

Due to the equation above, $Q_{r_0}(\cdot)$ is a probability measure on $\mathbb{N} \times \mathbb{Z}^{d-1}$. It is similar to the regeneration measure, defined for the subcritical bond percolation model in Section 4 of [6], and later used in [20] for derivation of Brownian bridge assymptotics for that model.

The mass gap condition implies the exponential decay of $Q_{r_0}(\vec{x})$.

7.4 The Result For $\vec{\mathbf{a}} = (1, 0, ..., 0)$.

We fix $\vec{\mathbf{a}} \in \mathbb{Z}^d$. We let for a supercritical constant β and all $n \in \mathbb{N}$, $P_n(\cdot)$ to be a law on a set of self-avoiding random paths ω, conditioned on ω being a bridge between 0 and $n\vec{\mathbf{a}}$ ($\omega : 0 -^b \to n\tilde{\mathbf{a}}$). More precisely, we define P_n as

$$P_n(\omega : 0 -^b \to n\tilde{\mathbf{a}}) \equiv \frac{\exp(-\beta|\omega|)}{\sum_{\tilde{\omega}:0-^b\to n\tilde{\mathbf{a}}} \exp(-\beta|\tilde{\omega}|)} = \frac{\exp(-\beta|\omega|)}{h(n\tilde{\mathbf{a}})}. \tag{7.4}$$

For now, we let $\vec{\mathbf{a}} = [1, 0] \equiv (1, 0, ..., 0)$ and $\vec{r}_o = (\mathbb{Z}^+, 0, ..., 0) \bigcap \partial\mathbf{K}^\beta$. Here, we define *the regeneration points* in a way, similar to that, used to define the regeneration points for the case of Bernoulli bond percolation model:

Definition 7. Given a self-avoiding path ω, and a break point b. We say that $\omega(T_b)$ is the corresponding **regeneration point** if $T_b = \max\{t : \omega_1(t) = b\}$.

In a very important step, we notice that given the Ornstein-Zernike equations (7.1) above and the definition of probability distribution $P_n(\cdot)$, we can explicitly write (in terms of measure Q_{r_o}) the probability of the walk passing through the particular regeneration points $s_1 \equiv x_1$, $(s_2 \equiv x_1 + x_2)$,..., $(s_{k-1} \equiv x_1 + ... + x_{k-1})$, where are all $x_i \in \mathbb{Z}^+ \times \mathbb{Z}^{d-1}$:

$$\begin{aligned} P_n[s_1, ..., s_{k-1} \text{ are reg. pts. }] &= \frac{1}{h(n\tilde{\mathbf{a}})} \left(\sum_{\omega: 0 -^{ib} \to s_1} e^{-\beta|\omega|} \right) ... \left(\sum_{\omega: s_{k-1} -^{ib} \to s_k} e^{-\beta|\omega|} \right) \\ &= \frac{1}{h(n\tilde{\mathbf{a}})} f(x_1) ... f(x_k) \\ &= \frac{Q_{r_o}(x_1) ... Q_{r_o}(x_k)}{\sum_\kappa \sum_{\varsigma_1 + ... + \varsigma_\kappa = n\tilde{\mathbf{a}}} Q_{r_o}(\varsigma_1) ... Q_{r_o}(\varsigma_\kappa)}, \end{aligned} \tag{7.5}$$

where $s_0 \equiv 0$ and $s_k \equiv x_1 + ... + x_{k-1} + x_k = n\tilde{\mathbf{a}}$.

We recall that the moment generating function (the Laplace transform) under the measure $Q_{r_o}(\cdot)$ is finite in a neighborhood of zero:

$$\mathbf{E}_{r_0}(e^{\theta \cdot X_1}) < \infty$$

for all $\theta \in B_{\bar{\lambda}}(0)$. We use the brackets $[\cdot, \cdot] \in \mathbb{R} \times \mathbb{R}^{d-1}$ to denote the coordinates of $\mathbb{R}^d$ vectors for the simplicity of notation. Obviously $\tilde{\mathbf{a}} = [1, 0]$. We want to prove that the process corresponding to the last $d-1$ coordinates in the new basis of the scaled ($\frac{1}{n}$ times along $\tilde{\mathbf{a}}$ and $\frac{1}{\sqrt{n}}$ times in the orthogonal $d-1$ dimensions) interpolation of regeneration points of the self-avoiding path ω conditioned on $\omega : 0 -^b \to n\tilde{\mathbf{a}}$ converges weakly to the Brownian bridge $B^o(t)$ (with variance that depends only on measure Q_{r_0}) where t represents the scaled down first coordinate.

Let $X_1, X_2, ...$ be i.i.d. random variables distributed according to Q_{r_0} law. We interpolate $0, X_1, (X_1 + X_2), ..., (X_1 + ... + X_k)$ and scale by $\frac{1}{n} \times \frac{1}{\sqrt{n}}$ along $< \tilde{\mathbf{a}} > \times < \tilde{\mathbf{a}} >^\perp$

to get the process $[t, Y_{n,k}(t)]$. The technical theorem (see chapter 5 or section 8.2) implies the following

Theorem 5. *The process*

$$\{Y_{n,k} \text{ for some } k \text{ such that } X_1 + ... + X_k = n\tilde{\mathbf{a}}\}$$

conditioned on the existence of such k converges weakly to the Brownian bridge (with variance that depends only on measure Q_{r_0}).

Now, we again let for $\vec{y}_1, ..., \vec{y}_k \in \mathbb{Z}^d$ with positive increasing first coordinates, $\gamma(\vec{y}_1, ..., \vec{y}_k)$ to be the last $(d-1)$ coordinates in the new basis of the scaled $(\frac{1}{n} \times \frac{1}{\sqrt{n}})$ interpolation of points $0, \vec{y}_1, ..., \vec{y}_k$ (where the first coordinate is time). Notice that $\gamma(\vec{y}_1, ..., \vec{y}_k) \in C_o[0,1]^{d-1}$ as a function of scaled first coordinate whenever $\vec{y}_k = n\vec{\mathbf{a}}$. By the important observation (7.5) that we have made before, for any function $F(\cdot)$ on $C[0,1]^{d-1}$,

$$\sum_k \sum_{\vec{x}_1+...+\vec{x}_k = n\vec{\mathbf{a}}} F(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \textstyle\sum_{i=1}^k \vec{x}_i))$$

$$\times P_n[0 \overset{h_{r_0}}{\longleftrightarrow} n\vec{\mathbf{a}} \text{ ; regeneration points: } \vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^{k-1} \vec{x}_i]$$

$$= \sum_k \sum_{\vec{x}_1+...+\vec{x}_k = n\vec{\mathbf{a}}} F(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) f(\vec{x}_1)...f(\vec{x}_k)$$

$$= e^{-r_0 \cdot n\vec{\mathbf{a}}} \sum_k \sum_{\vec{x}_1+...+\vec{x}_k = n\vec{\mathbf{a}}} F(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) Q_{r_0}(\vec{x}_1)...Q_{r_0}(\vec{x}_k).$$

Therefore, for any $A \subset C[0,1]^{d-1}$

$$P_p[\gamma(\text{regeneration points of } \omega) \in A \mid \omega : 0 \overset{b}{-} \rightarrow n\vec{\mathbf{a}}]$$

$$= \frac{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\tilde{\mathbf{a}}} I_A(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i))f(\vec{x}_1)...f(\vec{x}_k)}{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\tilde{\mathbf{a}}} f(\vec{x}_1)...f(\vec{x}_k)}$$

$$= \frac{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} I_A(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i))Q_{r_0}(\vec{x}_1)...Q_{r_0}(\vec{x}_k)}{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} Q_{r_0}(\vec{x}_1)...Q_{r_0}(\vec{x}_k)}$$

$= P[Y_{n,k} \in A$ for the k such that $X_1+...+X_k = n\tilde{\mathbf{a}} \mid \exists k$ such that $X_1+...+X_k = n\vec{\mathbf{a}}]$.

Hence, we have proved the following

Corollary. *The process corresponding to the last $d-1$ coordinates of the scaled $(\frac{1}{n} \times \frac{1}{\sqrt{n}})$ interpolation of regeneration points of the self-avoiding path ω (with the scaled first coordinate denoting the time interval) conditioned on $\omega : 0 -^b \to n\tilde{\mathbf{a}}$ converges weakly to the Brownian bridge (with variance that depends only on measure Q_{r_0}).*

7.5 Shrinking of the Self-Avoiding Walks.

Here we again let $\tilde{\mathbf{a}} = [1,0] \equiv (1,0,...,0)$ and $\vec{r}_o = [\mathbb{Z}^+, 0] \bigcap \partial\mathbf{K}^\beta$. In the way of proving that the scaled walk $\omega : 0 -^b \to n\tilde{\mathbf{a}}$ shrinks, we shell need to show that the consequent regeneration points are situated relatively close to each other:

Lemma.

$$P_p[\max_i |x_i - x_{i-1}| > n^{1/3},\ x_i\text{- } reg.\ points \mid 0 -^b \to n\tilde{\mathbf{a}}] < \frac{1}{n}$$

for n large enough.

Proof. Since $\vec{r}_o = [\|\vec{r}_o\|, 0] \in \partial\mathbf{K}^\beta$ and therefore

$$\vec{r}_o \cdot [v_1, v_2] = \vec{r}_o \cdot [v_1, 0] \le \tau_\beta([v_1, 0]) \le \tau_\beta([v_1, v_2])$$

for all $[v_1, v_2] \in Z^d$,

$$\vec{r}_o \cdot [1,0] = \tau_\beta([1,0]) \text{ and } \|\vec{r}_o\|^2 = \tau_\beta(\vec{r}_o).$$

Hence, by the pseudo-linearity of $\tau_\beta(\cdot)$,

$$\nabla\tau_\beta(\tilde{\mathbf{a}}) = [\tau_\beta(1), 0] = [\|\vec{r}_o\|, 0] = \vec{r}_o.$$

Now, by the convexity of $\tau_\beta(\cdot)$,

$$\frac{\tau_\beta(\tilde{\mathbf{a}}) - \tau_\beta(\tilde{\mathbf{a}} - \frac{\vec{x}}{n})}{(\frac{\|\vec{x}\|}{n})} \leq \frac{\vec{x}}{\|\vec{x}\|} \cdot \nabla\tau_\beta(\tilde{\mathbf{a}})$$

for $\vec{x} \in \mathbb{Z}^d$ ($\vec{x} \neq 0$), and therefore

$$\tau_\beta(n\tilde{\mathbf{a}}) - \tau_\beta(n\tilde{\mathbf{a}} - \vec{x}) = \|\vec{x}\| \frac{\tau_\beta(\tilde{\mathbf{a}}) - \tau_\beta(\tilde{\mathbf{a}} - \frac{\vec{x}}{n})}{(\frac{\|\vec{x}\|}{n})} \leq \vec{x} \cdot \nabla\tau_\beta(\tilde{\mathbf{a}}) = \tilde{\mathbf{r}} \cdot \vec{x}.$$

Thus, since $Q_{r_0}(\vec{x})$ decays exponentially and therefore

$$f(\vec{x})e^{\tau_\beta(n\tilde{\mathbf{a}}) - \tau_\beta(n\tilde{\mathbf{a}} - \vec{x})} \leq Q_{r_0}(\vec{x})$$

and also decays exponentially. Hence by Ornstein-Zernike result (Theorem 4),

$$P_p[n^{1/3} < |\vec{x}|,\ \vec{x}\text{-first reg. point } |0 \overset{b}{-} \to n\tilde{\mathbf{a}}] = \sum_{n^{1/3} < |\vec{x}|} f(\vec{x}) \frac{h(n\tilde{\mathbf{a}} - \vec{x})}{h(n\tilde{\mathbf{a}})} < \frac{1}{n^2}$$

for n large enough. So, since the number of the regeneration points is no greater than n,

$$P_p[\max_i |x_i - x_{i-1}| > n^{1/3},\ x_i\text{- reg. points } |\ 0 \overset{b}{-} \to n\tilde{\mathbf{a}}] < \frac{1}{n}$$

for n large enough.

□

Now, it is really easy to check that there is a constant $\lambda_f > 0$ such that

$$f(\vec{x}) > e^{-\lambda_f \|\vec{x}\|}$$

for all $\vec{x}$ such that $f(\vec{x}) \neq 0$ (here we only need to connect points zero and $\vec{x}$ with an "S"-shaped irreducible bridge). Hence, due to the exponential decay (4.1) of the two point function $g_\beta(x, y)$, for a given $\epsilon > 0$,

$$P_p[\text{ the walk } \{\omega(i)\}_{i=0,\dots,|\omega(\vec{x})|} \not\subset [\mathbb{R}, B^{d-1}_{\epsilon\sqrt{n}}(0)] \mid 0 -^{ib} \to \vec{x}] < C_\beta e^{\lambda_f \|\vec{x}\| - c_\beta \epsilon \sqrt{n}},$$

and therefore, summing over the regeneration points, we get

$$P_p[\text{ the scaled walk } \{\omega(i)\}_{i=0,\dots,|\omega(\vec{x})|} \not\subset \epsilon\text{-neighbd. of } [0,1]\times\gamma(\text{ reg. points }) \mid 0 -^{b} \to n\tilde{\mathbf{a}}]$$

$$< \frac{1}{n} + nC_\beta e^{\lambda_f \|\vec{x}\| - c_\beta \epsilon \sqrt{n}}$$

for n large enough due to the lemma above.

We can now state the main result for $\tilde{\mathbf{a}} = [1, 0]$:

Main Theorem (SAW). *The process corresponding to the last $d-1$ coordinates of the scaled $(\frac{1}{n} \times \frac{1}{\sqrt{n}})$ interpolation of regeneration points of the self-avoiding path ω (with the scaled first coordinate denoting the time interval) conditioned on $\omega : 0 -^{b} \to n\tilde{\mathbf{a}}$ converges weakly to the Brownian bridge (with variance that depends only on measure Q_{r_0}).*
Also for a given $\epsilon > 0$

$$P_p[\text{ the scaled walk } \{\omega(i)\}_{i=0,\dots,|\omega(\vec{x})|} \not\subset \epsilon\text{-neighbd. of } [0,1]\times\gamma(\text{ reg. points }) \mid 0 -^{b} \to n\tilde{\mathbf{a}}] \to 0$$

as $n \to \infty$.

7.6 General Case.

Now, we turn our attention to all $\vec{\mathbf{a}} \in \mathbb{Z}^d$ not on the axis. It turned out that the main theorem of section 7.5 holds for all $\vec{\mathbf{a}}$ in $\mathbb{Z}^d$. In a more direct approach used in the corresponding developments in percolation (see Section 4 of [6]) and finite range Ising models (see [7]), we can replicate the same recurrence structures, as those in section 7.1, in a given direction (say $\vec{\mathbf{a}}$), yielding the same renewal relations (as in section 7.2). The technique is simpler than that used in percolation and finite range Izing models. We choose a direction vector $\vec{\mathbf{r}} \in \partial \mathbf{K}^{\beta}$, we define the corresponding notions of "a bridge" in the direction $\vec{\mathbf{r}}$ and the $\vec{\mathbf{r}}$-regeneration points:

Definition 8. We call ω **an $\vec{\mathbf{r}}$-bridge** if

$$\omega(a) \cdot \vec{\mathbf{r}} < \omega(j) \cdot \vec{\mathbf{r}} \leq \omega(b) \cdot \vec{\mathbf{r}}$$

for all $a < j \leq b$. If $x = \omega(a)$ is the initial point and $y = \omega(b)$ is the final point, we write $\omega : x{-}^{b(\vec{\mathbf{r}})} \to y$.

Similarly, we define the *cylindrical* two-point function

$$h_{\vec{\mathbf{r}}}(\vec{x}) \equiv \sum_{\omega:0-^{b(\vec{\mathbf{r}})}\to\vec{x}} e^{-\beta|\omega|},$$

where $h_{\vec{\mathbf{r}}}(\vec{x}) = \delta_0(\vec{x})$ for all $\vec{x} \in <\vec{\mathbf{r}}>^{\perp}$.

Definition 9. We say that $\omega(k) \in \mathbb{Z}^d$ $(a < k < b)$ is **an $\vec{\mathbf{r}}$-regeneration point** of ω if there exists $N \in [a, b]$ such that $\omega(j) \cdot \vec{\mathbf{r}} \leq \omega(k) \cdot \vec{\mathbf{r}}$ whenever $j \leq N$ and $\omega(j) \cdot \vec{\mathbf{r}} > \omega(k) \cdot \vec{\mathbf{r}}$ whenever $j > N$.

Definition 10. An $\vec{\mathbf{r}}$-bridge $\omega : x{-}^{b} \to y$ (where, as before, $x = \omega(a)$ and $y = \omega(b)$) is called $\omega(k) \cdot$ **$\vec{\mathbf{r}}$-irreducible** if it has no $\vec{\mathbf{r}}$-regeneration points. In that case we write $\omega : x{-}^{ib(\vec{\mathbf{r}})} \to y$.

We again redefine the corresponding *irreducible* two-point function

$$f_{\vec{\mathbf{r}}}(\vec{x}) \equiv \sum_{\omega:0-^{ib(\vec{\mathbf{r}})}\to\vec{x}} e^{-\beta|\omega|},$$

with $f_{\vec{\mathbf{r}}}(\vec{x}) = \delta_0(\vec{x})$ for all $\vec{x} \in <\vec{\mathbf{r}}>^{\perp}$.

The generalized Ornstein-Zernike recurrence equations also hold here: by counting the $\vec{\mathbf{r}}$-bridges between the origin and a lattice point $\vec{k} \in <\vec{\mathbf{r}}> \times <\vec{\mathbf{r}}>^{\perp}$, we establish

$$h_{\vec{\mathbf{r}}}(\vec{k}) = \sum_{0<\vec{m}\cdot\vec{\mathbf{r}}\leq\vec{k}\cdot\vec{\mathbf{r}}} f_{\vec{\mathbf{r}}}(\vec{m}) h_{\vec{\mathbf{r}}}(\vec{k}-\vec{m}), \tag{7.6}$$

where, in the sum, all $\vec{m} \in \mathbb{Z}^d$.

As in [6], we can replicate all the regeneration structures, and in particular show the existence of a positive $\bar{\lambda}$ such that

$$Q^{\vec{\mathbf{r}}}_{r_0}(\vec{x}) \equiv f_{\vec{\mathbf{r}}}(\vec{x}) e^{\vec{x}\cdot\vec{r}_0}$$

is a probability measure whenever $\vec{r}_0 \in B_{\bar{\lambda}}(\vec{\mathbf{r}}) \bigcap \partial\mathbf{K}^{\beta}$. Taking an appropriate $\vec{\mathbf{r}}$ (say $\vec{\mathbf{r}} = \nabla\tau_{\beta}(\vec{\mathbf{a}})$), we can show, as it was done in [20] for percolation clusters in subcritical phase, the skeleton convergence and shrinking of the scaled self-avoiding walks, conditioned on arriving to $n\vec{\mathbf{a}}$. Whence the main theorem of section 7.5 would hold if we scale the walks by $\frac{1}{n\|\vec{\mathbf{a}}\|}$ along $<\vec{\mathbf{a}}>$ and by $\frac{1}{\sqrt{n}}$ in all orthogonal directions (along $<\vec{\mathbf{a}}>^{\perp}$).

Chapter 8

Convergence to Brownian Bridge

As it was mentioned before, in chapter 5, this chapter is entirely dedicated to proving the technical theorem that we have already used in the proof of the main results in percolation (see chapter 6) and self-avoiding walks (see chapter 7). First, we are going to mention the "directed" random walk implication of the theorem that was briefly introduced in chapter 5 followed by an outline of the proof presented in this chapter:

• We notice that the technical theorem establishes a Donsker-type asymptotics. Given a vector $\vec{\mathbf{a}}$ in $\mathbb{Z}^d$, we consider a scaled "directed" random walk, conditioned on arriving to a faraway point $n\vec{\mathbf{a}}$. By a directed random walk we mean a random walk in which the steps $\{\zeta_i\}$ are i.i.d. (with finite mean and variance) and the probability $P[\zeta_i \cdot \vec{\mathbf{a}} > 0] = 1$. We scale the interpolation trajectory of the walk $\frac{1}{n\|\vec{\mathbf{a}}\|}$ times along $< \vec{\mathbf{a}} >$ and $\frac{1}{\sqrt{n}}$ times in all of the orthogonal directions $< \vec{\mathbf{a}} >^{\perp}$. Changing the basis so that the scaled interval connecting $n\vec{\mathbf{a}}$ to the origin becomes a $[0,1]$-time interval, the technical theorem establishes a weak convergence of the remaining d-1 coordinates of the interpolation trajectories to a (d-1)-dimensional Brownian bridge (up to a constant multiple depending on the variance of ζ_i).

• In section 8.1, we prove theorem 6, which is just a case of the technical theorem when the probability $P[\vec{a} \cdot \zeta_i = c_\zeta] = 1$ for some fixed c_ζ, and all i. We would like to point out that the trick used in the proof of the appropriate (Brownian bridge) covariance for the Lemma 1(b) works in the general settings as well.

• In section 8.2 we use theorem 6 and a "truncation" argument to gradually extend the result from the case $P[\vec{a} \cdot \zeta_i = c_\zeta] = 1$ corresponding to the setting of theorem 6 to that with $\mathbf{E}[\vec{a} \cdot \zeta_i] = c_\zeta$ corresponding to the general setting of the technical theorem. The covariance coefficient for the general case followes from that of the simple case, established in section 8.1 (see Lemma 1).

8.1 Simple Case

Let $Z_1, Z_2, ...$ be i.i.d. random variables on $\mathbb{Z}$ with the span of the lattice distribution equal to one (see [10], section 2.5) and mean $\mu = \mathbf{E}Z_1 < \infty$, $\sigma^2 = Var(Z_1) < \infty$. Also let point zero be inside of the closed convex hull of $\{z : P[Z_1 = z] > 0\}$.

Consider a one dimensional plane and a walk X_j that starts with $X_0 = 0$ and for a given X_j, the (j+1)-st step to be $X_{j+1} = X_j + Z_{j+1}$. After interpolation we get

$$X(t) = X_{[t]} + (t - [t])(X_{[t]+1} - X_{[t]})$$

for $0 \le t < \infty$.
And define $\bar{X}(t) = (t, X(t))$ to be a two dimensional walk.

Now, if for a given integer $n > 0$ we define $X_n(t) \equiv \frac{X(nt)}{\sqrt{n}}$ for $0 \le t \le 1$, then $X_n(t)$ would belong to $C[0, 1]$ and $X_n(0) = 0$.

Theorem 6. *$X_n(t)$ conditioned on $X_n(1) = 0$ converges weakly to the Brownian bridge.*

First we need to prove the theorem when $\mu = 0$. For this, due to the theorem 9 of appendix B, we only need to prove the convergence of the finite-dimensional distributions (Lemma 1) together with the tightness of the family of probability distributions (lemma 2):

Lemma 1. *For $A_0 \subseteq C[0, 1]$, let $P_n(A_0) = P[X_n \in A_0 | X_n(1) = 0]$ to be the law of X_n conditioned on $X_n(1) = 0$. Then* *(a) For $\mu = 0$, the finite-dimensional distributions of P_n converge weakly to a Gaussian distributions.*

(b) There are positive $\{C_n\}_{n=1,2,\dots} \to C$ ($C = \sigma^2$ when $\mu = 0$) such that $0 < C < \infty$ and

$$Cov_{P_n}(X_n(s), X_n(t)) = C_n s(1-t) + O(\frac{1}{n})$$

for all $0 \le s \le t \le 1$. More precisely: $Cov_{P_n}(X_n(s), X_n(t)) = C_n s(1-t)$ if $[ns] < [nt]$ and
$Cov_{P_n}(X_n(s), X_n(t)) = C_n s(1-t) - C_n \frac{\epsilon_1(1-\epsilon_2)}{n}$ if $[ns] = [nt]$, where $\epsilon_1 = \frac{ns-[ns]}{n} \in [0,1)$ and $\epsilon_2 = \frac{nt-[nt]}{n} \in [0,1)$.

and

Lemma 2. *For $\mu = 0$, the probability measures P_n induced on the subspace of $X_n(t)$ trajectories in $C[0,1]$ are tight.*

Proof of Lemma 1: (a) Though it is not difficult to show that a finite-dimensional distribution of P_n converges weakly to a Gaussian distribution, here we only show the convergence for one and two points on the interval (in case of one point $t \in [0,1]$, we show that the limit variance has to be equal to $t(1-t)\sigma^2$). Take $t \in \frac{1}{n}\mathbb{Z} \cap (0,1)$ and let $\alpha = \frac{k}{\sqrt{n}}$, then by the Local CLT (see appendix A),

$$P[X(tn) = k] = \frac{1}{\sqrt{n}}\Phi_{\sigma\sqrt{t}}(\alpha) + o(\frac{1}{\sqrt{n}}), \text{ where } \Phi_v(x) \equiv \frac{1}{v\sqrt{2\pi}}e^{-\frac{|x|^2}{2v^2}} \tag{8.1}$$

is the normal density function, and the error term is uniformly bounded by a $o(\frac{1}{\sqrt{n}})$ function independent of k.
Therefore, substituting (8.1),

$$P_n[X_n(t) = \alpha] = \frac{(\frac{1}{\sqrt{n}}\Phi_{\sigma\sqrt{t}}(\alpha) + o(\frac{1}{\sqrt{n}}))(\frac{1}{\sqrt{n}}\Phi_{\sigma\sqrt{1-t}}(\alpha) + o(\frac{1}{\sqrt{n}}))}{\frac{1}{\sqrt{n}}\Phi_\sigma(0) + o(\frac{1}{\sqrt{n}})} = \frac{1}{\sqrt{n}}\Phi_{\sigma\sqrt{t(1-t)}}(\alpha) + o(\frac{1}{\sqrt{n}}).$$

Thus for a set A in $\mathbb{R}$,

$$P_n[X_n(t) \in A] = \sum_{k \in \sqrt{n}A} [\frac{1}{\sqrt{n}} \Phi_{\sigma\sqrt{t(1-t)}}(\alpha) + o(\frac{1}{\sqrt{n}})] = \mathbf{N}[0, t(1-t)\sigma^2](A) + o(1)$$

-here the limit variance is equal to $t(1-t)\sigma^2$. Given that the variance $\sigma^2 < 0$, the convergence follows.

The same method works for more than one point, here we do it for two: Let $\alpha_1 = \frac{k_1}{\sqrt{n}}$ and $\alpha_2 = \frac{k_2}{\sqrt{n}}$, then as before, for $t_1 < t_2$ in $\frac{1}{n}\mathbb{Z} \cap (0,1)$, writing the conditional probability as a ratio of two probabilities, and representing the probabilities according to (8.1), we get

$$P_n[X_n(s) = \alpha_1, X_n(t) = \alpha_2] = \frac{\sqrt{|\mathcal{A}|}}{2\pi\sigma^2} \exp\left\{ -\frac{(\alpha_1, \alpha_2)\mathcal{A}(\alpha_1, \alpha_2)^T}{2\sigma^2} \right\} + o(\frac{1}{n}).$$

where

$$\mathcal{A} = \begin{pmatrix} \frac{t_2}{(t_2-t_1)t_1} & -\frac{1}{t_2-t_1} \\ -\frac{1}{t_2-t_1} & \frac{1-t_1}{(t_2-t_1)(1-t_2)} \end{pmatrix}.$$

Thus for sets A_1 and A_2 in $\mathbb{R}$,

$$\begin{aligned} P_n[X_n(t_1) \in A_1, X_n(t_2) \in A_2] &= \sum_{k_1 \in \sqrt{n}A_1, k_2 \in \sqrt{n}A_2} [\frac{\sqrt{|\mathcal{A}|}}{2\pi\sigma^2} \exp\left\{ -\frac{(\alpha_1, \alpha_2)\mathcal{A}(\alpha_1, \alpha_2)^T}{2\sigma^2} \right\} + o(\frac{1}{n})] \\ &= \mathbf{N}[0, \mathcal{A}^{-1}](A_1 \times A_2) + o(1) \end{aligned}$$

Observe that $(\sigma^2 \mathcal{A}^{-1}) = \begin{pmatrix} t_1(1-t_1)\sigma^2 & t_1(1-t_2)\sigma^2 \\ t_1(1-t_2)\sigma^2 & t_2(1-t_2)\sigma^2 \end{pmatrix}$ is the covariance matrix, and the part (b) of the lemma follows in case $\mu = 0$.

(b) Though the estimate above produces the needed variance in case when the mean $\mu = 0$, in general, we need to apply the following approach: We first consider the case when $s < t$ and both $s, t \in \frac{1}{n}\mathbb{Z} \cap (0,1)$ where

$$\mathbf{E}[X_n(s) \mid X_n(t) = y] = \mathbf{E}[Z_1 + ... + Z_{sn} | Z_1 + ... + Z_{tn} = y] = \frac{s}{t} y,$$

and therefore

$$Cov_{P_n}(X_n(s), X_n(t)) \quad = \quad \frac{s}{t} \mathbf{E}[X_n^2(t) | X_n(1) = 0]$$

as $\{-X_n(1-t) \mid X_n(1) = 0\}$ and $\{X_n(t) \mid X_n(1) = 0\}$ are identically distributed.

Now, by symmetry (time reversal),

$$Cov_{P_n}(X_n(s), X_n(t)) = Cov_{P_n}(X_n(1-t), X_n(1-s)) = \frac{1-t}{1-s} \mathbf{E}[X_n^2(s) | X_n(1) = 0],$$

and therefore

$$\frac{\mathbf{E}[X_n^2(s) | X_n(1) = 0]}{\mathbf{E}[X_n^2(t) | X_n(1) = 0]} = \frac{s(1-s)}{t(1-t)}.$$

Hence, there exists a constant C_n such that for all $t \in \frac{1}{n}\mathbb{Z} \cap (0,1)$

$$\frac{\mathbf{E}[X_n^2(t) | X_n(1) = 0]}{t(1-t)} \equiv C_n.$$

Thus we have shown that for $s \leq t$ in $\frac{1}{n}\mathbb{Z} \cap [0,1]$,

$$Cov_{P_n}(X_n(s), X_n(t)) = \frac{s}{t} \mathbf{E}[X_n^2(t) | X_n(1) = 0] = \frac{s}{t} C_n t(1-t) = C_n s(1-t).$$

Now, consider the general case: $s = s_0 + \frac{\epsilon_1}{n} \leq t = t_0 + \frac{\epsilon_2}{n}$, where $ns_0, nt_0 \in \mathbb{Z}$ and $\epsilon_1, \epsilon_2 \in [0,1)$. Then the covariance

$$
\begin{aligned}
Cov_{P_n}(X_n(s), X_n(t)) &= (1-\epsilon_1)(1-\epsilon_2)Cov_{P_n}(X_n(s_0), X_n(t_0)) \\
&+ (1-\epsilon_1)\epsilon_2 Cov_{P_n}(X_n(s_0), X_n(t_0+\frac{1}{n})) \\
&+ \epsilon_1(1-\epsilon_2)Cov_{P_n}(X_n(s_0+\frac{1}{n}), X_n(t_0)) \\
&+ \epsilon_1\epsilon_2 Cov_{P_n}(X_n(s_0+\frac{1}{n}), X_n(t_0+\frac{1}{n}))
\end{aligned}
$$

Therefore

$$
Cov_{P_n}(X_n(s), X_n(t)) = C_n s(1-t) \text{ when } s_0 < t_0 \ ([ns] < [nt]),
$$

and

$$
Cov_{P_n}(X_n(s), X_n(t)) = C_n s(1-t) - C_n \frac{\epsilon_1(1-\epsilon_2)}{n} \text{ when } s_0 = t_0 \ ([ns] = [nt]).
$$

Now, plugging in $s = t = \frac{1}{2}$ we get

$$
C_n = 4\mathbf{E}[X_n^2(\frac{1}{2})|X_n(1) = 0] \text{ when } n \text{ is even,}
$$

and

$$
C_n = 4\mathbf{E}[X_n^2(\frac{1}{2})|X_n(1) = 0](\frac{n}{n-1}) \text{ when } n \text{ is odd.}
$$

Therefore

$$
C_n = 4\mathbf{E}[X_n^2(\frac{1}{2})|X_n(1) = 0](1 + O(\frac{1}{n})) \to C = \sigma^2
$$

as $\{X_n(\frac{1}{2}), P_n\}$ converges in distribution as $n \to +\infty$.

□

Proof of Lemma 2: Before we begin the proof of tightness, we notice that the only real obstacle we face is that the process is conditioned on $X_n = 0$. The tightness for the case without the conditioning has been proved years ago as part of the Donsker's Theorem (see Chapter 10 in [3]). With the help of the local CLT (see appendix A) we are essentially removing the difference between the two cases.

Given a $\lambda > 0$ and let $m = [n\delta]$ for a given $0 < \delta \leq 1$, then for any $\mu > 0$,

$$\begin{aligned}
P_\lambda &\equiv P[\max_{0\leq i\leq m} X_i \geq \lambda\sqrt{n} > X_m > -\lambda\sqrt{n}|X_n = 0] \\
&= \sum_{a=-[\lambda\sqrt{n}]}^{[\lambda\sqrt{n}]} \frac{P[\max_{0\leq i\leq m} X_i > [\lambda\sqrt{n}] \ ; \ X_m = a \ ; \ X_n = 0]}{P[X_n = 0]} \\
&= \sum_{a=-[\lambda\sqrt{n}]}^{[\lambda\sqrt{n}]} \frac{P[\max_{0\leq i\leq m} X_i > [\lambda\sqrt{n}] \ ; \ X_m = a]P[X_{n-m} = -a]}{P[X_n = 0]} \\
&\leq \max_{-[\lambda\sqrt{n}]\leq a\leq[\lambda\sqrt{n}]} (\frac{P[X_{n-m} = -a]}{P[X_n = 0]}) \times \sum_{a=-[\lambda\sqrt{n}]}^{[\lambda\sqrt{n}]} P[\max_{0\leq i\leq m} X_i > [\lambda\sqrt{n}] \ ; \ X_m = a] \\
&\leq 2P[\max_{0\leq i\leq m} X_i \geq \lambda\sqrt{n} \geq X_m \geq -\lambda\sqrt{n}]
\end{aligned}$$

for n large enough, where by the local CLT,

$$\max_{-[\lambda\sqrt{n}]\leq a\leq[\lambda\sqrt{n}]} (\frac{P[X_{n-m} = -a]}{P[X_n = 0]}) \leq 2$$

for n large enough as $n - m$ linearly depends on n.

Therefore, the probability

$$P[\max_{0\leq i\leq m} |X_i| \geq \lambda\sqrt{n}|X_n = 0] \leq 2P_\lambda + P[|X_m| \geq \lambda\sqrt{n}|X_n = 0],$$

where

$$P_\lambda \leq 2P[\max_{0\leq i\leq m} X_i \geq \lambda\sqrt{n}].$$

Now, due to the point-wise convergence, we can proceed as in Chapter 10 of [3] by bounding the two remaining probabilities:

$$P[\max_{0\leq i\leq m} |X_i| \geq \lambda\sqrt{n}] \leq 2P[|X_m| \geq \frac{1}{2}\lambda\sqrt{n}] \to 2P[|\sqrt{\delta}N| \geq \frac{\lambda}{2\sigma}] \leq \frac{16\delta^{3/2}\sigma^3}{\lambda^3}\mathbf{E}[|N|^3]$$

and similarly

$$P[|X_m| \geq \lambda\sqrt{n}|X_n = 0] \to P[|\sqrt{\delta(1-\delta)}N| \geq \frac{\lambda}{\sigma}] \leq \frac{\delta^{3/2}\sigma^3}{\lambda^3}\mathbf{E}[|N|^3].$$

Thus, for all integer $k \in [0, n-m]$,

$$P[\max_{0\leq i\leq m}|X_{k+i}-X_k| \geq \lambda\sqrt{n}|X_n = 0] = P[\max_{0\leq i\leq m}|X_i| \geq \lambda\sqrt{n}|X_n = 0] \leq 70\frac{\delta^{3/2}\sigma^3}{\lambda^3}\mathbf{E}[|N|^3]$$

for n large enough, (see Chapter 10 in [3]). Therefore $\{P_n\}$ are tight (see Theorem 11).

□

Proof of Theorem 6: The lemmas above imply the convergence when the mean $\mu = 0$. Now, for $\mu \neq 0$, there exists a $\rho \in \mathbb{R}$ such that $\sum_{z\in\mathbb{Z}} ze^{\rho z}P[Z_1 = z] = 0$. Then we let $\hat{Z}_1, \hat{Z}_2, ...$ be i.i.d. random variables with their distribution defined in the following fashion:

$$P[\hat{Z}_j = z] \equiv \frac{e^{\rho z}}{C_\rho}P[Z_j = z]$$

for all j and $z \in \mathbb{R}$, where $C_\rho \equiv \sum_{z\in\mathbb{Z}} P[\hat{Z}_1 = z] = \sum_{z\in\mathbb{Z}} e^{\rho z}P[Z_1 = z]$. Then the law of $Z_1, ..., Z_n$ conditioned on $Z_1 + ... + Z_n = 0$ is the same as that of $\hat{Z}_1, ..., \hat{Z}_n$ conditioned on $\hat{Z}_1 + ... + \hat{Z}_n = 0$, and the case is reduced to that of $\mu = 0$ as $\mathbf{E}\hat{Z}_j = 0$. We also estimate the covariance equal to $\hat{C}s(1-t)$ for all $0 \leq s \leq t \leq 1$, where as before

$$\hat{C} = \lim_{n\to+\infty}\mathbf{E}[Z_1^2 \mid Z_1 + ... + Z_n = 0].$$

□

Observe that the result can be modified for $X_1, X_2, ...$ defined on a multidimensional lattice $\mathbb{L} \subset \mathbb{R}^d, d > 1$, if we condition on $X_n(1) = \mathbf{a}(n) = \mathbf{a} + \mathbf{o}(\mathbf{1}) \in \{z\sqrt{n} : z \in \bigoplus_1^n \mathbb{L}\}$. We again let point zero be inside the closed convex hull of $\{z : P[Z_1 = z] > 0\}$. In this case the process $\tilde{X}_n(t) = X_n(t) + (\mathbf{a} - \mathbf{a}(n))t$ converges to the Brownian bridge $B^{0,\mathbf{a}}$, and convergence is uniform whenever $\mathbf{a}(n)$ uniformly converges to zero thanks to the Local CLT (see appendix A).

Theorem 7. *$\tilde{X}_n(t)$ conditioned on $X_n(1) = \mathbf{a}(n) = \mathbf{a} + o(1)$ converges weakly to the Brownian bridge.*

Here, as before, if we take $t \in \frac{1}{n}\mathbb{Z} \cap [0,1]$ and let $\alpha = \frac{k}{\sqrt{n}}$, then

$$\begin{aligned} P[X_n(t) = \alpha \mid X_n(1) = \mathbf{a}(n)] &= \frac{(\frac{1}{\sqrt{n}}\Phi_{\sigma\sqrt{t}}(\alpha) + o(\frac{1}{\sqrt{n}}))(\frac{1}{\sqrt{n}}\Phi_{\sigma\sqrt{1-t}}(\mathbf{a}(n) - \alpha) + o(\frac{1}{\sqrt{n}}))}{\frac{1}{\sqrt{n}}\Phi_{\sigma}(\mathbf{a}(n)) + o(\frac{1}{\sqrt{n}})} \\ &= \frac{1}{\sqrt{n}}\Phi_{\sigma\sqrt{t(1-t)}}(\alpha - \mathbf{a}(n)t) + o(\frac{1}{\sqrt{n}}). \end{aligned}$$

8.2 General Case

As before, for a given non-zero vector $\tilde{\mathbf{a}} \in \mathbb{Z}^d$, we let $X_1, X_2, ...$ be i.i.d. random variables on $\mathbb{Z}^d$ with the span of the lattice distribution equal to one (see [10]) such that the probability $P[\tilde{\mathbf{a}} \cdot X_1 > 0] = 1$, the mean $\mu = \mathbf{E}X_1 < \infty$ and there is a constant $\bar{\lambda} > 0$ such that the moment-generating function

$$\mathbf{E}[e^{\theta \cdot X_1}] < \infty$$

for all $\theta \in B_{\bar{\lambda}}$. Also we let $\mathbf{P}_{\tilde{\mathbf{a}}}$ denote the projection map on $< \tilde{\mathbf{a}} >$ and $\mathbf{P}_{\tilde{\mathbf{a}}}^{\perp}$ denote the orthogonal projection on $< \tilde{\mathbf{a}} >^{\perp}$. Now we can decompose the mean $\mu = \mu_a \times \mu_{or}$, where $\mu_a \equiv \mathbf{P}_{\tilde{\mathbf{a}}}\mu$ and $\mu_{or} \equiv \mathbf{P}_{\tilde{\mathbf{a}}}^{\perp}\mu$.

As before we introduce a new basis $\{\vec{f_1}, \vec{f_2}, ..., \vec{f_d}\}$, where $\vec{f_1} = \frac{\tilde{\mathbf{a}}}{\|\tilde{\mathbf{a}}\|}$. We again use $[\cdot,\cdot]_f \in \mathbb{R} \times \mathbb{R}^{d-1}$ to denote the coordinates of a vector with respect to the new basis. We denote $X_i = [T_i, Z_i]_f \in \mathbb{Z} \times \mathbb{Z}^{d-1}$, where $[T_i, 0]_f = \mathbf{P}_{\tilde{\mathbf{a}}}X_i$ and $[0, Z_i]_f = \mathbf{P}_{\tilde{\mathbf{a}}}^{\perp}X_i$, and we let $X_1 + ... + X_i = [t_i, Y_i]_f \in \mathbb{Z} \times \mathbb{Z}^{d-1}$. Note: T_i and Z_i don't have to be independent. Interpolating Y_i, we get

$$Y(t) = Y_{[t]} + (t - [t])(Y_{[t]+1} - Y_{[t]})$$

for $0 \le t \le \infty$ and if we now define $Y_n(t) \equiv \frac{Y(nt)}{\sqrt{n}}$ for $0 \le t \le 1$, then the following theorem easily follows from the previous result:

Corollary. *$Y_n(t)$ conditioned on $Y_n(1) = 0$ converges weakly to the Brownian bridge.*

Since the first coordinate T_i is positive with probability one, the next step will be to interpolate $[t_i, Y_i]_f$, and prove that if scaled and conditioned on $[t_n, Y_n]_f = X_1 + ... + X_n = [n\|\tilde{\mathbf{a}}\|, 0]_f = n\tilde{\mathbf{a}}$ it will converge weakly to the Brownian bridge (with the first coordinate being the time axis). Now, the last theorem (theorem 7) implies the result for $P[[T_i, 0]_f = \mu_a] = 1$, we want the same result for $\mathbf{E}T_i = \|\mu_a\|$ with $VarT_i < \infty$.
We first let $\bar{X}_i \equiv X_i - \mu_a$, then $\mathbf{E}\bar{X}_i = \mu_{or}$ and $Var\bar{X}_i < \infty$. We again interpolate:

$$\bar{X}(t) = \bar{X}_{[t]} + (t - [t])(\bar{X}_{[t]+1} - \bar{X}_{[t]})$$

for $0 \leq t \leq \infty$, and scale $\bar{X}_k(t) \equiv \frac{\bar{X}(kt)}{\sqrt{k}}$. Note: the last $d-1$ coordinates of $\bar{X}_k(t)$ w.r.t. the new basis are $Y_k(t)$ (e.g. $\mathbf{P}_{\tilde{\mathbf{a}}}^{\perp}\bar{X}_k(t) = [0, Y_k(t)]_f$).
From here on we denote $S_j \equiv [t_j, Y_j]_f = X_1 + ... + X_j$ and $\bar{S}_j \equiv \bar{X}_1 + ... + \bar{X}_j = S_j - j\mu_a$ for any positive integer j. As a first important step, we state another important

Corollary. *For $k = k(n) = [\frac{n\|\tilde{\mathbf{a}}\|}{\|\mu_a\|} + k_0\sqrt{n}]$, $\{\bar{X}_k(t) - (k_0\sqrt{\frac{\|\mu_a\|}{\|\tilde{\mathbf{a}}\|}}\mu_a + \frac{n\tilde{\mathbf{a}} - k\mu_a}{\sqrt{k}})t\}$ conditioned on $\bar{X}_k(1) = n\tilde{\mathbf{a}} - k\mu_a$ (e.g.$[t_k, Y_k]_f = n\tilde{\mathbf{a}}$) converges weakly to the Brownian bridge $B^{0, -k_0\sqrt{\frac{\|\mu_a\|}{\|\tilde{\mathbf{a}}\|}}\mu_a}$.*

Observe that $n\tilde{\mathbf{a}} - k\mu_a = -k_0\sqrt{n}\mu_a + o(\sqrt{n})$ and that the convergence is uniform for all k_0 in a compact set . Now, looking only at the last $d-1$ coordinates of $\bar{X}_k(t)$, w.r.t. the new basis the last Corollary implies:

Lemma 3. *For $k = k(n) = [\frac{n\|\tilde{\mathbf{a}}\|}{\|\mu_a\|} + k_0\sqrt{n}]$, $Y_k(t)$ conditioned on $t_k = n\|\tilde{\mathbf{a}}\|$ and $Y_k(1) = 0$ converges weakly to the Brownian bridge.*

Note that convergence is uniform for k_0 in a compact set.
What the Lemma above says is the following: the interpolation of $[\frac{i}{k}, \frac{1}{\sqrt{k}}Y_i]_f$ conditioned on $[t_k, Y_k]_f = n\tilde{\mathbf{a}}$ converges to Time$\times$Brownian bridge. Now, define the process $[t, Y^*_{n,k}(t)]_f$ to be the interpolation of $[\frac{1}{n\|\tilde{\mathbf{a}}\|}t_i, \frac{1}{\sqrt{n}}Y_i]_f^{i=0,1,\ldots,k}$, then

Theorem 8. *For $k = k(n) = [\frac{n\|\tilde{\mathbf{a}}\|}{\|\mu_a\|} + k_0\sqrt{n}]$, $\sqrt{\frac{n}{k}}Y^*_{n,k}(t)$ conditioned on $t_k = n\|\tilde{\mathbf{a}}\|$ and $Y_k(1) = 0$ converges weakly to the Brownian bridge.*

Proof: Here we observe that the mean $\mathbf{E}[\frac{t_i}{n\|\tilde{\mathbf{a}}\|} - \frac{t_{i-1}}{n\|\tilde{\mathbf{a}}\|}]$ is actually equal to $\frac{\|\mu_a\|}{n\|\tilde{\mathbf{a}}\|} = \frac{1}{k-k_0\sqrt{n}} + o(\frac{1}{n})$, and that for a given $\epsilon > 0$, the probability of the $\|[\frac{1}{n\|\tilde{\mathbf{a}}\|}t_i, \frac{1}{\sqrt{n}}Y_i]_f - [\frac{i}{k}, \frac{1}{\sqrt{k}}Y_i]_f\| = |\frac{t_j}{n\|\tilde{\mathbf{a}}\|} - \frac{j}{k}|$ exceeding ϵ for some $j \leq k$,

$$\begin{aligned} P[\max_{0\leq j\leq k}|t_j - \frac{n\|\tilde{\mathbf{a}}\|}{k}j| \geq n\epsilon \mid S_n = n\tilde{\mathbf{a}}] &\leq P[\max_{0\leq j\leq k}\|S_j - \frac{n\|\tilde{\mathbf{a}}\|j}{k}\mu_a\| \geq n\epsilon \mid S_k = n\tilde{\mathbf{a}}] \\ &\leq P[\max_{0\leq j\leq k}|\bar{S}_j| \geq n\frac{\epsilon}{2} \mid \bar{S}_k = [n\|\tilde{\mathbf{a}}\| - k\|\mu_a\|, 0]_f] \\ &\to 0 \end{aligned}$$

as $n \to +\infty$ since $n\|\tilde{\mathbf{a}}\| - k\|\mu_a\| = -\|\mu_a\|k_0\sqrt{n} + o(\sqrt{n})$.

□

Now, the next step is to prove that the process

$$\{Y^*_{n,k} \text{ for some } k \text{ such that } [t_k, Y_k]_f = n\tilde{\mathbf{a}}\}$$

conditioned on the existence of such k converges weakly to the Brownian bridge.

First of all the last theorem implies

Lemma 4. *For given $k = k(n) = [\frac{n\|\tilde{\mathbf{a}}\|}{\|\mu_a\|} + k_0\sqrt{n}]$, $Y^*_{n,k}(t)$ conditioned on $t_k = n\|\tilde{\mathbf{a}}\|$ and $Y_k(1) = 0$ converges weakly to the Brownian bridge.*

For a fixed $M > 0$, convergence is also uniform on $k \in [\frac{n\|\tilde{\mathbf{a}}\|}{\|\mu_a\|} - M\sqrt{n}, \frac{n\|\tilde{\mathbf{a}}\|}{\|\mu_a\|} + M\sqrt{n}]$. For the future purposes we denote $\kappa \equiv \frac{\|\mu_a\|}{\|\tilde{\mathbf{a}}\|}$ and $I_M \equiv [\frac{n}{\kappa} - M\sqrt{n}, \frac{n}{\kappa} + M\sqrt{n}] \bigcap \mathbb{Z}$.

Finally, we want to prove the following technical result, in which we use the uniformity of convergence for all $k = k(n) \in I_M$ and the truncation techniques to show the convergence of $Y^*_{n,k}$ to the Brownian bridge in case when we condition only on the existence of such k.

Technical Theorem. *The process*

$$\{Y^*_{n,k} \textit{ for some } k \textit{ such that } [t_k, Y_k]_f = n\tilde{\mathbf{a}}\}$$

conditioned on the existence of such k converges weakly to the Brownian bridge.

Proof: Take M large, notice that for $A \subset C^{d-1}[0,1]$,

$$\max_{k \in I_M} |P[Y_k^* \in A \mid [t_k, Y_k]_f = n\tilde{\mathbf{a}}] - P[B^o \in A]| = o(1),$$

where the brownian bridge B^o is scaled up to the same constant for all those k.

Hence,

$$lim_{n\to+\infty} \frac{\sum_{k\in I_M} P[S_k = n\tilde{\mathbf{a}}]P[Y^*_{n,k} \in A|S_k = n\tilde{\mathbf{a}}]}{\sum_{k\in I_M} P[S_k = n\tilde{\mathbf{a}}]} = P[B^o \in A].$$

Therefore we are only left to prove the truncation argument as $M \to +\infty$. Now, for any $\epsilon > 0$ there exists $M > 0$ such that

$$(1+\epsilon)\sum_{k\in I_M} P[S_k = n\tilde{\mathbf{a}}] \le \sum_k P[S_k = n\tilde{\mathbf{a}}] \le (1+2\epsilon)\sum_{k\in I_M} P[S_k = n\tilde{\mathbf{a}}]$$

for n large enough, as by the large deviation upper bound, there is a constant $\bar{C}_{LD} > 0$ such that

$$P[S_k = n\tilde{\mathbf{a}}] \le e^{-\bar{C}_{LD}\frac{(n-k\kappa)^2}{k} \wedge |n-k\kappa|},$$

and therefore $\exists C_{LD} > 0$ such that

$$\sum_{|n-k\kappa|>n^{2/3}} P[S_k = n\tilde{\mathbf{a}}] < e^{-C_{LD}n^{1/3}}.$$

Also, by the local CLT (see appendix A),

$$P[S_k = n\tilde{\mathbf{a}}] = P[\bar{S}_k = (n-k\kappa)\tilde{\mathbf{a}}] = \frac{1}{k^{d/2}\sqrt{Var\bar{X}_1(2\pi)^d}} e^{-\frac{1}{2Var\bar{X}_1}\frac{(n-k\kappa)^2}{k}} + o(\frac{1}{k^{d/2}})$$

implying

$$\sum_{|n-k\kappa|\le n^{2/3}} P[S_k = n\tilde{\mathbf{a}}] = \frac{1}{n^{\frac{d-1}{2}}}[\int_{-\infty}^{+\infty} \frac{1}{\sqrt{Var\bar{X}_1(2\pi)^d}} e^{-\frac{x^2}{2Var\bar{X}_1}} dx + o(1)]$$

where

$$\sum_{k\in I_M} P[S_k = n\tilde{\mathbf{a}}] = \frac{1}{n^{\frac{d-1}{2}}}[\int_{-M}^{M} \frac{1}{\sqrt{Var\bar{X}_1(2\pi)^d}} e^{-\frac{x^2}{2Var\bar{X}_1}} dx + o(1)].$$

Therefore

$$\frac{1}{1+2\epsilon}\frac{\sum_{k\in I_M} P[S_k = n\tilde{\mathbf{a}}]P[Y^*_{n,k} \in A|S_k = n\tilde{\mathbf{a}}]}{\sum_{k\in I_M} P[S_k = n\tilde{\mathbf{a}}]} \leq \frac{\sum_k P[S_k = n\tilde{\mathbf{a}}]P[Y^*_{n,k} \in A|S_k = n\tilde{\mathbf{a}}]}{\sum_k P[S_k = n\tilde{\mathbf{a}}]}$$

$$\leq \frac{1}{1+\epsilon}\frac{\sum_{k\in I_M} P[S_k = n\tilde{\mathbf{a}}]P[Y^*_{n,k} \in A|S_k = n\tilde{\mathbf{a}}]}{\sum_{k\in I_M} P[S_k = n\tilde{\mathbf{a}}]}$$

for all $A \subset C^{d-1}[0,1]$. Taking the lim inf and lim sup of the fraction in the middle completes the proof.

□

Chapter 9

Conclusions

As for the current develoments in the field, first, we know of an interesting research done under supervision of D.Ioffe to produce a Brownian bridge asymptotics for some cases of 2D Ising model in lieu of [7].

In case of the subcritical bond percolation model, a cluster conditioned on connecting three or more faraway points is studied (see [22]), and some precise asymptotes are produced.

As yet another possible extension of the subject studied in this thesis, we conjecture that the result of chapter 6 holds (up to a scalar multiple) for the subcritical Fortuin-Kasteleyn (FK) random-cluster models with $q \geq 1$. The random-cluster percolation models which, as it was shown in [11], are just a recast of the Potts models can be characterized with the corresponding FK random-cluster measures. That is for a finite graph $G = (V, E)$, we let $\Omega_E = \{0, 1\}^E$ be the set of all possible outcomes of the percolation on the edges of the graph, namely for an $\omega \in \Omega_E$, we let $\omega(e) = 1$ if the edge e is open, and $\omega(e) = 0$ otherwise. Then the random-cluster measure of an $\omega \in \Omega_E$ for percolation probability p and a parameter $q > 0$ would be

$$\phi_{p,q}(\omega) = \frac{1}{Z_{G,p,q}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)},$$

where $k(\omega)$ is the number of open connected components on the graph and $Z_{G,p,q}$ is a normalization factor. We notice that on a square lattice, the case of $q = 1$ corresponds to the regular bond percolation. The measure satisfies the FKG inequality if $q \geq 1$. We conjecture that inside a large box with sides of order n, the result of chapter 6 holds in some form for all $q \geq 1$.

We would also like to mention an important new direction in the study of rare events primarily in 3D supercritical percolation replicating some well known results concerning the interfaces in 3D Ising and Potts models, and various disordered spin systems. For the edges of the 3-dimensional square lattice, one can define the corresponding "dual faces" (or "plaquettes") as an analogue of the dual edges in two dimensions. So, in case of the 3D supercritical bond percolation model, the specific interface consisting of the connected plaquettes corresponding to the closed edges inside a cube is studied. The edges on the boundary of the cube are subject to the so called "Dobrushin boundary" conditions opening all the edges on the boundary of the cube except for a belt of closed edges (connecting points $(x_1, x_2, 0)$ and $(x_1, x_2, 1)$ outside the cube), splitting the surface of the cube in two open "semi-spheres". The interface appears if we condition on the two semi-spheres being disconnected inside the cube, so that there is a closed dual interface separating them. We want to point out that the problem of finding a precise assymptotics for the interface is similar to that studied in chapter 6, though a different approach might be needed.

Appendix A

Local Limit Theorem

In this thesis we use the version of the local CLT borrowed from [10]: Let $X_1, X_2, ... \in \mathbb{R}$ be i.i.d. with $\mathbf{E}X_i = 0$, $\mathbf{E}X_i^2 = \sigma^2 \in (0, \infty)$, and having a common lattice distribution with span h. If $S_n = X_1 + ... + X_n$ and $P[X_i \in b + h\mathbb{Z}] = 1$ then $P[S_n \in nb + h\mathbb{Z}] = 1$. We put

$$p_n(x) = P[S_n/\sqrt{n} = x] \text{ for } x \in \Lambda_n = \{(nb + hz)/\sqrt{n} : z \in \mathbb{Z}\}$$

and

$$n(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2) \text{ for } x \in (-\infty, \infty)$$

Local CLT. *Under the above hypotheses,* $\sup_{x \in \Lambda_n} |\frac{\sqrt{n}}{h} p_n(x) - n(x)| \to 0$ *as* $n \to \infty$.

Appendix B

Weak Convergence

Here we briefly outline the results on weak convergence and tightness in $C = C[0, 1]$ necessary for the material presented in the thesis. The uniform topology on C is defined by the distance norm $d(x, y) \equiv \sup_t |x(t) - y(t)|$ for all functions x and y in C. Please see [3] for a consistent and elaborate presentation of the material of this chapter. The content of the chapter is in fact based on [3].

We are given probability measures $\{P_n\}_n$ and P on $(S, \mathcal{F})$, where S is a metric space (in our case it is $S = C[0, 1]$) and $\mathcal{F}$ is the corresponding σ-algebra. If $\int_S f dP_n \to \int_S f dP$ for every bounded, continuous real function $f(\cdot)$ on S, we say that P_n **converges weakly** to P and write $P_n \Rightarrow P$. We say that a sequence of random variables $\{X_n\}_n$ **converges in distribution** to the random variable X if the corresponding distributions P_n of the $\{X_n\}_n$ converge weakly to the distribution P of X (e.g. $P_n \Rightarrow P$).

Here is another crucial definition: A probability measure P on $(S, \mathcal{F})$ is **tight** if for every $\varepsilon \in [0, 1]$ there exists a compact set K such that $P(K) > 1 - \varepsilon$. Similarly, for a family Π of probability measures on $(S, \mathcal{F})$, Π is **tight** if for every $\varepsilon \in [0, 1]$ there exists a compact set K such that $P(K) > 1 - \varepsilon$ for all P in Π.

Now, we are ready to state the classical results crucial for the weak convergence argument that we have used here to derive the Brownian Bridge asymptotics (8.2) for a simple walk with a drift which later led us to proving the main results of convergence in case of percolation, self-avoiding walks and other models of statistical mechanics.

We say that a family Π of probability measures on $(S, \mathcal{F})$ is **relatively compact** if in every sequence of elements of Π there is a weakly convergent subsequence. The theorem of Prohorov stated below says that tightness is necessary and sufficient for relative compactness, and therefore (as it is going to be shown in the theorem to follow) convergence of finite-dimensional distributions together with tightness are enough to show the weak convergence of probability measures. Here is the statement of the theorem:

Prohorov's Theorem. • *If Π is tight, then it is relatively compact.*
• *Suppose S is separable and complete. If Π is relatively compact, then it is tight.*

The statement of the next theorem concerns the case of $S = C \equiv C[0,1]$ and $\mathcal{F} = \mathcal{C}$.

Theorem 9. *Let P_n,P be probability measures on $(C, \mathcal{C})$. If the finite-dimensional distributions of P_n converge weakly to those of P, and if $\{P_n\}_n$ is tight, then $P_n \Rightarrow P$.*

As it was mentioned before, the theorem follows directly from Prohorov's theorem. It is worth noticing that Arzelà-Ascoli theorem plays a fundamental role in proving the tightness argument of the theorem below.

Theorem 10. *The sequence $\{P_n\}_n$ of probability measures on $(C, \mathcal{C})$ is tight if and only if the following two conditions hold:* ***(a)*** *For each positive η, there exists an a such that*

$$P_n[x : |x(0)| > a] \leq \eta, \text{ for all } n \geq 1.$$

(b) *For each positive ε and η, there exist a $\delta \in (0,1)$, and an integer N such that*

$$P_n[x : \sup_{|s-t|<\delta} |x(s) - x(t)| \geq \varepsilon] \leq \eta, \text{ for all } n \geq N.$$

Here is another tightness result which follows directly from the previous theorem. The setting is similar to that of the Donsker's theorem, theorem 6 of this thesis and the technical theorem proved in section 8.2. We define $\{X_n(t, \omega)\}_n$ in $(C, \mathcal{C})$ in the following way: we let $\xi_1, \xi_2, ...$ to be a sequence of random variables on some probability space $(\Omega, \mathcal{B}, P)$ with finite variance $\sigma^2 > 0$ ($\xi_1, \xi_2, ...$ need not be stationary or independent). We also define $S_0 \equiv 0$ and $S_n \equiv \xi_1 + ... + \xi_n$. For points $\frac{i}{n} \in [0, 1]$, we set $X_n(\frac{i}{n}, \omega) = \frac{1}{\sigma\sqrt{n}} S_i(\omega)$, and for the rest of the points

$$X_n(t, \omega) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}(\omega) + (nt - [nt]) \frac{1}{\sigma\sqrt{n}} \xi_{[nt]+1}(\omega).$$

Theorem 11. *The sequence $\{X_n\}_n$ is tight if for each positive ε there exist a $\lambda > 1$ and an integer N such that, if $n \geq N$, then*

$$P[\max_{i \leq n} |S_{k+i} - S_k| \geq \lambda\sigma\sqrt{n}] \leq \frac{\varepsilon}{\lambda^2}$$

holds for all k.

The above theorems were significant for proving various modifications of Donsker's Theorem, and as it was mentioned before, for proving similar theorems: Theorem 6 and Technical Theorem of section 8.2, which together constituted one of the major instruments for the research work described in this thesis.

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