Discrete and continuous quantum walks

Yevgeniy Kovchegov
Oregon State University

(based on joint work with R.Burton, Z.Dimcovic and T.Nguyen)

Introduction

Stochastic processes: $\ell^1(\mathbb{R})$ norm preserving linear evolution

$$\frac{d}{dt}\mu_t = \mu_t Q$$

For $\mu_t = (\mu_1(t), \mu_2(t), \dots)$ being $\ell^1(\mathbb{R})$ norm preserving means

$$\mu_1(t) + \mu_2(t) + \cdots = 1$$

at all times.

Introduction

Quantum evolution: $\ell^2(\mathbb{C})$ norm preserving linear evolution

$$\frac{d}{dt}\psi_t = -iH\psi_t, \quad H \text{ is self-adjoint}$$

For $\psi_t = (\psi_1(t), \psi_2(t), \dots)$ being $\ell^2(\mathbb{C})$ norm preserving means

$$|\psi_1(t)|^2 + |\psi_2(t)|^2 + \cdots = 1$$

at all times.

Dirac notations: $\frac{d}{dt}|\psi_t >= -iH|\psi_t >$

Introduction

Shrödinger Eq.

$$\frac{d}{dt}\psi_t = -iH\psi_t, \quad H \text{ is self-adjoint}$$

Dirac notations: $\frac{d}{dt}|\psi_t >= -iH|\psi_t >$

Hamiltonian operator H: Eigenvalues must be real $\lambda_j \in \mathbb{R}$, and the eigenvectors v_j are orthonormal.

Operator $U_t = e^{-itH}$ will have eigenvectors $e^{-it\lambda_j}$ of unit magnitude, and the same orthonormal eigenvectors v_j

Introduction

Operator $U_t = e^{-itH}$ will have eigenvectors $e^{-it\lambda_j}$ of unit magnitude, and the same orthonormal eigenvectors v_j

Take $\psi = \sum_j a_j v_j$ s.t. $\sum_j |a_j|^2 = 1$, then

$$U_t \psi = \sum_j a_j e^{-it\lambda_j} v_j,$$

where $\sum_j |a_j e^{-it\lambda_j}|^2 = 1$

Dirac notation: $|\psi >= \sum_j a_j |v_j >$, then

$$U_t |\psi >= \sum_j a_j e^{-it\lambda_j} |v_j >$$

Classical randomized algorithms

Randomized algorithms is an effective tool for speeding up computations and is an important field for applications of stochastic processes, e.g. Markov chain Monte Carlo (MCMC).

In short: classical computation makes use of the $\ell^1(\mathbb{R})$ norm preserving linear Markov evolution $\frac{d}{dt}\mu_t = \mu_t Q$

Randomized algorithms

In short: classical computation makes use of the $\ell^1(\mathbb{R})$ norm preserving linear Markov evolution$\frac{d}{dt}\mu_t = \mu_t Q$

Quantum computation: analogous tool is being developed, called the **quantum walk**. Idea: make use of the $\ell^2(\mathbb{C})$ norm preserving linear Shrödinger evolution $\frac{d}{dt}|\psi_t >= -iH|\psi_t >$

Both the classical and quantum computers provide the framework for implementation.

Quantum computation: qubits

One qubit system: two basis vectors $|0>$ and $|1>$

Two qubit system: four basis vectors $|00>$, $|01>$, $|10>$ and $|11>$

Another notation: $|0>$, $|1>$, $|2>$ and $|3>$

Tensor notation: $|0>\otimes|0>$, $|0>\otimes|1>$, $|1>\otimes|0>$ and $|1>\otimes|1>$

Quantum walk

Hilbert space $\mathcal{H}_C = \{|\downarrow>, |\uparrow>\}$ represents the outcome of a "coin toss"

Hilbert space $\mathcal{H}_P$ represents the position of the walker

Distribution

$$|\psi >= \sum_j a_j |\uparrow> \otimes |j> + b_j |\downarrow> \otimes |j>$$

means the walker is at site j with probability

$$|a_j|^2 + |b_j|^2$$

Quantum walk

Hilbert space: $\mathcal{H}_C \otimes \mathcal{H}_P$

Shrödinger Eq. $\frac{d}{dt}|\psi_t >= -iH|\psi_t >$

Discrete time: $U = e^{-iH}$, $|\psi_{t+1} >= U|\psi_t >$

Quantum evolution:

$$|\psi_t >= \sum_j a_j(t)|\uparrow> \otimes |j> + b_j(t)|\downarrow> \otimes |j>$$

means the walker is at site j with probability

$$|a_j(t)|^2 + |b_j(t)|^2$$

Hadamard quantum walk

Conside the following (Hdamard) coin on the two qubit space $\mathcal{H}_C = \{|\downarrow>, |\uparrow>\}$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Let the transition matrix for the Hadamard quantum walk be the following operator on $\mathcal{H}_C \otimes \mathcal{H}_P$

$$U = S(C \otimes I),$$

where

$$S = |\uparrow><\uparrow| \otimes \sum_s |s{+}1><s| + |\downarrow><\downarrow| \otimes \sum_s |s{-}1><s|$$

Example: Hadamard quantum walk

Take $|\psi_0 >= |\downarrow> \otimes|0 >$. First iteration:

$$(C \otimes I)|\psi_0 >= C|\downarrow> \otimes I|0 >$$

$$= \frac{1}{\sqrt{2}}|\uparrow> \otimes|0 > - \frac{1}{\sqrt{2}}|\downarrow> \otimes|0 >$$

Now,

$$S = |\uparrow><\uparrow| \otimes \sum_s |s+1 >< s| + |\downarrow><\downarrow| \otimes \sum_s |s-1 >< s|$$

and $|\psi_1 >= U|\psi_0 >= S(C \otimes I)|\psi_0 >$

$$= \frac{1}{\sqrt{2}}|\uparrow> \otimes|1 > - \frac{1}{\sqrt{2}}|\downarrow> \otimes|-1 >$$

$$|\psi_0 >= |\downarrow> \otimes |0>$$

$$|\psi_1 >= \frac{1}{\sqrt{2}}|\uparrow> \otimes |1> - \frac{1}{\sqrt{2}}|\downarrow> \otimes |-1>$$

Next iteration:

$$(C \otimes I)|\psi_1 >= \frac{1}{\sqrt{2}} C|\uparrow> \otimes |1> - \frac{1}{\sqrt{2}} C|\downarrow> \otimes |-1>$$

$= \frac{1}{2}|\uparrow> \otimes |1> + \frac{1}{2}|\downarrow> \otimes |1> - \frac{1}{2}|\uparrow> \otimes |-1>$
$+ \frac{1}{2}|\downarrow> \otimes |-1>$ Now,

$$S = |\uparrow><\uparrow| \otimes \sum_s |s+1><s| + |\downarrow><\downarrow| \otimes \sum_s |s-1><s|$$

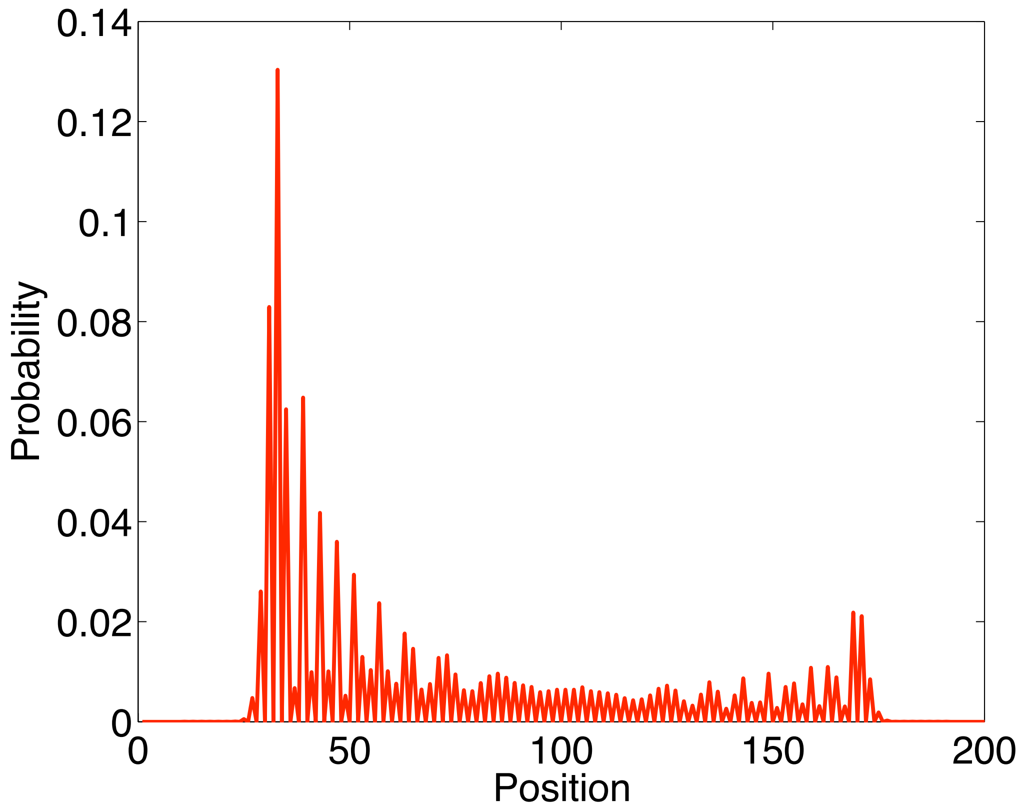
and $|\psi_2 >= \frac{1}{2}|\uparrow> \otimes |2> + \frac{1}{2}|\downarrow> \otimes |0>$
$-\frac{1}{2}|\uparrow> \otimes |0> + \frac{1}{2}|\downarrow> \otimes |-2>$

Now $|\psi_2>= \frac{1}{2}|\uparrow> \otimes|2> +\frac{1}{2}|\downarrow> \otimes|0>$
$-\frac{1}{2}|\uparrow> \otimes|0> +\frac{1}{2}|\downarrow> \otimes|-2>$

$$(C\otimes I)|\psi_2>= \frac{1}{2}C|\uparrow> \otimes|2> +\frac{1}{2}C|\downarrow> \otimes|0>$$
$$-\frac{1}{2}C|\uparrow> \otimes|0> +\frac{1}{2}C|\downarrow> \otimes|-2>$$
$$= \frac{1}{2\sqrt{2}}|\uparrow> \otimes|2> +\frac{1}{2\sqrt{2}}|\downarrow> \otimes|2>$$
$$-\frac{2}{2\sqrt{2}}|\downarrow> \otimes|0> +\frac{1}{2\sqrt{2}}|\uparrow> \otimes|-2>$$
$$-\frac{1}{2\sqrt{2}}|\downarrow> \otimes|-2>$$

$$|\psi_2>= \frac{1}{2\sqrt{2}}|\uparrow> \otimes|3> +\frac{1}{2\sqrt{2}}|\downarrow> \otimes|1>$$
$$-\frac{2}{2\sqrt{2}}|\downarrow> \otimes|-1> +\frac{1}{2\sqrt{2}}|\uparrow> \otimes|-1>$$
$$-\frac{1}{2\sqrt{2}}|\downarrow> \otimes|-3>$$

Quantum Walk with Hadamard coin



Markov chain with internal states

$U(\pm| \uparrow> \otimes |s >)$

$$= \frac{1}{\sqrt{2}}(\pm| \uparrow> \otimes |s+1 >) + \frac{1}{\sqrt{2}}(\pm| \downarrow> \otimes |s-1 >)$$

and

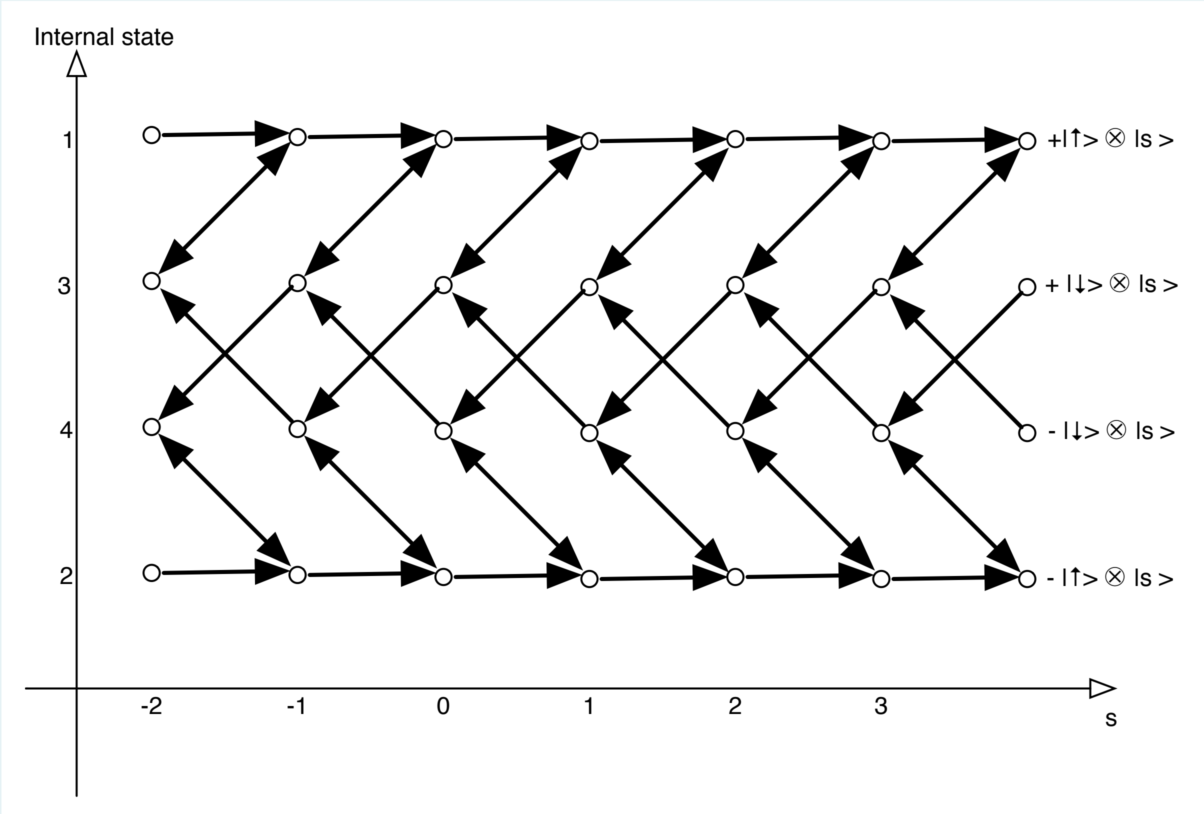
$U(\pm| \downarrow> \otimes |s >)$

$$= \frac{1}{\sqrt{2}}(\pm| \uparrow> \otimes |s+1 >) + \frac{1}{\sqrt{2}}(\mp| \downarrow> \otimes |s-1 >)$$

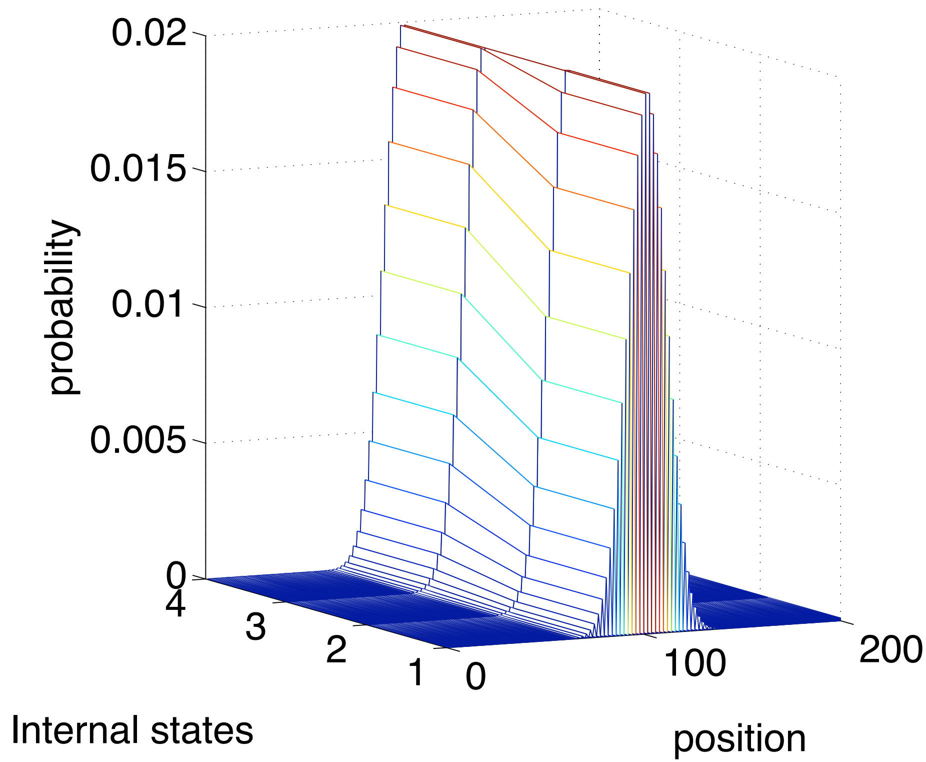
Four internal states:

$(| \uparrow> \otimes |s >), \quad -(| \uparrow> \otimes |s >), \quad (| \downarrow> \otimes |s >)$
and $\quad -(| \downarrow> \otimes |s >)$

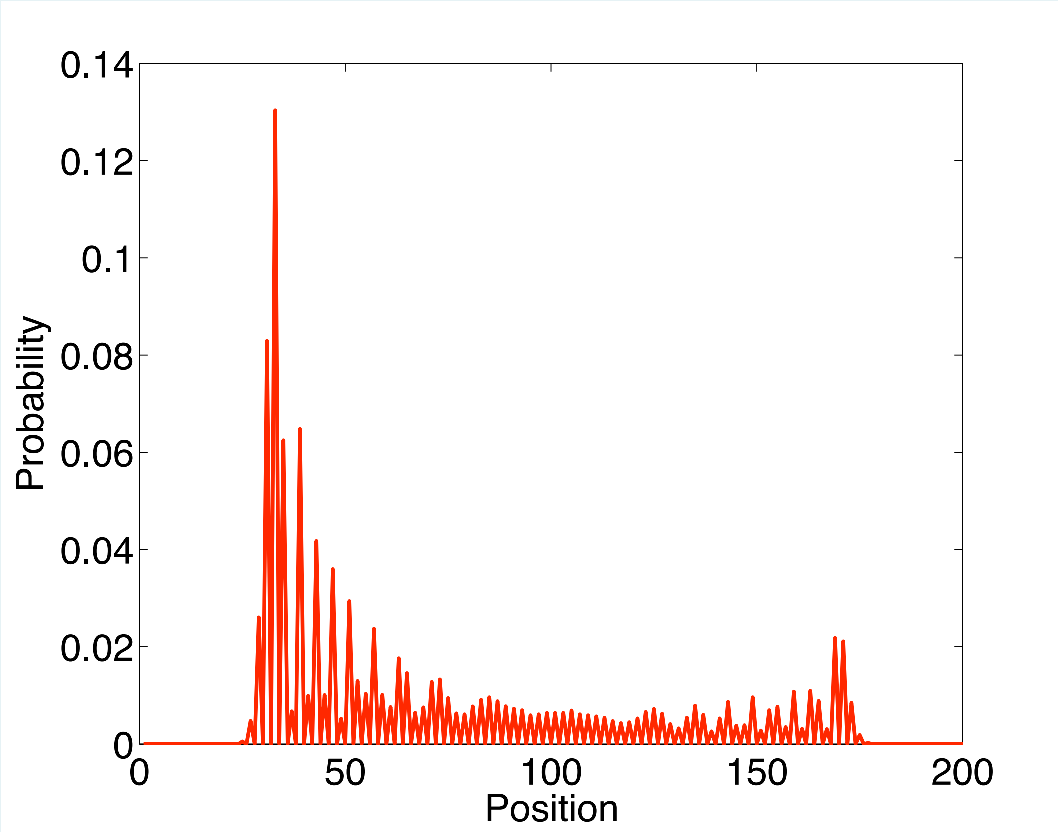
Markov Chain



Markov chain



$$2^n[(P_n(s,1)-P_n(s,2))^2+(P_n(s,3)-P_n(s,4))^2]$$



Four eigenvalues in the Fourier space:

$$\lambda_1 = 0, \quad \lambda_2 = 2\cos(k)$$

$$\text{and} \quad \lambda_{3,4} = \pm\sqrt{1+\cos^2(k)} + i\sin(k)$$

The distribution of the Hadamard quantum walk is expressed in the closed form as

$$\mu_n(s) = \frac{D^2_{\uparrow,n}(s) + D^2_{\downarrow,n}(s)}{2^n},$$

where

$$D_{\uparrow,n}(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-i(s+1)k}}{2\sqrt{1+\cos^2(k)}}\left[\lambda_3^n - \lambda_4^n\right] dk$$

and $D_{\downarrow,n}(s)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-isk}\left[(\lambda_4^n - \lambda_3^n)\cos(k) + (\lambda_3^n + \lambda_4^n)\sqrt{1+\cos^2(k)}\right]}{2\sqrt{1+\cos^2(k)}} dk$$