A note on adiabatic theorem for Markov chains and adiabatic quantum computation

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Introduction

Max Born and Vladimir Fock in 1928: "a physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum" (see Wikipedia page on adiabatic theorem).

Introduction

A way of expressing Hamiltonians via reversible Markov chains $\Rightarrow$ Corresponding theorem for Markov chains.

Quantum adiabatic theorem

Given two Hamiltonians, $H_{initial}$ and H_{final}, acting on a quantum system. Let

$$H(s) = (1 - s)H_{initial} + sH_{final}$$

System evolves according to $H(t/T)$ on t : 0 $\rightarrow T$ (*adiabatic evolution*).

Quantum adiabatic theorem

$$H(s) = (1-s)H_{initial} + sH_{final}$$

System evolves according to $H(t/T)$ on $t : 0 \rightarrow T$ (*adiabatic evolution*).

Adiabatic thm of quantum mechanics: For T large enough, the final state of the system will be close to the ground state of H_{final}, i.e. ϵ close in l_2 norm whenever $T \geq \frac{C}{\epsilon\beta^3}$, where β is the least spectral gap of $H(s)$ over all $s \in [0,1]$.

Mixing and relaxation times

Definition. If μ and ν are two probability distributions over Ω, the *total variation distance* is

$$\|\mu-\nu\|_{TV} = \frac{1}{2}\sum_{x\in\Omega} |\mu(x)-\nu(x)| = \sup_{A\subset\Omega} |\mu(A)-\nu(A)|$$

Observe that the total variation distance measures the coincidence between the distributions on a scale from zero to one.

Mixing and relaxation times

Definition. Suppose P is an irreducible and aperiodic Markov chain with stationary distribution π, i.e. $\pi P = \pi$. Given an $\epsilon > 0$, the mixing time $t_{mix}(\epsilon)$ is defined as

$$t_{mix}(\epsilon) = \inf \left\{ t \ : \ \|\nu P^t - \pi\|_{TV} \leq \epsilon, \quad \text{for all } \nu \right\}$$

Mixing and relaxation times

Suppose $P = \Big(p(x,y)\Big)_{x,y}$ is *reversible*:

$$\pi(x)p(x,y) = \pi(y)p(y,x) \quad x,y \in \Omega$$

P is reversible $\Rightarrow$ P is self-adjoint w.r.t. π
$\Rightarrow$ All real eigenvalues.

$\beta = 1 - |\lambda_2|$ is the *spectral gap* of P.

The *relaxation time* is defined as

$$\tau_{rlx} = \frac{1}{\beta}$$

Mixing and relaxation times

Theorem. If P is a reversible, irreducible and aperiodic Markov chain. Then

$$(\tau_{rlx} - 1) \log(2\epsilon)^{-1} \leq t_{mix}(\epsilon) \leq \tau_{rlx} \log(\epsilon \min_{x \in \Omega} \pi(x))^{-1}$$

Adiabatic time Given $P_{initial}$ and P_{final}, where P_{final} is irreducible and aperiodic. Let

$$P_s = (1-s)P_{initial} + sP_{final}$$

π_f is the stationary distribution for P_{final}.

Definition. Given $\epsilon > 0$, a time T_ϵ is called the *adiabatic time* if it is the least T such that

$$\max_\nu \|\nu P_{\frac{1}{T}} P_{\frac{2}{T}} \cdots P_{\frac{T-1}{T}} P_1 - \pi_f\|_{TV} \leq \epsilon,$$

where the maximum is taken over all probability distributions ν over Ω.

Adiabatic theorem for Markov chains

Theorem A (K.2008) Let t_{mix} denote the mixing time for P_{final}. Then the adiabatic time

$$T_\epsilon = O\left(\frac{t_{mix}(\epsilon/2)^2}{\epsilon}\right)$$

Proof:

$$\nu P_{\frac{1}{T}} P_{\frac{2}{T}} \cdots P_{\frac{T-1}{T}} P_1 = \frac{T!}{N! T^{T-N}} \;\; u_N P_{final}^{T-N} + \mathcal{E},$$

where $u_N = \nu P_{\frac{1}{T}} P_{\frac{2}{T}} \cdots P_{\frac{N}{T}}$ and $\mathcal{E}$ is the rest of the terms.

Hence, by triangle inequality,

$\max_\nu \|\nu P_{\frac{1}{T}} P_{\frac{2}{T}} \cdots P_{\frac{T-1}{T}} P_1 - \pi_f\|_{TV}$

$$\leq \max_\nu \|\nu P_{final}^{T-N} - \pi_f\|_{TV} \cdot \frac{T!}{N! T^{T-N}} + S_N,$$

where $0 \leq S_N \leq 1 - \frac{T!}{N! T^{T-N}}$.

Proof:

Let $T = Kt_{mix}(\epsilon/2)$ and $N = (K-1)t_{mix}(\epsilon/2)$

Then

$$\max_{\nu} \|\nu P_{final}^{T-N} - \pi_f\|_{TV} \leq \epsilon/2$$

and

$$e^{\int_N^T \log x dx + (T-N)\log T} \leq \frac{T!}{N!T^{T-N}}$$

simplifies to

$$\left(\frac{\left(1+\frac{1}{K-1}\right)^{K-1}}{e}\right)^{t_{mix}(\epsilon/2)} \leq \frac{T!}{N!T^{T-N}} \leq 1$$

Proof:

$\max_\nu \|\nu P_{\frac{1}{T}} P_{\frac{2}{T}} \cdots P_{\frac{T-1}{T}} P_1 - \pi_f\|_{TV}$

$$\leq \max_\nu \|\nu P_{final}^{T-N} - \pi_f\|_{TV} \cdot \frac{T!}{N! T^{T-N}} + S_N,$$

where $0 \leq S_N \leq 1 - \frac{T!}{N!T^{T-N}}$ and

$$0 \leq S_N \leq 1 - \frac{T!}{N!T^{T-N}} \leq 1 - \left(\frac{\left(1 + \frac{1}{K-1}\right)^{K-1}}{e} \right)^{t_{mix}(\epsilon/2)}$$

Proof:

Need the least K such that

$$1 - \left(\frac{\left(1 + \frac{1}{K-1}\right)^{K-1}}{e} \right)^{t_{mix}(\epsilon/2)} \leq \epsilon/2$$

and since $\log(1+x) = x - \frac{x^2}{2} + O(x^3)$, the least such K is approximated as follows

$$K \approx \frac{t_{mix}(\epsilon/2)}{-2\log\left(1-\epsilon/2\right)} \approx \frac{t_{mix}(\epsilon/2)}{\epsilon}$$

Thus for $T = K t_{mix}(\epsilon/2) \approx \frac{t_{mix}(\epsilon/2)^2}{\epsilon}$,

$$\max_{\nu} \|\nu P_{\frac{1}{T}} P_{\frac{2}{T}} \cdots P_{\frac{T-1}{T}} P_1 - \pi_f\|_{TV} \leq \epsilon$$

Mixing and relaxation times

Theorem. If P is a reversible, irreducible and aperiodic Markov chain. Then

$$(\tau_{rlx}-1)\log(2\epsilon)^{-1} \leq t_{mix}(\epsilon) \leq \tau_{rlx}\log(\epsilon \min_{x\in\Omega} \pi(x))^{-1}$$

Theorem A (K.2008) Let t_{mix} denote the mixing time for P_{final}. Then the adiabatic time

$$T_\epsilon = O\left(\frac{t_{mix}(\epsilon/2)^2}{\epsilon}\right)$$

A Corollary

If P_{final} is reversible, irreducible and aperiodic with its spectral gap $\beta > 0$, then

$$T_\epsilon = O\left(\frac{\log\frac{2}{\epsilon} + \log\frac{1}{\min_{x\in\Omega}\pi_f(x)}}{\epsilon\beta^2}\right)$$

Continuous time Markov processes

Suppose Q is a bounded Markov generator for a continuous time Markov chain $P(t)$, and

$$\lambda \geq \max_{i \in \Omega} \sum_{j : j \neq i} q(i,j)$$

Uniformization:

$$P(t) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} P_\lambda^n, \qquad \text{where} \quad P_\lambda = I + \frac{1}{\lambda} Q$$

Mixing times: continuous time

Definition. $P(t)$ is an irreducible continuous time Markov chain with stationary distribution π. Given an $\epsilon > 0$, the mixing time $t_{mix}(\epsilon)$ is defined as

$$t_{mix}(\epsilon) = \inf\{t \;:\; \|\nu P(t) - \pi\|_{TV} \leq \epsilon, \quad \text{all } \nu\}$$

Continuous time

Suppose $Q_{initial}$ and Q_{final} are bounded Markov generators, and π_f is the only stationary distribution for Q_{final}. Let

$$Q[s] = (1-s)Q_{initial} + sQ_{final}$$

be a time non-homogeneous generator.

Given $T > 0$, let $P_T(s,t)$ $(0 \leq s \leq t \leq T)$ denote a matrix of transition probabilities of a Markov process generated by $Q\left[\frac{t}{T}\right]$.

Adiabatic time: continuous time

Definition. Given $\epsilon > 0$, a time T_ϵ is called the *adiabatic time* if it is the least T such that

$$\max_{\nu} \|\nu P_T(0,T) - \pi_f\|_{TV} \leq \epsilon,$$

where the maximum is taken over all probability distributions ν over Ω.

Adiabatic theorem for Markov processes.

Let

$$\lambda \geq \max_{i \in \Omega} \sum_{j : j \neq i} q_{initial}(i, j)$$

and

$$\lambda \geq \max_{i \in \Omega} \sum_{j : j \neq i} q_{final}(i, j)$$

$q_{initial}(i, j)$ and $q_{final}(i, j)$ are the rates in $Q_{initial}$ and Q_{final} respectively.

Adiabatic theorem for Markov processes.

$$\lambda \geq \max_{i\in\Omega} \sum_{j:j\neq i} q_{initial}(i,j), \lambda, \quad \max_{i\in\Omega} \sum_{j:j\neq i} q_{final}(i,j)$$

Then

Theorem B.(K. 2008) Let t_{mix} denote the mixing time for Q_{final}. Then adiabatic time

$$T_\epsilon \leq \frac{\lambda t_{mix}(\epsilon/2)^2}{\epsilon}$$

Proof:

$\lambda \geq \max_{i\in\Omega}\sum_{j:j\neq i} q_t(i,j)$, where $q_t(i,j)$ are rates in $Q\left[\frac{t}{T}\right]$ $(0 \leq t \leq T)$.

Let $T = Kt_{mix}(\epsilon/2)$ and $N = (K-1)t_{mix}(\epsilon/2)$, so that

$$\max_{\nu} \|\nu P_{final}(T-N) - \pi_f\|_{TV} \leq \epsilon/2,$$

where $P_{final}(t) = e^{tQ_{final}}$, transition probability generated by Q_{final}.

Let $P_0 = I + \frac{1}{\lambda}Q_{initial}$ and $P_1 = I + \frac{1}{\lambda}Q_{final}$.

Proof:

$$\nu P_T(0,T) = u_N \left(\sum_{n=0}^{\infty} \frac{(\lambda(T-N)^n}{n!} e^{-\lambda(T-N)} I_n \right),$$

where $u_N = \nu P_T(0,N)$ and

$$I_n = \tfrac{n!}{(T-N)^n} \iint [(1 - x_1/T)P_0 + (x_1/T)P_1] \ldots$$

$$\ldots [(1-x_n/T)P_0 + (x_n/T)P_1] dx_1 \ldots dx_n,$$

over $N < x_1 < x_2 < \cdots < x_n < T$.

Proof: Hence

$$\nu P_T(0,T)$$

$$= u_N \left(\sum_{n=0}^{\infty} \frac{(\lambda(T-N)^n e^{-\lambda(T-N)}}{(T-N)^n T^n n!} P_1^n \iint_N^T x_1 \ldots x_n dx_1 \ldots dx_n\right)$$

$$+\mathcal{E}$$

$$= e^{-\lambda t_{mix}(\epsilon/2)} u_N \left(\sum_{n=0}^{\infty} \frac{\lambda^n [(1-\frac{1}{2K}) t_{mix}(\epsilon/2)]^n}{n!} P_1^n\right)$$

$$+\mathcal{E}$$

$$= e^{-\lambda t_{mix}(\epsilon/2)} u_N P_{final} \left(\lambda \left(1 - \frac{1}{2K}\right) t_{mix}(\epsilon/2)\right)$$

$$+\mathcal{E}$$

where $\mathcal{E}$ is the rest of the terms.

Proof:

$$\nu P_T(0,T)$$
$$= e^{-\lambda t_{mix}(\epsilon/2)} u_N P_{final}\left(\lambda\left(1-\frac{1}{2K}\right)t_{mix}(\epsilon/2)\right)$$
$$+\mathcal{E}$$

where $\mathcal{E}$ is the rest of the terms.

Thus, the total variation distance,

$$\max_\nu \|\nu P_T(0,T) - \pi_f\|_{TV} \le e^{-\lambda t_{mix}(\epsilon/2)}\epsilon/2 + S_N$$

whenever $\lambda\left(1-\frac{1}{2K}\right) \ge 1$, i.e. $K \ge \frac{\lambda}{2(\lambda-1)}$, easy condition.

Proof:

$\max_{\nu} \|\nu P_T(0,T) - \pi_f\|_{TV} \leq e^{-\lambda t_{mix}(\epsilon/2)} \epsilon/2 + S_N$

Here

$S_N = \|\mathcal{E} - \pi_f\|_{TV}$

$$\leq 1 - e^{-\lambda t_{mix}(\epsilon/2)} \sum_{n=0}^{\infty} \frac{\lambda^n [(1 - \frac{1}{2K}) t_{mix}(\epsilon/2)]^n}{n!}$$

$$= 1 - e^{-\frac{\lambda t_{mix}(\epsilon/2)}{2K}} \leq \epsilon/2$$

whenever $K \geq \frac{\lambda t_{mix}(\epsilon/2)}{\epsilon}$, and therefore

$$T_\epsilon \leq \frac{\lambda t_{mix}(\epsilon/2)^2}{\epsilon}$$