Exclusion Processes with Multiple Interactions.

Yevgeniy Kovchegov *

Abstract

We introduce the mathematical theory of the particle systems that interact via permutations, where transition rates are assigned not to the jumps from a site to a site, but to the permutations themselves. These permutation processes can be viewed as the natural generalization of symmetric exclusion processes, where particles interact via transpositions. We develop a number of innovative coupling techniques for the permutation processes and establish the needed conditions for them to apply. We use duality, couplings and other tools to explore the stationary distributions of the permutation processes with translation invariant rates.

1 Introduction.

We begin by reformulating the general setup of symmetric exclusion processes. Let S be a general countable set, and $p(x, y)$ be transition probabilities for a Markov chain on S. Let η_t denote a continuous time Feller process with values in $\{0, 1\}^S$, where $\eta_t(x) = 1$ when the site $x \in S$ is occupied by a particle at time t while $\eta_t(x) = 0$ means the site is empty at time t. Exclusion process is an important example of a Markovian interacting particle system, with the name justified by the following condition on when a transition can occur

$$\eta \to \eta_{x,y} \quad \text{at rate} \quad p(x, y) \quad \textbf{if} \quad \eta(x) = 1, \eta(y) = 0,$$

where for $\eta \in \{0, 1\}^S$, $\eta_{x,y}(u) = \eta(u)$ when $u \notin \{x, y\}$, $\eta_{x,y}(x) = \eta(y)$ and $\eta_{x,y}(y) = \eta(x)$. The condition

$$\sup_{y \in S} \sum_x p(x, y) < \infty$$

is sufficient to guarantee that the exclusion process η_t is indeed a well defined Feller process. We refer the reader to [5] and [6] for a complete and rigorous treatment of the subject.

An exclusion process is said to be *symmetric* if $p(y, x) = p(x, y)$ for all $x, y \in S$. In the symmetric case we can reformulate the process by considering all the transpositions $\tau_{x,y}$. For each transposition $\tau_{x,y}$ $(x, y \in S,\ x \neq y)$ we will assign the corresponding rate $q(\tau_{x,y}) = p(y, x) = p(x, y)$ at which the transposition occurs:

$$\eta \to \tau_{x,y}(\eta) \quad \text{at rate} \quad q(\tau_{x,y}),$$

*This research was supported in part by NSF VIGRE Grant DMS 9983726 at UCLA

where $\tau_{x,y}(\eta) := \eta_{x,y}$. It was suggested to the author by T. Liggett to study the natural generalization of the symmetric exclusion process that arises with the above reformulation. Liggett's idea was to assign the rates not to the particles inhabiting the space S, but to the various permutations of finitely many points of S. Namely, we can consider other permutations besides the transpositions. We let Σ be the set of all such permutations with **positive** rates. If $\sigma \in \Sigma$, we let

$$Range(\sigma) = \{x \in S : \quad \sigma(x) \neq x\}.$$

For each $\eta \in \{0,1\}^S$, let $\sigma(\eta)$ be the new configuration of particles after the permutation σ was applied to η, i.e.

$$\sigma(\eta)(x) = \eta(\sigma^{-1}(x)) \quad \text{for all} \quad x \in S.$$

Observe that we only permute the particles inside $Range(\sigma)$.

Now, we want to construct a continuous time Feller process, where rates $q(\sigma)$ $(\sigma \in \Sigma)$ are assigned so that

$$\eta \to \sigma(\eta) \quad \text{at rate} \quad q(\sigma).$$

Example. Let $S = \mathbb{Z}$, and $\Sigma = \bigcup_{x \in \mathbb{Z}} \left\{ \sigma_x := \overline{(x, x+1, x+2)}, \sigma_x^2 = \sigma_x^{-1} \right\}$ consists of all the three-cycles of consecutive integers. As we will see later, the three-cycles are very special for the theory of "permutation" processes described in this manuscript.

First we would like to mention some of the results from the theory of exclusion processes that we will extend to the newly introduced permutation processes. For consistency we will use the notations of [5] and [6]. We let $\mathcal{I}$ denote the class of stationary distributions for the given Feller process. As the set $\mathcal{I}$ is convex, we will denote by $\mathcal{I}_e$ the set of all the extreme points of $\mathcal{I}$. The results that we want to generalize are the two theorems given below. Consider the case of $S = \mathbb{Z}^d$ with shift-invariant random walk rates (e.g. $p(x,y) = p(0, y-x)$). The first theorem was proved by F.Spitzer (see [9]) in the recurrent case and by T.Liggett in the transient case (see [2]).

Theorem. *For the symmetric exclusion process, $\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$, where ν_ρ is the homogeneous product measure on $\{0,1\}^S$ with marginal probability ρ (e.g. $\nu_\rho\{\eta : \eta := 1 \text{ on } A\} = \rho^{|A|}$ for any $A \subset S$).*

Let $\mathcal{S}$ denote the class of the shift invariant probability measures on $\{0,1\}^S$, and $(\mathcal{I} \cap \mathcal{S})_e$ the set of all extreme points of $(\mathcal{I} \cap \mathcal{S})$. Next theorem was proved in [4] by T.Liggett. A special case of it was proved by R.Holley in [1].

Theorem. *For the general exclusion process, $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$.*

As it was the case with the exclusion processes, coupling method will play the crucial role in proving the analogues of the above results for the permutation processes. The difficult part is to construct the right types of couplings for the corresponding proofs to work.

1.1 Existence of the process. The permutation law.

We need to formalize the construction of the permutation process. For a configuration $\eta \in \{0,1\}^S$ and a permutation $\sigma \in \Sigma$, $\sigma(\eta)$ defined as

$$\sigma(\eta)(x) := \eta(\sigma^{-1}(x)) \quad \text{for all } x \in S$$

is the resulting configuration after the permutation σ is applied. For any cylinder function f (i.e. a function $f(\eta)$ from $\{0,1\}^S$ to $\mathbb{R}$ that depends on finitely many sites in S), let

$$\Omega f(\eta) := \sum_{\sigma\in\Sigma} q(\sigma)[f(\sigma(\eta)) - f(\eta)].$$

Now, we have to guarantee that the permutation process η_t with generator Ω is a well defined Feller process. For this, by Theorem 3.9 of Chapter I in [5] (see also the conditions (3.3) and (3.8) there), it is sufficient to assume that the rates $q(\sigma)$ are such that for every $x \in S$,

$$M_{PL} := \sup_{x\in S} \sum_{\sigma: x\in Range(\sigma)} q(\sigma) < \infty. \tag{1}$$

Then the semigroup Ω_t of the permutation process η_t, generated by such Ω, is well defined. Such process will then be said to obey the permutation law (1).

Throughout the paper we require that the random walk generated by the permutations $\{\sigma \in \Sigma\}$ is **irreducible**, i.e. for every x and y in S there is a sequence $\sigma_1, \dots, \sigma_k \in \Sigma$ such that $\sigma_k \circ \dots \circ \sigma_1(x) = y$.

1.2 Duality.

For a nonnegative continuous function $H(\eta, \zeta)$ of two variables, the Markov processes η_t and ζ_t are said to be dual with respect to $H(\cdot,\cdot)$ if

$$E^\eta H(\eta_t, \zeta) = E^\zeta H(\eta, \zeta_t)$$

for all η, ζ and all $t \geq 0$.
For a configuration $\eta \in \{0,1\}^S$ and a set $A \subset S$ let

$$H(\eta, A) = \prod_{x\in A} \eta(x).$$

Applying the generating operator Ω to $H(\eta, A)$ as a function of η get

$$\begin{aligned}
\Omega H(\cdot, A)(\eta) &= \sum_{\sigma\in\Sigma} q(\sigma)[H(\sigma(\eta), A) - H(\eta, A)] \\
&= \sum_{\sigma\in\Sigma} q(\sigma)[H(\eta, \sigma^{-1}(A)) - H(\eta, A)] \\
&= \sum_{\sigma\in\Sigma} q(\sigma)[H(\eta, \sigma(A)) - H(\eta, A)], \qquad (2)
\end{aligned}$$

where the last equality holds whenever

$$q(\sigma) = q(\sigma^{-1}) \text{ for all } \sigma \in \Sigma. \tag{3}$$

Then the right hand side of (2) is equal to the generator Ω applied to $H(\eta, A)$ as a function of A, i.e.

$$\Omega H(\cdot, A)(\eta) = \Omega H(\eta, \cdot)(A),$$

and permutation processes η_t and A_t having the same transition rates that satisfy (3) with initial conditions $\eta_0 = \eta$ and $A_0 = A$ are dual with respect to $H(\cdot, \cdot)$. So a permutation process satisfying (3) is *self-dual*. Therefore

$$P^{\eta}[\eta_t \equiv 1 \text{ on } A] = P^{A}[\eta \equiv 1 \text{ on } A_t]. \tag{4}$$

Condition (3) is essential in order to have a useful duality. From now on we will say that a permutation process is *symmetric* whenever the above condition (3) is satisfied. Observe that the self-duality of symmetric permutation processes is analogous to that of symmetric exclusion processes, where the corresponding self-duality was indispensable and is the reason why symmetric exclusion was so successfully studied (see Chapter VIII of [5] and Part III of [6]).

2 Symmetric permutation processes.

Throughout the rest of the paper we restrict ourselves to studying **irreducible permutation processes on $S = \mathbb{Z}^d$ with translation invariant rates.** We will also assume that the rates $q(\sigma)$, for all $\sigma \in \Sigma$, satisfy the following two conditions.
Condition I.

$$M_I := \sup_{\sigma \in \Sigma} \big|Range(\sigma)\big| < \infty, \tag{5}$$

where $|\cdot|$ denotes the cardinality.

The second condition consists of two parts.
Condition II.

> **1)** If σ_1 is a finite permutation of elements in S such that $Range(\sigma_1) = Range(\sigma_2)$ for some $\sigma_2 \in \Sigma$, then $\sigma_1 \in \Sigma$.
> **2)**
>
> $$M_{II} := \sup_{\sigma_1, \sigma_2 \in \Sigma : Range(\sigma_1) = Range(\sigma_2)} \left|\frac{q(\sigma_1)}{q(\sigma_2)}\right| < \infty. \tag{6}$$

It should be mentioned that the first part of the above second condition is slightly stronger than it needs to be. We only need Σ to be the class of permutations where for any subset $R \subset$

S with $R = Range(\sigma)$ for some $\sigma \in \Sigma$, any ordering (word) of 1's and 0's can be permuted into any other ordering with the same number of 1's and 0's by applying a permutation from that class. Once again, assume that the permutation process satisfies the symmetry condition (3) that guarantees its self-duality. We will prove the following result.

Theorem 1. *For the symmetric permutation processes, $\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$, where ν_ρ is the homogeneous product measure on $\{0,1\}^S$ with marginal probability ρ (i.e. $\nu_\rho\{\eta : \eta := 1 \text{ on } A\} = \rho^{|A|}$).*

The notion of a bounded harmonic function for a Markov chain can be adapted to permutation processes. We will say that a bounded function $f : S \to \mathbb{R}$ is *harmonic* if for a permutation process η_t and each $t > 0$, $f(\eta) = \sum_{\zeta \in \{0,1\}^S} P^\eta[\eta_t = \zeta] f(\zeta)$. We refer the reader to Chapter I of [5] for more on Markov processes, their semigroups and construction of interacting particle systems. We will need the following

Theorem 2. *If f is a bounded harmonic function for the well defined finite permutation process A_t, then f is constant on $\{A : |A| = n\}$ for each given integer $n \geq 1$.*

As it was the case for symmetric exclusion, Theorem 1 follows from Theorem 2 and the duality of the process (see [5], Chapter VIII). The proof of Theorem 1 echos the corresponding proof in case of the symmetric exclusion processes. However, we will briefly go through it. Assume that we already have Theorem 2.

Proof of Theorem 1: A probability measure μ on $\{0,1\}^S$ is called *exchangeable* if for any finite $A \subset S$, $\mu\{\eta : \eta \equiv 1 \text{ on } A\}$ is a function of cardinality $|A|$ of A. By de Finetti's theorem, if S is infinite, then every exchangeable measure is a mixture of the homogeneous product measures ν_ρ. Therefore Theorem 1 holds if and only if $\mathcal{I}$ agrees with the set of exchangeable probability measures.

The duality equation (4) implies

$$\begin{aligned}
\mu\Omega_t\{\eta : \eta \equiv 1 \text{ on } A\} &= \int P^\eta[\eta_t \equiv 1 \text{ on } A] d\mu \\
&= \int P^A[\eta \equiv 1 \text{ on } A_t] d\mu \\
&= \sum_B P^A[A_t = B]\mu\{\eta : \eta \equiv 1 \text{ on } B\}.
\end{aligned}$$

Thus every exchangeable measure is stationary. Now, if $\mu \in \mathcal{I}$, then $\mu\Omega_t = \mu$ (for all t), so by the above equation, $f(A) = \mu\{\eta : \eta \equiv 1 \text{ on } A\}$ is harmonic for A_t. Hence Theorem 2 implies that μ is exchangeable. $\square$

Lets summarize the above construction: we used self-duality of symmetric permutation processes in order to show that if $\mu \in \mathcal{I}$, then $f(A) = \mu\{\eta : \eta \equiv 1 \text{ on } A\}$ is harmonic. We need to prove Theorem 2 in order to show that $f(A)$ is a function of cardinality $|A|$, i.e. μ is exchangeable. By de Finetti's theorem, μ is exchangeable if and only if it is a mixture of the homogeneous product measures ν_ρ. The proof of Theorem 2 is different for the processes with recurrent and transient rates. We will do both.

2.1 Recurrent case.

By **recurrence** here we mean the recurrence of $I_1(t) - I_2(t)$, where $I_1(t)$ and $I_2(t)$ are two independent one-point processes moving according to the permutation law as described in the introduction. For the rest of the subsection we will assume that the process is recurrent.

As it was the case with the symmetric exclusion processes, in order to prove Theorem 2 in the recurrent case it is enough to construct a *successful* coupling of two copies A_t and B_t of the permutation process with initial states A_0 and B_0 of the same cardinality n that coincide at all but two sites of S (i.e. $|A_0 \cap B_0| = n - 1$). By successful coupling we mean

$$P[A_t = B_t \text{ for all } t \text{ beyond some time }] = 1.$$

If f is a bounded harmonic function for the finite permutation process for which we can construct a successful coupled process (see above), then

$$|f(A_0) - f(B_0)| = |Ef(A_t) - Ef(B_t)| \leq E|f(A_t) - f(B_t)|$$

$$\leq \|f\| P[A_t \neq B_t].$$

We need to construct a successful coupling of A_t and B_t in order to have $P[A_t \neq B_t] \to 0$ as $t \to \infty$. Letting t go to infinity, we get $f(A_0) = f(B_0)$ thus proving Theorem 2 for the case when there are only two discrepancies between A_0 and B_0, i.e. the cardinalities $|A_0|$=$|B_0|$, and $|A_0 \cap B_0| = |A_0| - 1$. Then by induction Theorem 2 holds for all A_0 and B_0 of the same cardinality.

The points in $\{(A_t \cup B_t) \backslash (A_t \cap B_t)\}$ are called the "discrepancies". The coupled process $\begin{pmatrix} A_t \\ B_t \end{pmatrix}$ has two discrepancies at time $t = 0$. Our challenge is to couple the two permutation processes A_t and B_t so that the number of discrepancies never increases and in fact decreases with time (from two to zero). So we can have at most two discrepancies: one $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ discrepancy (that we denote by d_t^+) and one $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ discrepancy (that we denote by d_t^-). Here is an example:

$$\begin{array}{lccccccccccc}
A_t: & \dots & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\
B_t: & \dots & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \dots \\
 & & & & \uparrow & & & & \uparrow & & \\
 & & & & d_t^+ & & & & d_t^- & &
\end{array}$$

We recall a similar coupling construction by F.Spitzer that was implemented in the recurrent case for symmetric exclusion processes. There, whenever the two discrepancies happened to be inside the range of a transposition with positive rate, applying the transposition to either A_t or B_t we were canceling the discrepancies (see [9]). In our situation, the **tricky** component of such coupling construction is that when the two discrepancies happen to be inside the range of a permutation in Σ, applying the permutation to either A_t or B_t, even if canceling the original two discrepancies, might create new discrepancies. This is the challenge that we have to overcome in this subsection.

2.1.1 Coupling of two-point processes.

We postpone constructing a successful coupling of A_t and B_t up until 2.1.2 and instead concentrate on creating tools necessary for such a construction. Here we will consider three two-point processes I_t, J_t and E_t in S with the same initial configuration $x = (x_1, x_2)$ such that $x_1 \neq x_2$. We will construct two couplings, one of I_t and J_t, and one of E_t and J_t. First we need to define the processes.

We assume that the permutation rates $\{q(\sigma)\}_{\sigma\in\Sigma}$ are known. We define $I_t = \{I_1(t), I_2(t)\}$ to be the process that consists of two **independent** one-point permutation processes $I_1(t)$ and $I_2(t)$ on S, that is two independent one-point permutation processes (random walks) projected on the same space.

Now, we let $J_t = \{J_1(t), J_2(t)\} \subset S$ be the two-point process that depends on I_t in the following way. The initial configuration must be the same: $(I_1(0), I_2(0)) = (J_1(0), J_2(0)) = x$. The above one-point processes $I_1(t)$ and $I_2(t)$ live separate lives. For each of the two of them, every $\sigma \in \Sigma$ is enacted with frequency $q(\sigma)$. The total frequency will be $2q(\sigma)$. However, the permutations acting on one of the one-point processes will not affect the other. When constructing J_t, of all the permutations acting on $I_1(t)$ and $I_2(t)$ separately, we will apply to J_t only those of them that actually displace one of the two random walkers I_1 or I_2 . Hence, at every moment of time, we are waiting for the permutations that contain at least one of the two points (I_1 and I_2), assigning the corresponding q-rate to those containing exactly one of them in the range, and twice the q-rate to those containing both in the range.

Observe, that I_t and J_t are naturally coupled until the "decoupling" time T_{dec} when a permutation containing both $J_1(T_{dec}-)$ and $J_2(T_{dec}-)$ occurs ("$t-$" signifies time that precedes t such that no changes occur in $[t-, t)$ time interval). So $J_1(T_{dec}) \neq J_1(T_{dec}-)$ and $J_2(T_{dec}) \neq J_2(T_{dec}-)$. Such permutation should happen before $I_1(t) - I_2(t)$ visits zero for the first time. Thus

$$P^x\Big\{\exists T_{dec} \in (0,\infty) \text{ s.t. } J_1(T_{dec}) \neq J_1(T_{dec}-) \text{ and } J_2(T_{dec}) \neq J_2(T_{dec}-)\Big\} \tag{7}$$

$$\geq P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } I_1(t) = I_2(t)\Big\},$$

where P^x denotes the probability measure, given the initial configuration $x \in S^2$, for the corresponding two-point process (I_t or J_t, and later E_t). We recall that $I_1(t) - I_2(t)$ is recurrent. Hence the left hand side probability above is equal to one. As it will be seen soon, this is the primary reason why conditions (5) and (6) are necessary for the coupling construction in 2.1.2 that follows.

Now, on the time interval from zero until the decoupling time T_{dec} the process J_t behaves almost as a two-point permutation process. The only difference being the double rates applied to the permutations containing together J_1 and J_2 in the range at the moment. Thus, we find it natural to couple J_t with a two-point exclusion process $E_t = \{E_1(t), E_2(t)\}$ obeying the same fixed q-rates. Lets do that and on the way clarify the whole construction. Define sets $\Sigma_1(t) := \{\sigma \in \Sigma : I_1(t) \in Range(\sigma)\}$ and $\Sigma_2(t) := \{\sigma \in \Sigma : I_2(t) \in Range(\sigma)\}$. Each permutation in each of the two sets occurs with the corresponding q-rate, where each permutation in $\Sigma_1(t) \bigcap \Sigma_2(t)$ is counted twice as if two different permutations. Think of Σ_1 and Σ_2 as two sets of permutations, of which some are identical, but we do not know it and assign separate rates anyways. If the first permutation to occur is from $\Sigma_1(t)$, it will act on I_1 but not I_2, and if it is from $\Sigma_2(t)$, it will act on I_2, but not I_1. Regardless as to which of the two sets it belongs, the same permutation will act on both J_1 and J_2 even if both are in its range (in the later case the processes decouple, and T_{dec} is set to be equal to the action time of such permutation). The same permutation will act on both E_1 and E_2 but only if it comes from $\Sigma_1(t)$ or $\Sigma_2(t)\backslash\Sigma_1(t) := \{\sigma \in \Sigma_2(t) : I_2(t) \in Range(\sigma), I_1(t) \notin Range(\sigma)\}$. Of course, Σ_1 and Σ_2 evolve after each transformation of I_t. After decoupling, the processes I_t, J_t and E_t evolve independently, where I_t is the process consisting of two one-point permutation processes, E_t is a two-point permutation process and J_t is a two-point process where the corresponding q-rates are assigned to all permutations in Σ except for those containing both points J_1 and J_2 in the range at the moment, assigning the double rates to them.

For each $\sigma \in \Sigma$, the corresponding Poisson process with frequency $q(\sigma)$ can be embedded into a Poisson process with twice the frequency (that is $2q(\sigma)$). Let $T_{\frac{1}{2}}(\sigma)$ denote the set of jump times for the double-frequency Poisson process, then at each point in the set $T_{\frac{1}{2}}(\sigma)$, the σ permutation is either applied to E_t with probability $\frac{1}{2}$, or not applied with probability $\frac{1}{2}$. When $\sigma \in \Sigma_1 \bigcap \Sigma_2$, that determines whether the permutation comes from Σ_1 or from Σ_2. Now, before E_t and J_t decouple, if $\sigma \in \Sigma$ and $t \in T_{\frac{1}{2}}(\sigma)$ are such that $E_1(t), E_2(t) \in Range(\sigma)$, then $J(t) = \sigma(J(t-))$. Thus

$$P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } t \in T_{\frac{1}{2}}(\sigma) \text{ and } E_1(t), E_2(t) \in Range(\sigma) \text{ for some } \sigma \in \Sigma\Big\} \tag{8}$$

$$= P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } J_1(t) \neq J_1(t-) \text{ and } J_2(t) \neq J_2(t-)\Big\}.$$

At such t, either $E(t) = E(t-)$ with probability $\frac{1}{2}$ or $E(t) = \sigma\big(E(t-)\big)$. In the first case the processes decouple. Since the right hand side of (8) is equal to one in the recurrent case (see (7)),

$$P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } t \in T_{\frac{1}{2}}(\sigma) \text{ and } E_1(t), E_2(t) \in Range(\sigma) \text{ for some } \sigma \in \Sigma\Big\} = 1$$

regardless of what the starting point $x = (x_1, x_2)$ (s.t. $x_1 \neq x_2$) is. So, such t should arrive infinitely often. Hence, in the recurrent case,

$$P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } E_1(t) \neq E_1(t-) \text{ and } E_2(t) \neq E_2(t-)\Big\} = 1. \tag{9}$$

It is natural to compare processes I_t, E_t and J_t since all three of them coincide up until a certain decoupling time.

2.1.2 The coupling.

Returning to A_t and B_t, we will now try to reconstruct the Spitzer's coupling proof (see [9]) in the case when conditions (5) and (6) are satisfied by a symmetric permutation process. Recall that a permutation consisting of one cycle is itself called a cycle. Lets denote by Σ_{cyclic} the set of all cycles in Σ. We will say that a subset $R \subset S$ is a "range set" if there is a $\sigma \in \Sigma$ with $Range(\sigma) = R$. Consider a range set R. Let

$$m(R) = \min_{\sigma \in \Sigma : Range(\sigma) = R} \{q(\sigma)\}$$

and

$$Z(R) = \sum_{\sigma \in \Sigma : Range(\sigma) = R} q(\sigma).$$

First, observe that for all range sets R that contain both discrepancies (d_t^- and d_t^+) at the same time, the sum

$$z_d(t) := \sum_{\substack{\text{range sets } R: \\ d_t^-, d_t^+ \in R}} Z(R) \leq M_{PL}.$$

We let the coupled process $\begin{pmatrix} A_t \\ B_t \end{pmatrix}$ evolve according to the following transition rates. For each range set R containing both discrepancies at time t we pick a **cycle** $\sigma_R \in \Sigma_{cyclic}$ of range R such that $\sigma_R(A_t) = B_t$ (there must be at least one such cycle). For each range set we can order all cycles, and pick the first one that satisfies the description. Then

$$\begin{pmatrix} A_t \\ B_t \end{pmatrix} \text{ transforms into } \begin{cases} \begin{pmatrix} \sigma_R^2(A_t) \\ \sigma_R(B_t) \end{pmatrix} = \begin{pmatrix} \sigma_R(B_t) \\ \sigma_R(B_t) \end{pmatrix} & \text{with rate } m(R), \\[2ex] \begin{pmatrix} \sigma_R^3(A_t) \\ \sigma_R^2(B_t) \end{pmatrix} = \begin{pmatrix} \sigma_R^2(B_t) \\ \sigma_R^2(B_t) \end{pmatrix} & \text{with rate } m(R), \\ \dots & \\ \begin{pmatrix} \sigma_R^{|R|-1}(A_t) \\ \sigma_R^{|R|-2}(B_t) \end{pmatrix} = \begin{pmatrix} \sigma_R^{|R|-2}(B_t) \\ \sigma_R^{|R|-2}(B_t) \end{pmatrix} & \text{with rate } m(R), \\[2ex] \begin{pmatrix} \sigma_R(A_t) \\ \sigma_R^{|R|-1}(B_t) \end{pmatrix} = \begin{pmatrix} B_t \\ A_t \end{pmatrix} & \text{with rate } m(R), \\[2ex] \begin{pmatrix} \sigma_R(A_t) \\ \sigma_R(B_t) \end{pmatrix} & \text{with rate } q(\sigma_R) - m(R), \\[2ex] \begin{pmatrix} \sigma_R^2(A_t) \\ \sigma_R^2(B_t) \end{pmatrix} & \text{with rate } q(\sigma_R^2) - m(R), \\ \dots & \\ \begin{pmatrix} \sigma_R^{|R|-1}(A_t) \\ \sigma_R^{|R|-1}(B_t) \end{pmatrix} & \text{with rate } q(\sigma_R^{|R|-1}) - m(R), \\[2ex] \begin{pmatrix} \sigma(A_t) \\ \sigma(B_t) \end{pmatrix} & \text{with rate } q(\sigma) \text{ if } Range(\sigma) = R \text{ and } \sigma \neq \sigma_R^i, \text{ all } i. \end{cases}$$

The coupled process $\begin{pmatrix} A_t \\ B_t \end{pmatrix}$ will transform into $\begin{pmatrix} \sigma(A_t) \\ \sigma(B_t) \end{pmatrix}$ with rate $q(\sigma)$ if $Range(\sigma)$ does not contain both discrepancies. We observe that the rates are well defined. We also observe that the transformations that we have allowed to have non-zero rates do not increase the number of discrepancies. Moreover there could be a positive probability of the discrepancies disappearing, in which case we let A_t and B_t evolve simultaneously as a single permutation process. The rates sum up enabling us to conclude that the above process is a well-defined coupling of processes A_t and B_t.

2.1.3 The coupling is successful. Example.

The coupling is successful because, according to (9), if waiting with rate $z_d(t)$ for a permutation that contains both discrepancies in its range, though $z_d(t)$ changes with time, we are guaranteed to have a finite holding time. Now, (5) and (6) imply $m(R)M_{II}\mathcal{P}(M_I) \geq Z(R)$, where $\mathcal{P}(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! < n!$ denotes the number of permutations of $n > 1$ distinct elements such that each element is displaced, i.e. element k is not in the k-th position for all $k \in \{1, 2, \dots, n\}$. At the holding time, the discrepancies will cancel with probability

$$\geq \sum_{\substack{\text{range sets } R: \\ d_t^-, d_t^+ \in R}} \frac{m(R)}{z_d(t)} \geq \sum_{\substack{\text{range sets } R: \\ d_t^-, d_t^+ \in R}} \frac{Z(R)}{\mathcal{P}(M_I)M_{II}z_d(t)} = \frac{1}{\mathcal{P}(M_I)M_{II}}.$$

The coupled process will keep arriving to such holding times up until the discrepancies cancel. Theorem 2 is proved in the recurrent case.

Remark. A simple but beautiful argument by Euler proves the following identity $\mathcal{P}(n) = (n-1)(\mathcal{P}(n-1) + \mathcal{P}(n-2))$ that was used to derive $\mathcal{P}(n)$. Notice that $\mathcal{P}(n)$ is obviously increasing with n. Finding the expression for $\mathcal{P}(n)$ is a case of a famous problem, known in the history of mathematics as "problème des rencontres". The number $\mathcal{P}(n)$ is also called the number of "derangements" of n elements. We refer the reader to Chapters 3 and 8 of [7] for more on the subject.

Example. The author wishes to thank the referee for suggesting the following simple example that illustrates how the above coupling works. Let $S = \mathbb{Z}$,
$\Sigma = \bigcup_{x\in\mathbb{Z}} \left\{\sigma_x := \overline{(x, x+1, x+2)}, \sigma_x^2 = \sigma_x^{-1}\right\}$ and $q(\sigma_x) = q(\sigma_x^{-1}) = q$ for all $x \in \mathbb{Z}$, where $q > 0$ is fixed. Then one gets $M_{PL} = 6q$, $M_I = 3$ and $M_{II} = 1$ (see (5) and (6)). Relevant range sets are $R_x = \{x, x+1, x+2\}$ for $x \in \mathbb{Z}$. There $m(R_x) = q$ and $Z(R_x) = 2q$ since $R_x = Range(\sigma_x) = Range(\sigma_x^{-1})$. Suppose the discrepancies are for instance at y and $y+1$, say $d_t^+ = y$ and $d_t^- = y+1$, and the rest of the points around y are occupied in the following way:

$$\begin{array}{lccccccccccc} A_t: & \dots & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ B_t: & \dots & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ & & & \uparrow & \uparrow & \uparrow & & & & & \\ & & & y-1 & y & y+1 & & & & & \end{array}$$

There are exactly two range sets that contain both discrepancies $d_t^+ = y$ and $d_t^- = y+1$, those are R_{y-1} and R_y. For the range set $R = R_{y-1}$ there is a unique choice of σ_R: $\sigma_R = \sigma_{y-1}^{-1}$ is the only cycle in Σ with range R such that $\sigma_R(A_t) = B_t$. Similarly for $R = R_y$, the choice $\sigma_R = \sigma_y$ for σ_R is unique. Thus the coupling in 2.1.2 reads

$$\begin{pmatrix} A_t \\ B_t \end{pmatrix} \text{ transforms into } \begin{cases} \begin{pmatrix} \sigma_{y-1}(A_t) \\ \sigma_{y-1}^{-1}(B_t) \end{pmatrix} = \begin{array}{cccccc} \dots & 0 & 1 & 1 & 0 & \dots \\ \dots & 0 & 1 & 1 & 0 & \dots \\ & \uparrow & \uparrow & \uparrow & & \\ & y-1 & y & y+1 & & \end{array} \text{ with rate } q, \\[2em] \begin{pmatrix} \sigma_{y-1}^{-1}(A_t) \\ \sigma_{y-1}(B_t) \end{pmatrix} = \begin{array}{cccccc} \dots & 1 & 0 & 1 & 0 & \dots \\ \dots & 1 & 1 & 0 & 0 & \dots \\ & \uparrow & \uparrow & \uparrow & & \\ & y-1 & y & y+1 & & \end{array} \text{ with rate } q, \\[2em] \begin{pmatrix} \sigma_{y}^{-1}(A_t) \\ \sigma_{y}(B_t) \end{pmatrix} = \begin{array}{cccccc} \dots & 1 & 0 & 0 & 1 & \dots \\ \dots & 1 & 0 & 0 & 1 & \dots \\ & \uparrow & \uparrow & \uparrow & & \\ & y-1 & y & y+1 & & \end{array} \text{ with rate } q, \\[2em] \begin{pmatrix} \sigma_{y}(A_t) \\ \sigma_{y}^{-1}(B_t) \end{pmatrix} = \begin{array}{cccccc} \dots & 1 & 0 & 1 & 0 & \dots \\ \dots & 1 & 1 & 0 & 0 & \dots \\ & \uparrow & \uparrow & \uparrow & & \\ & y-1 & y & y+1 & & \end{array} \text{ with rate } q. \end{cases}$$

The four permutations that contain both $d_t^+ = y$ and $d_t^- = y+1$ have total rate $z_d(t) = Z(R_y) + Z(R_{y-1}) = 4q$. If, after waiting with rate $z_d(t)$, the holding time arrives (before any changes within $R_y \bigcup R_{y-1}$ occur), the discrepancies will cancel with probability equal to $\frac{2q}{z_d(t)} = \frac{1}{2}$. In general, in all such cases when the discrepancies are within distance ≤ 2 from each other and the holding time for all the permutations containing the two discrepancies rings, the probability of cancelation of discrepancies should be no less than $\frac{1}{\mathcal{P}(M_I)M_{II}} = \frac{1}{2}$ as $\mathcal{P}(3) = 2$. In this example, it will always be equal to $\frac{1}{2}$.

The case is obviously recurrent as the difference of corresponding one-point processes $I_1(t) - I_2(t)$ is a recurrent random walk on $\mathbb{Z}$. One can show (see the argument in 2.1.1) that for the two-point permutation process $E_t = \{E_1(t), E_2(t)\}$ with the rates given in the beginning of the example, the above recurrence implies that $E_1(t)$ will come within distance ≤ 2 of $E_2(t)$ infinitely often insuring that the coupling is successful.

2.2 Transient, translation invariant case.

The following functions will denote various probabilities, some of which we already used in the preceding subsections. We let

$$\bar{g}_2(x) := P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } E_1(t) \neq E_1(t-) \text{ and } E_2(t) \neq E_2(t-)\Big\},$$

$$g_2(x) := P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } I_1(t) = I_2(t)\Big\}$$

and

$$\bar{\bar{g}}_2(x) := P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } J_1(t) \neq J_1(t-) \text{ and } J_2(t) \neq J_2(t-)\Big\}.$$

Therefore (7) is equivalent to

$$\bar{\bar{g}}_2(x) \geq g_2(x).$$

Moreover, by construction, $\bar{\bar{g}}_2(x) \geq \bar{g}_2(x) \geq g_2(x)$. Conversely, (8) implies $\bar{g}_2(x) \geq \frac{1}{2}\bar{\bar{g}}_2(x)$, and one similarly obtains $\frac{1}{M_{II}\mathcal{P}(M_I)}\bar{\bar{g}}_2(x) \leq g_2(x)$, where as before, $\mathcal{P}(N)$ denotes the number of permutations of N elements such that each element is displaced. Hence, taking all the above inequalities together, we conclude that

$$g_2(x) \sim \bar{g}_2(x) \sim \bar{\bar{g}}_2(x). \tag{10}$$

Now, let

$$T_n = \{x = (x_1, \ldots, x_n) \in S^n \,:\, x_i \neq x_j \text{ for all } i \neq j\},$$

and let Ω_t, U_t and V_t be the semigroups of respectively E_t, I_t and J_t. We can extend E_t to denote an n-point permutation process (when applies) and similarly extend I_t to be an n-point process such that each particle moves independently of the others as a one-point permutation process. Moreover, Ω_t and U_t will still denote the corresponding semigroups of extended E_t and I_t. We can also redefine

$$\bar{g}_n(x) := P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } E_i(t) \neq E_i(t-) \text{ and } E_j(t) \neq E_j(t-) \text{ for some } i \neq j \in \{1,\ldots,n\}\Big\}$$

and

$$g_n(x) := P^x\Big\{\exists t \in (0,\infty) \text{ s.t. } I_t = (I_1(t),\ldots,I_n(t)) \notin T_n\Big\},$$

where $E_t = (E_1(t),\ldots,E_n(t))$ is the n-point permutation process and $I_1(t),\ldots,I_n(t)$ are independent random walk processes. The properties of g_n were thoroughly studied before (see for example [5]). In particular, for $x = (x_1,\ldots,x_n) \in S^n$,

$$g_n(x) \leq \sum_{1\leq i<j\leq n} g_2(x_i,x_j) \leq \binom{n}{2} g_n(x).$$

Therefore, redoing the first part of (10) for a general n, get

$$\bar{g}_n(x) \sim g_n(x) \sim \sum_{1\leq i<j\leq n} g_2(x_i,x_j) \sim \sum_{1\leq i<j\leq n} \bar{g}_2(x_i,x_j) \tag{11}$$

2.2.1 Case $n = 2$.

Here for the case of two particles we need to prove that a bounded harmonic function is constant. By following Liggett's proof (see Theorem 1.24 in Chapter VIII of [5], [2] and [3]) of Theorem 1 for transient symmetric exclusion processes, we observe that if f is a function such that $0 \le f \le 1$, then, by construction,

$$|V_t f(x) - U_t f(x)| \le \bar{\bar{g}}_2(x), \qquad x \in T_2.$$

Processes J_t and I_t agree until the first time t such that $J_1(t) \ne J_1(t-)$ and $J_2(t) \ne J_2(t-)$. Now, J_t agrees with E_t up until at least such t. Thus

$$|V_t f(x) - \Omega_t f(x)| \le \bar{\bar{g}}_2(x), \qquad x \in T_2.$$

and, by (10),

$$|\Omega_t f(x) - U_t f(x)| \le 2\bar{\bar{g}}_2(x) \le 4\bar{g}_2(x), \qquad x \in T_2. \tag{12}$$

Suppose f is also symmetric on T_2, and $\Omega_t f = f$ for all $t \ge 0$, i.e. f is harmonic with respect to semigroup Ω_t. It can be extended to all of S^2 by setting $f = 0$ on $T_2^c := S^2 \backslash T_2$. Then, by (12),

$$|f(x) - U_t f(x)| \le 4\bar{g}_2(x), \qquad x \in S^2 \tag{13}$$

as $\bar{g}_2 := 1$ on T_2^c.
Now, in the transient case, the independently moving particles $I_1(t), \ldots, I_n(t)$ tend to scatter away from each other. This can be expressed with the following limit (see [5])

$$\lim_{t \to \infty} U_t g_n(x) = 0, \qquad x \in S^n.$$

Thus, by (10),

$$\lim_{t \to \infty} U_t \bar{g}_2(x) = 0, \qquad x \in S^2. \tag{14}$$

The inequality (13) implies

$$|U_s f(x) - U_{s+t} f(x)| \le 4 U_s \bar{g}_2(x), \qquad x \in S^2,$$

where, by (14), the right hand side goes to zero. So, the limit of $U_s f$ exists and is a harmonic function of a two particle process with semigroup U_t (where each particle moves independently), whence it is a constant

$$\lim_{t \to \infty} U_t f(x) = C, \qquad x \in S^2.$$

Thus (13) implies

$$|f(x) - C| \le 4\bar{g}_2(x), \qquad x \in S^2.$$

Since we know that $\Omega_t f = f$,

$$|f(x) - C| = |\Omega_t f(x) - C| \le 4\Omega_t \bar{g}_2(x), \qquad x \in T_2. \tag{15}$$

We want to use (15) in order to show $|f(x) - C| = 0$.

Three-cycles. If we only allow transpositions and three-cycles then the situation will be much simpler. First consider the case when Σ contains only three-cycles. So, we only have to consider the permutations σ_z, indexed by $z \neq x_1$ or x_2 in S such that $\sigma_z :\ z \to x_1 \to x_2 \to z$, as well as σ_z^{-1}. Let Ω, U and V be the generators of the corresponding semigroups Ω_t, U_t and V_t. By construction for a cylinder function $h : S \times S \to \mathbb{R}$ and $x = (x_1, x_2) \in S^2$,

$$(\mathsf{U} - \mathsf{V})h(x) = \sum_{\sigma : x_1, x_2 \in Range(\sigma)} q(\sigma)\Big[h(\sigma(x_1), x_2) + h(x_1, \sigma(x_2)) - 2h(\sigma(x_1), \sigma(x_2))\Big]$$

and

$$(\mathsf{V} - \Omega)h(x) = \sum_{\sigma : x_1, x_2 \in Range(\sigma)} q(\sigma)\Big[h(\sigma(x_1), \sigma(x_2)) - h(x_1, x_2)\Big].$$

Thus

$$(\mathsf{U} - \Omega)h(x) = \sum_{\sigma : x_1, x_2 \in Range(\sigma)} q(\sigma)\Big[h(\sigma(x_1), x_2) + h(x_1, \sigma(x_2)) - h(\sigma(x_1), \sigma(x_2)) - h(x_1, x_2)\Big].$$

Letting $S_{x_1,x_2} := \Big\{z \in S \backslash \{x_1, x_2\} : \sigma_z \in \Sigma\Big\}$, and summing over all three-cycles σ_z and σ_z^{-1} for $z \in S_{x_1,x_2}$ we obtain the following equality:

$$\begin{aligned}
(\mathsf{U} - \Omega)h(x) = & \sum_{\sigma : x_1, x_2 \in Range} q(\sigma)\Big[h(\sigma(x_1), x_2) + h(x_1, \sigma(x_2)) - h(\sigma(x_1), \sigma(x_2)) - h(x_1, x_2)\Big] \\
= & \sum_{z \in S_{x_1,x_2}} q(\sigma_z)\Big[h(x_2, x_2) + h(x_1, z) - h(x_2, z) - h(x_1, x_2)\Big] \\
& + \sum_{z \in S_{x_1,x_2}} q(\sigma_z)\Big[h(z, x_2) + h(x_1, x_1) - h(z, x_1) - h(x_1, x_2)\Big] \\
= & \left(\sum_{z \in S_{x_1,x_2}} q(\sigma_z)\right)\Big[h(x_1, x_1) + h(x_2, x_2) - 2h(x_1, x_2)\Big].
\end{aligned}$$

A bounded symmetric function F on S^2 is said to be positive definite if

$$\sum_{u_1, u_2 \in S} F(u_1, u_2)\beta(u_1)\beta(u_2) \ge 0 \tag{16}$$

whenever $\sum_{u \in S} |\beta(u)| < \infty$ and $\sum_{u \in S} \beta(u) = 0$. A bounded symmetric function F on S^n is said to be positive definite if it is a positive definite function of each pair of its variables. Now, $h(x) = U_s g_2(x)$ is positive definite (see the proof of Lemma 1.23 in Chapter VIII of

[5]). Taking $\beta(u) = \begin{cases} +1 \text{ if } u = x_1 \\ -1 \text{ if } u = x_2 \\ 0 \text{ otherwise} \end{cases}$ in (16) we conclude that $(\mathsf{U} - \Omega)U_s g_2(x) \geq 0$.

Thus

$$\Omega_t g_2(x) \leq U_t g_2(x). \tag{17}$$

follows from the integration by parts formula for semigroups

$$U_t - \Omega_t = \int_0^t \Omega_{t-s}(\mathsf{U} - \Omega)U_s ds.$$

(17) together with (15) and (10) complete the argument in the case when we only allow three-cycles. The proof can be easily extended to allow Σ to contain both transpositions and three-cycles by incorporating the proof of Proposition 1.7 in Chapter VIII of [5].

For the general case the inequalities like (17) are hard to prove. However (17) is stronger than what we really need.

By transience, $\lim_{x\to\infty} g_2(0,x) = 0$, $x \in S$. Thus the equivalence relation (10) implies $\lim_{x\to\infty} \bar{g}_2(0,x) = 0$. So for any $\epsilon > 0$, $\exists R_\epsilon > 0$ such that $\bar{g}_2(0,x) \leq \epsilon$ whenever $|x| > R_\epsilon$. Now , we claim that there exists $\Delta < 1$ such that $\bar{g}_2(x_1,x_2) = \bar{g}_2(0,x_2 - x_1) \leq \Delta$ for all $x_1, x_2 \in S$. To prove this, we fix $\epsilon = \frac{1}{2}$ and denote $R = R_{\frac{1}{2}}$. We only need to prove that $\bar{g}_2(0,x) < 1$ whenever $|x| \leq R$, $x \in S$. Suppose there is a point x inside the ball B_R of radius R around the origin such that $\bar{g}_2(0,x) = 1$. If there is a permutation $\sigma_1 \in \Sigma$ with $\sigma_1(x) \in B_R^c$ and $0 \notin Range(\sigma_1)$, then

$$1 - \bar{g}_2(0,x) \geq (1 - \bar{g}_2(0,\sigma_1(x))tq(\sigma_1)e^{-3M_{PL}t} > 0$$

for t small enough, where $tq(\sigma_1)e^{-3M_{PL}t}$ is a lower bound for the probability of the event that no permutation containing at least one of the three sites 0, x and $\sigma_1(x)$ in its range is applied within $(0,t)$ time interval with the exception of σ_1 (recall how M_{PL} was defined in (1)).

Thus $\exists \Delta_1 < 1$ such that $\bar{g}_2(0,x) \leq \Delta_1$ whenever

$$x \in B_R^c \cup \{x \in B_R : \exists \sigma_1 \in \Sigma \text{ s.t. } \sigma_1(0) = 0,\ \sigma_1(x) \in B_R^c\}.$$

We iterate the above argument in order to show that since there are finitely many points of S inside B_R, $\exists \Delta < 1$ such that $\bar{g}_2(0,x) \leq \Delta$ whenever

$$x \in B_R^c \cup \{x \in B_R : \exists k \geq 1, \sigma_1, ..., \sigma_k \in \Sigma \text{ s.t. } \sigma_1(0) = ... = \sigma_k(0) = 0,\ \sigma_k \circ \sigma_{k-1} \circ ... \circ \sigma_1(x) \in B_R^c\}.$$

By irreducibility assumption, the above set is all of S, proving the claim that $\bar{g}_2(x_1,x_2) \leq \Delta < 1$ for all $x_1, x_2 \in S$. Thus $\forall M > 0$, $P^{(0,x)}[\liminf_{t\to\infty} |E_1(t) - E_2(t)| \leq M] = 0$, i.e. $|E_1(t) - E_2(t)| \to \infty$ a.s. as $t \to \infty$. Then $\lim_{x\to\infty} \bar{g}_2(0,x) = 0$ implies

$$\lim_{x\to\infty} \Omega_t \bar{g}_2(0,x) = 0.$$

Hence, by (15), if f is a bounded symmetric function and $\Omega_t f = f$ then $f(x)$ is a constant for all $x \in T_2$, i.e. a bounded harmonic function for a transient permutation process is constant for all sets of cardinality $n = 2$, proving Theorem 2 in this case.

2.2.2 General n.

The proof that if f is a bounded symmetric function on T_n, and if $\Omega_t f = f$, then

$$|f(x) - C| = |\Omega_t f(x) - C| \leq c\Omega_t \bar{g}_n(x), \qquad x \in T_n \tag{18}$$

for some constants C and c is the same for $n \geq 2$ as for the case when $n = 2$. However, here we do not have to redo the rest of the computation again. Since $\lim_{x\to\infty} \Omega_t \bar{g}_2(0, x) = 0$ for all $x \neq 0$ in S, (11) implies that the right side of (18) converges to zero. Thus, for any given $n \geq 2$, a bounded harmonic function for a transient permutation process must be constant for all sets of cardinality n. **Theorem 2 is proved.**

3 General case: shift invariant stationary measures.

Once again, we assume that the conditions (5) and (6) are satisfied, though, as it was mentioned in the previous section, it is possible to obtain some of the same results with slightly weaker conditions.

Let $\mathcal{S}$ again denote the class of the shift invariant probability measures on $\{0, 1\}^S$. In this section we will prove the following important result.

Theorem 3. *For the general permutation process,* $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$.

3.1 Modifying the coupling.

First we have to modify the coupling of two permutation processes A_t and B_t on S, where we are no longer constrained to only two discrepancies at time $t = 0$. We should find the way of coupling the processes so that the number of discrepancies is at least not increasing with time. We will adapt the following (generally accepted) notation: for two configurations η and ζ in $\{0, 1\}^S$, we say that $\eta \leq \zeta$ if

$$\eta(x) \leq \zeta(x) \text{ for every } x \in S.$$

We will say that $\eta \leq \zeta$ **on** a subset Ω_{sub} of S if $\eta(x) \leq \zeta(x)$ for every $x \in \Omega_{sub}$.

At a given time t, for every range set R in S, there must be at least one $\sigma_R \in \Sigma_{cyclic}$ of range R (i.e. $Range(\sigma_R) = R$) such that either $\sigma_R(A_t) \geq B_t$ **on** R or $\sigma_R(B_t) \geq A_t$ **on** R. In the case when

$$|\{x \in R : A_t(x) = 1\}| \geq |\{x \in R : B_t(x) = 1\}|,$$

we can only pick σ_R so that $\sigma_R(A_t) \geq B_t$ on R. Then we let $\begin{pmatrix} A_t \\ B_t \end{pmatrix}$ transform into either $\begin{pmatrix} \sigma_R(A_t) \\ B_t \end{pmatrix}$, $\begin{pmatrix} \sigma_R^2(A_t) \\ \sigma_R(B_t) \end{pmatrix}$, $\begin{pmatrix} \sigma_R^3(A_t) \\ \sigma_R^2(B_t) \end{pmatrix}$, $\ldots$, or $\begin{pmatrix} \sigma_R^{|R|}(A_t) \\ \sigma_R^{|R|-1}(B_t) \end{pmatrix} = \begin{pmatrix} A_t \\ \sigma_R^{-1}(B_t) \end{pmatrix}$ with rate $m(R)$ each, where $m(R)$ was defined in 2.1.2. For all permutations $\sigma \in \Sigma$ of range R, we will apply

$\begin{pmatrix}\sigma\\ \sigma\end{pmatrix}$ with the remaining rates: $\begin{pmatrix}A_t\\ B_t\end{pmatrix}$ transforms into $\begin{pmatrix}\sigma(A_t)\\ \sigma(B_t)\end{pmatrix}$ with rate $= q(\sigma) - m(R)$ if $\sigma = \sigma_R^i$ for some $i \in \{1, ..., |R| - 1\}$, and with rate $= q(\sigma)$ if otherwise. The case when

$$|\{x \in R : A_t(x) = 1\}| \leq |\{x \in R : B_t(x) = 1\}|$$

is dealt with symmetrically. The way we select σ_R among the cycles of range R is by initially ordering all the cycles of range R in Σ_{cyclic} and each time selecting the one of the highest order such that either $\sigma_R(A_t) \geq B_t$ or $\sigma_R(A_t) \leq B_t$ on R. It is **important** that the ordering of all the cycles of range R should be done parallel to ordering of all the cycles of range $R + y$ for each $y \in S$, i.e. σ_R selected for $\begin{pmatrix}A_t(x)\\ B_t(x)\end{pmatrix} = \begin{pmatrix}\eta(x)\\ \zeta(x)\end{pmatrix}$ for all $x \in S$ should be the $(-y)$-shift of σ_{R+y} selected for $\begin{pmatrix}A_t(x)\\ B_t(x)\end{pmatrix} = \begin{pmatrix}\eta(x-y)\\ \zeta(x-y)\end{pmatrix}$ for all $x \in S$. We **observe** that the number of discrepancies here can only decrease.

We will denote by $\mathcal{I}^*$ the class of stationary distributions for the coupled process, and by $\mathcal{S}^*$ we will denote the class of translation invariant distributions for the coupled process. We will also write $\mathcal{I}_e^*$ for the set of all the extreme points of $\mathcal{I}^*$, and $(\mathcal{I}^* \cap \mathcal{S}^*)_e$ for the set of all the extreme points of $(\mathcal{I}^* \cap \mathcal{S}^*)$. Let ν^* be the measure on $\{0,1\}^S \times \{0,1\}^S$ with the marginals ν_1 and ν_2. Our next theorem is a case of Theorem 2.15 in Chapter III of [5].

Theorem 4. *(a) If ν^* is in $\mathcal{I}^*$, then its marginals are in $\mathcal{I}$.*
(b) If $\nu_1, \nu_2 \in \mathcal{I}$, then there is a $\nu^* \in \mathcal{I}^*$ with marginals ν_1 and ν_2.
(c) If $\nu_1, \nu_2 \in \mathcal{I}_e$, then the ν^* in part (b) can be taken to be in $\mathcal{I}_e^*$.
(d) In parts (b) and (c), if $\nu_1 \leq \nu_2$, then ν^* can be taken to concentrate on $\{\eta \leq \zeta\}$.
(e) In the translation invariant case, parts (a)-(d) hold if $\mathcal{I}$ and $\mathcal{I}^*$ are replaced by $(\mathcal{I} \cap \mathcal{S})$ and $(\mathcal{I}^* \cap \mathcal{S}^*)$ respectively.

3.2 Case $\nu^* \in (\mathcal{I}^* \cap \mathcal{S}^*)$: the two types of discrepancies do not coexist.

For permutations $\sigma_1, \sigma_2 \in S$ of a given range R, let $q^*(\sigma_1, \sigma_2; \eta(R), \zeta(R))$ denote the rate of the newly defined coupled process assigned to $\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$ transformation if given the values $\begin{pmatrix}\eta(R)\\ \zeta(R)\end{pmatrix} = \left\{\begin{pmatrix}\eta(x)\\ \zeta(x)\end{pmatrix} \text{ for all } x \in R\right\}$. We also let $\Omega^*(t)$ denote the semigroup of the coupled process. The following definition will be useful as we proceed:

Definition. For $\sigma \in \Sigma$ and $x \in Range(\sigma)$, the subset

$$O(\sigma, x) = \{\sigma^i(x) : \quad i = 0, 1, \dots\}$$

of $Range(\sigma)$ is called the **orbit** of x under σ.

Theorem 5. *If $\nu^* \in (\mathcal{I}^* \cap \mathcal{S}^*)$, then*

$$\nu^*\big\{(\eta, \zeta) : \eta(u) = \zeta(v) = 0,\ \zeta(u) = \eta(v) = 1\big\} = 0$$

for every x and y in S.

Proof: Here we reconstruct a clever trick from the theory of exclusion processes. If the coupled measure ν^* is in $(\mathcal{I}^* \cap \mathcal{S}^*)$ then

$$\begin{aligned}
0 &= \frac{d}{dt}\nu^*\Omega^*(t)\{(\eta,\zeta) : \eta(x) \neq \zeta(x)\}\Big|_{t=0} \qquad (19)\\
&= \sum_{\substack{\text{range sets } R:\\ x\in R}} \sum_{\substack{\sigma\in\Sigma:\\ Range(\sigma)=R}} \sum_{\substack{\ddot\eta,\ddot\zeta\in\{0,1\}^R:\\ \ddot\eta(x)=\ddot\zeta(x),\\ \ddot\eta(\sigma^{-1}(x))\neq\ddot\zeta(\sigma^{-1}(x))}} q^*(\sigma,\sigma;\ddot\eta,\ddot\zeta)\cdot\nu^*\Big\{\begin{pmatrix}\eta\\ \zeta\end{pmatrix} : \begin{pmatrix}\eta(R)\\ \zeta(R)\end{pmatrix} = \begin{pmatrix}\ddot\eta\\ \ddot\zeta\end{pmatrix}\Big\}\\
&- \sum_{\substack{\text{range sets } R:\\ x\in R}} \sum_{\substack{\sigma\in\Sigma:\\ Range(\sigma)=R}} \sum_{\substack{\ddot\eta,\ddot\zeta\in\{0,1\}^R:\\ \ddot\eta(x)\neq\ddot\zeta(x),\\ \ddot\eta(\sigma^{-1}(x))=\ddot\zeta(\sigma^{-1}(x))}} q^*(\sigma,\sigma;\ddot\eta,\ddot\zeta)\cdot\nu^*\Big\{\begin{pmatrix}\eta\\ \zeta\end{pmatrix} : \begin{pmatrix}\eta(R)\\ \zeta(R)\end{pmatrix} = \begin{pmatrix}\ddot\eta\\ \ddot\zeta\end{pmatrix}\Big\}\\
&+ \sum_{\substack{\text{range sets } R:\\ x\in R}} \sum_{\substack{\ddot\eta,\ddot\zeta\in\{0,1\}^R:\\ |\{x\in R:\ddot\eta(x)=1\}|\\ \geq |\{x\in R:\ddot\zeta(x)=1\}|}} m(R)\big[D(\sigma_R(\ddot\eta),\ddot\zeta) - D(\ddot\eta,\ddot\zeta)\big]\cdot\nu^*\Big\{\begin{pmatrix}\eta\\ \zeta\end{pmatrix} : \begin{pmatrix}\eta(R)\\ \zeta(R)\end{pmatrix} = \begin{pmatrix}\ddot\eta\\ \ddot\zeta\end{pmatrix}\Big\}\\
&+ \sum_{\substack{\text{range sets } R:\\ x\in R}} \sum_{\substack{\ddot\eta,\ddot\zeta\in\{0,1\}^R:\\ |\{x\in R:\ddot\eta(x)=1\}|\\ \geq |\{x\in R:\ddot\zeta(x)=1\}|}} m(R)\big[D(\ddot\eta,\sigma_R(\ddot\zeta)) - D(\ddot\eta,\ddot\zeta)\big]\cdot\nu^*\Big\{\begin{pmatrix}\eta\\ \zeta\end{pmatrix} : \begin{pmatrix}\eta(R)\\ \zeta(R)\end{pmatrix} = \begin{pmatrix}\ddot\eta\\ \ddot\zeta\end{pmatrix}\Big\},
\end{aligned}$$

where for each a range set R and configuration $\ddot\eta,\ddot\zeta\in\{0,1\}^R$ of the coupled process on R, σ_R is uniquely defined, i.e. $\sigma_R = \sigma_R[\ddot\eta,\ddot\zeta]$. Also $D^+(\ddot\eta,\ddot\zeta)$ is the number of $\begin{pmatrix}1\\0\end{pmatrix}$ discrepancies of $\begin{pmatrix}\ddot\eta\\ \ddot\zeta\end{pmatrix}$, $D^-(\ddot\eta,\ddot\zeta)$ is the number of $\begin{pmatrix}0\\1\end{pmatrix}$ discrepancies of $\begin{pmatrix}\ddot\eta\\ \ddot\zeta\end{pmatrix}$ and

$$D(\ddot\eta,\ddot\zeta) := D^+(\ddot\eta,\ddot\zeta) + D^-(\ddot\eta,\ddot\zeta)$$

is the total number of discrepancies on R; $\sigma_R(\ddot\eta)$ above denotes the new configuration of particles on R that we obtain after applying permutation σ_R to the original $\ddot\eta$, i.e. $\sigma_R(\ddot\eta)(x) := \ddot\eta(\sigma_R^{-1}(x))$ for all $x\in R$. $\sigma_R(\ddot\zeta)$ is defined by analogy.

Now, we need to explain (19). The third sum on the right hand side (RHS) of (19) represents the contribution to the derivative by all transformations $\begin{pmatrix}\sigma_R^i\\ \sigma_R^{i-1}\end{pmatrix}$ whenever

$$|\{x\in R:\eta(x)=1\}| \geq |\{x\in R:\zeta(x)=1\}|$$

(equivalently $\sigma_R(\eta)\geq\zeta$ on R). Symmetrically, the fourth sum on the RHS of (19) represents the contribution to the derivative by all transformations $\begin{pmatrix}\sigma_R^{i-1}\\ \sigma_R^i\end{pmatrix}$ whenever

$$|\{x\in R:\zeta(x)=1\}| \geq |\{x\in R:\eta(x)=1\}|$$

(equivalently $\sigma_R(\zeta) \geq \eta$ on R).

Now, lets show that the third sum is correct. We fix a range set R that contains x. Notice that since σ_R is a cycle, for each $y \in R$ there is a unique corresponding $i \in \{0, 1, ..., |R|-1\}$ such that $y = \sigma_R^{-i}(x)$. So $\sigma_R^{i+1}(\ddot{\eta})(x) = \ddot{\eta}(\sigma_R^{-(i+1)}(x)) = \ddot{\eta}(\sigma_R^{-1})(y) = \sigma_R(\ddot{\eta})(y)$ and similarly $\sigma_R^i(\ddot{\zeta})(x) = \ddot{\zeta}(y)$. Then counting all contributions to the derivative in (19) by $\begin{pmatrix} \sigma_R^{i+1} \\ \sigma_R^i \end{pmatrix}$ for all values of $i \in \{0, 1, ..., |R|-1\}$ given that $\begin{pmatrix} \eta(R) \\ \zeta(R) \end{pmatrix} = \begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix}$, one obtains the following product

$$\nu^*\Big\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} : \begin{pmatrix} \eta(R) \\ \zeta(R) \end{pmatrix} = \begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix} \Big\} \cdot \sum_{i=0}^{|R|-1} m(R) \Big[\mathbf{1}_{\{\sigma_R^{i+1}(\eta)(x) \neq \sigma_R^i(\zeta)(x)\}} - \mathbf{1}_{\{\eta(x) \neq \zeta(x)\}} \Big]$$

$$= \nu^*\Big\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} : \begin{pmatrix} \eta(R) \\ \zeta(R) \end{pmatrix} = \begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix} \Big\} \cdot m(R) \sum_{y \in R} \Big[\mathbf{1}_{\{\sigma_R(\eta)(y) \neq \zeta(y)\}} - \mathbf{1}_{\{\eta(x) \neq \zeta(x)\}} \Big]$$

$$= \nu^*\Big\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} : \begin{pmatrix} \eta(R) \\ \zeta(R) \end{pmatrix} = \begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix} \Big\} \cdot m(R) \Big[D(\sigma_R(\ddot{\eta}), \ddot{\zeta}) - |R| \mathbf{1}_{\{\eta(x) \neq \zeta(x)\}} \Big],$$

where $\mathbf{1}_{\{\lambda \neq \mu\}} := \begin{cases} 1 \ , \\ \lambda \neq \mu \text{ otherwise.} \end{cases}$

Next step is to consider all the shifts $R_{x-z} = \{R + x - z\}$ of R for all $z \in R$ together with the corresponding shifts $\begin{pmatrix} \ddot{\eta}^{x-z} \\ \ddot{\zeta}^{x-z} \end{pmatrix}$ of configuration $\begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix} \in \{0,1\}^R \times \{0,1\}^R$. Since ν^* is shift invariant, the contribution to the derivative in (19) coming from all transitions $\begin{pmatrix} \sigma_{R_{x-z}}^{i+1} \\ \sigma_{R_{x-z}}^i \end{pmatrix}$ for all values of $i \in \{0, 1, ..., |R|-1\}$ and $z \in R$, given that $\begin{pmatrix} \eta(R_{x-z}) \\ \zeta(R_{x-z}) \end{pmatrix} = \begin{pmatrix} \ddot{\eta}^{x-z} \\ \ddot{\zeta}^{x-z} \end{pmatrix}$, is equal to

$$\nu^*\Big\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} : \begin{pmatrix} \eta(R) \\ \zeta(R) \end{pmatrix} = \begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix} \Big\} \cdot m(R) \sum_{z \in R} \Big[D(\sigma_R(\ddot{\eta}), \ddot{\zeta}) - |R| \mathbf{1}_{\{\eta(z) \neq \zeta(z)\}} \Big]$$

$$= |R| \cdot m(R) \big[D(\ddot{\eta}, \sigma_R(\ddot{\zeta})) - D(\ddot{\eta}, \ddot{\zeta}) \big] \cdot \nu^*\Big\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} : \begin{pmatrix} \eta(R) \\ \zeta(R) \end{pmatrix} = \begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix} \Big\}.$$

Now, the above is the total contribution corresponding to all $|R|$ shifts of R that contain x and respective shifts of $\begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix}$. Hence we can count in $\frac{1}{|R|}$-th fraction of the total each time, thus verifying the correctness of the third sum on the RHS of (19). The fourth sum is obtained by symmetry.

Naturally, the first and the second sums on the RHS of (19) represent the contributions made to the derivative by all $\begin{pmatrix} \sigma \\ \sigma \end{pmatrix}$ transformations. We claim that because $\nu^* \in \mathcal{S}^*$, the first and the second sums on the RHS of (19) must cancel each other. We repeat the same trick: for a range set R containing x and $\ddot{\eta}, \ddot{\zeta} \in \{0,1\}^R$ we consider all shifts $R_{x-\sigma^i(x)} := R + x - \sigma^i(x)$

of R together with the respective shifts $\begin{pmatrix} \ddot{\eta}^{x-\sigma^i(x)} \\ \ddot{\zeta}^{x-\sigma^i(x)} \end{pmatrix}$ of $\begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix}$, for all $i \in \{1, 2, \ldots, |O(\sigma, x)|\}$. For a permutation $\sigma \in \Sigma$ of range R, let σ_i denote the corresponding $(x - \sigma^i(x))$-shift of σ. Then $Range(\sigma_i) = R_{x-\sigma^i(x)}$. Now, due to the shift-invariant way in which the coupling was constructed,

$$q^*(\sigma_i, \sigma_i; \ddot{\eta}^{x-\sigma^i(x)}, \ddot{\zeta}^{x-\sigma^i(x)}) = q^*(\sigma, \sigma; \ddot{\eta}, \ddot{\zeta})$$

for each $i \in \{1, 2, \ldots, |O(\sigma, x)|\}$. The following are trivial identities. For all $i \in \{1, 2, \ldots, |O(\sigma, x)|\}$,

$$\ddot{\eta}^{x-\sigma^i(x)}(x) = \ddot{\eta}(\sigma^i(x)), \qquad \ddot{\zeta}^{x-\sigma^i(x)}(x) = \ddot{\zeta}(\sigma^i(x)),$$

$$\sigma_i(\ddot{\eta})^{x-\sigma^i(x)}(x) = \sigma(\ddot{\eta})(\sigma^i(x)) = \ddot{\eta}(\sigma^{i-1}(x))$$

$$\text{and } \sigma_i(\ddot{\zeta})^{x-\sigma^i(x)}(x) = \sigma(\ddot{\zeta})(\sigma^i(x)) = \ddot{\zeta}(\sigma^{i-1}(x)).$$

The total contribution to both first and the second sums on the RHS of (19) made by the transformations $\begin{pmatrix} \sigma_i \\ \sigma_i \end{pmatrix}$ for all values of $i \in \{1, 2, \ldots, |O(\sigma, x)|\}$ is equal to

$$\sum_{i=1}^{|O(\sigma,x)|} q^*(\sigma_i, \sigma_i; \ddot{\eta}^{x-\sigma^i(x)}, \ddot{\zeta}^{x-\sigma^i(x)}) \left[\mathbf{1}_{\{\sigma_i(\ddot{\eta})^{x-\sigma^i(x)}(x) \neq \sigma_i(\ddot{\zeta})^{x-\sigma^i(x)}(x)\}} - \mathbf{1}_{\{\ddot{\eta}^{x-\sigma^i(x)}(x) \neq \ddot{\zeta}^{x-\sigma^i(x)}(x)\}} \right]$$

$$\times \nu^* \left\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} : \begin{pmatrix} \eta(R_{x-\sigma^i(x)}) \\ \zeta(R_{x-\sigma^i(x)}) \end{pmatrix} = \begin{pmatrix} \ddot{\eta}^{x-\sigma^i(x)} \\ \ddot{\zeta}^{x-\sigma^i(x)} \end{pmatrix} \right\}$$

$$= q^*(\sigma, \sigma; \ddot{\eta}, \ddot{\zeta}) \cdot \nu^* \left\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} : \begin{pmatrix} \eta(R) \\ \zeta(R) \end{pmatrix} = \begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix} \right\}$$

$$\times \sum_{i=1}^{|O(\sigma,x)|} \left[\mathbf{1}_{\{\ddot{\eta}(\sigma^{i-1}(x)) \neq \ddot{\zeta}(\sigma^{i-1}(x))\}} - \mathbf{1}_{\{\ddot{\eta}(\sigma^i(x)) \neq \ddot{\zeta}(\sigma^i(x))\}} \right] = 0.$$

Thus the difference of the first two sums on the RHS of (19) should add up to zero.

Returning to the third and fourth sums on the RHS of (19), since the second sum cancels the first, and since the LHS there is $= 0$, the third and the fourth sums should also add up to zero. We notice that since inside the third sum $\sigma_R(\ddot{\eta}) \geq \ddot{\zeta}$, implying $D(\sigma_R(\ddot{\eta}), \ddot{\zeta}) \leq D(\ddot{\eta}, \ddot{\zeta})$, where the equality holds only when $D^-(\ddot{\eta}, \ddot{\zeta}) = 0$. Similarly $D(\ddot{\eta}, \sigma_R(\ddot{\zeta})) \leq D(\ddot{\eta}, \ddot{\zeta})$ inside the fourth sum, where the equality holds only when $D^+(\ddot{\eta}, \ddot{\zeta}) = 0$. That is the number of discrepancies inside R does not change if initially all the discrepancies are of the same type, and decreases otherwise. So,

$$D(\sigma_R(\ddot{\eta}), \ddot{\zeta}) < D(\ddot{\eta}, \ddot{\zeta}) \text{ in the third sum, and } D(\ddot{\eta}, \sigma_R(\ddot{\zeta})) < D(\ddot{\eta}, \ddot{\zeta})$$

in the fourth sum whenever both types of discrepancies are present inside R, that is $D^+(\ddot{\eta}, \ddot{\zeta}) \neq 0$ and $D^-(\ddot{\eta}, \ddot{\zeta}) \neq 0$. Hence for any range set R, and any configuration $(\ddot{\eta}, \ddot{\zeta}) \in \{0,1\}^R \times \{0,1\}^R$ of the coupled process on R such that $D^+(\ddot{\eta}, \ddot{\zeta}) \neq 0$ and $D^-(\ddot{\eta}, \ddot{\zeta}) \neq 0$,

$$\nu^* \left\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} : \begin{pmatrix} \eta(R) \\ \zeta(R) \end{pmatrix} = \begin{pmatrix} \ddot{\eta} \\ \ddot{\zeta} \end{pmatrix} \right\} = 0.$$

Therefore, for all range sets R,

$$\nu^*\Big\{ \begin{pmatrix}\eta\\ \zeta\end{pmatrix} : \quad D^+(\eta(R),\zeta(R)) \neq 0, \quad D^-(\eta(R),\zeta(R)) \neq 0\Big\} = 0$$

implying

$$\nu^*\big\{(\eta,\zeta) : \eta(x) = \zeta(y) \neq \zeta(x) = \eta(y)\big\} = 0.$$

for every x and y in S that both belong to the same range set, i.e. $\exists\sigma \in \Sigma$ such that $x, y \in Range(\sigma)$.

The above identity is the first step of induction. For two points x and y in S, we let $n(x,y)$ be the least integer n such that there is a sequence

$$x = x_0, x_1, ..., x_n = y$$

of points in S such that $\{\sigma \in \Sigma : x_{i-1}, x_i \in Range(\sigma)\} \neq \emptyset$ for all $i = 1,\dots,n$. Observe that $\{\sigma \in \Sigma : x_i, x_j \in Range(\sigma)\} = \emptyset$ for all $0 \leq i,j \leq n$ with $|i-j| \neq 1$. We have just proved the basis step $n(x,y) = 1$. For the general step we assume that Theorem 5 is true for $n(x,y) = 1,2,\dots,n-1$. We need to prove Theorem 5 for $n(x,y) = n$. We will adapt the notation that was used in many papers on interacting particle systems:

$$\nu^*\begin{Bmatrix}1 & 0\\ 0 & 1\\ u & v\end{Bmatrix} = \nu^*\big\{(\eta,\zeta) : \eta(u) = \zeta(v) = 0,\ \zeta(u) = \eta(v) = 1\big\},$$

for example. Now, for x and y in S with $n(x,y) = n$, we can expand

$$\begin{aligned}\nu^*\begin{Bmatrix}1 & 0\\ 0 & 1\\ x & y\end{Bmatrix} &= \nu^*\begin{Bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ x & x_1 & y\end{Bmatrix} + \nu^*\begin{Bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ x & x_1 & y\end{Bmatrix}\\ &+ \nu^*\begin{Bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ x & x_1 & y\end{Bmatrix} + \nu^*\begin{Bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ x & x_1 & y\end{Bmatrix},\end{aligned}$$

where the last two terms on the right are equal to zero by the induction hypothesis. Here $n(x,x_1) = 1$ and $n(x_1,y) = n-1$. Thus, we can show that the first two terms on the RHS are also equal to zero since, by the preceding induction step,

$$0 = \nu^*\begin{Bmatrix}a_1 & 1 & 0\\ a_2 & 0 & 1\\ x & x_1 & y\end{Bmatrix} = \nu^*\Omega^*(t)\begin{Bmatrix}a_1 & 1 & 0\\ a_2 & 0 & 1\\ x & x_1 & y\end{Bmatrix}.$$

Now due to conditions (5) and (6) there is a $\sigma \in \Sigma$ with $x = x_0, x_1 \in Range(\sigma)$ and $x_2,\dots,x_n = y \notin Range(\sigma)$ such that $\sigma(x_0) = x_1$ and $\sigma(x_1) = x_0$ among other things. So,

$$\nu^*\Omega^*(t)\begin{Bmatrix}a_1 & 1 & 0\\ a_2 & 0 & 1\\ x & x_1 & y\end{Bmatrix} \geq \nu^*\begin{Bmatrix}1 & a_1 & 0\\ 0 & a_2 & 1\\ x & x_1 & y\end{Bmatrix} te^{-ct}q(\sigma),$$

where the constant c is greater than the sum of the rates of all other permutations in Σ containing any of the x_i's in their ranges.

Observe that one does not really need $\sigma(x_1) = x_0$ when doing this proof with weaker conditions than (5) and (6) that were mentioned in 2.1.1.

So

$$\nu^* \begin{Bmatrix} 1 & 0 \\ 0 & 1 \\ x & y \end{Bmatrix} = 0$$

for all x and y in S with all values of $n(x,y)$, and Theorem 5 is proved. $\square$

3.3 Proof of Theorem 3.

Since Theorem 4 and Theorem 5 are now proved, the proof of Theorem 3 is word to word identical to the analogous case in the theory of exclusion processes and is a part of the system of results developed by T.Liggett for the exclusion processes that we are trying to redo for the permutation processes. However the proof is short and we need to inform the reader of why Theorem 4 and Theorem 5 are important parts of the proof of Theorem 3.

Proof of Theorem 3: Since $\int \Omega f d\nu_\rho = 0$, $\nu_\rho \in \mathcal{I}$ and obviously $\nu_\rho \in \mathcal{S}$ for all $0 \leq \rho \leq 1$. Furthermore, $\nu_\rho \in \mathcal{S}_e$, since it is spatially ergodic. Therefore, $\nu_\rho \in (\mathcal{I} \cap \mathcal{S})_e$.

For the converse, take $\nu \in (\mathcal{I} \cap \mathcal{S})_e$. By Theorem 4(e), for any $0 \leq \rho \leq 1$, there is a $\nu^* \in (\mathcal{I}^* \cap \mathcal{S}^*)_e$ with marginals ν_ρ and ν. By Theorem 5,

$$\nu^*\big\{(\eta,\zeta) : \eta \leq \zeta \quad \eta \neq \zeta\big\} + \nu^*\big\{(\eta,\zeta) : \zeta \leq \eta \quad \eta \neq \zeta\big\} + \nu^*\big\{(\eta,\zeta) : \eta = \zeta\big\} = 1.$$

Since the three sets above are closed for the evolution and translation invariant, and since ν^* is extremal, it follows that one of the three sets has full measure. Therefore, for every $0 \leq \rho \leq 1$, either $\nu \leq \nu_\rho$ or $\nu_\rho \leq \nu$. It follows that $\nu = \nu_{\rho_0}$ where ρ_0 is determined by

$$\nu \leq \nu_\rho \quad \text{for } \rho > \rho_0,$$

$$\nu \geq \nu_\rho \quad \text{for } \rho < \rho_0.$$

$\square$

Acknowledgments

The author wishes to thank Tom Liggett who suggested that the author investigates the model, and who was the main source of help and inspiration along the way.

References

[1] R.Holley, *Pressure and Helmholtz free energy in a dynamic model of a lattice gas.* Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability **III** (1972), 565-578.

[2] T.M.Liggett, *A characterization of the invariant measures for an infinite particle system with interactions.* Trans. Amer. Math. Soc. **179** (1973), 433-453.

[3] T.M.Liggett, *A characterization of the invariant measures for an infinite particle system with interactions, II.* Trans. Amer. Math. Soc. **198** (1974), 201-213.

[4] T.M.Liggett, *Coupling the simple exclusion process.* Ann. Probab. **4** (1976), 339-356.

[5] T.M.Liggett, *Interacting Particle Systems.* Springer, New York 1985.

[6] T.M.Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Springer, Berlin 1999.

[7] J.Riordan, *Introduction to Combinatorial Analysis.* John Wiley & Sons, Inc., New York 1958 (republished by Dover Publications, Inc., Mineola, New York, 2002).

[8] F.Spitzer, *Interaction of Markov Processes.* Adv. Math. **5** (1970), 246-290.

[9] F.Spitzer, *Recurrent random walk of an infinite particle system.* Trans. Amer. Math. Soc. **198** (1974), 191-199.

Yevgeniy Kovchegov
Department of Mathematics, UCLA
Email: yevgeniy@math.ucla.edu
Fax: 1-310-206-6673