Orthogonality and probability: beyond nearest neighbor transitions

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Introduction

Karlin-McGreogor diagonalization is used for

- recurrence/transience questions
- prob. harmonic functions (martingales)
- occupation times and hitting times

other...

Drawbacks (according to Grünbaum 2007)

- (a) “typically one cannot get either the polynomials or the measure explicitly”

- (b) “the method is restricted to ‘nearest neighbour’ transition probability chains that give rise to tridiagonal matrices and thus to orthogonal polynomials”

We will give possible answers to (b). Also consider newer methods, e.g. Riemann-Hilbert

Karlin-McGregor refresher

$$P = \begin{pmatrix} r_0 & p_0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & \ddots \\ 0 & 0 & q_3 & r_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \quad r_j = 1 - p_j - q_j$$

Let $\mathbf{q}^T(\lambda) = \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{pmatrix}$ s.t. $Q_0 = 1$, $\lambda \mathbf{q}^T = P\mathbf{q}^T$

Thus $\lambda Q_j(\lambda) = q_j Q_{j-1}(\lambda) + r_j Q_j(\lambda) + p_j Q_{j+1}(\lambda)$

Karlin-McGregor refresher

Reversible (thus stationary) distribution π:
$\pi_0 = 1$, $\pi_j = \frac{p_0 \cdots p_{j-1}}{q_1 \cdots q_j}$ satisfies $\pi_j p_j = \pi_{j+1} q_{j+1}$

Let $b_k = \sqrt{\frac{\pi_k}{\pi_{k+1}}} p_k = \sqrt{\frac{\pi_{k+1}}{\pi_k}} q_{k+1}$, then

$$\lambda \sqrt{\pi_k} Q_k = b_{k-1} \sqrt{\pi_k} Q_{k-1} + r_k \sqrt{\pi_k} Q_k + b_k \sqrt{\pi_k} Q_{k+1}$$

So $\tilde{\mathbf{q}} = (\sqrt{\pi_0} Q_0, \sqrt{\pi_1} Q_1, \dots)$ solves $\widetilde{P} \tilde{\mathbf{q}} = \lambda \tilde{\mathbf{q}}$, where

$$\widetilde{P} = \begin{pmatrix} r_0 & b_0 & 0 & \dots \\ b_0 & r_1 & b_1 & \ddots \\ 0 & b_1 & r_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Karlin-McGregor refresher

Spectrums of P and $\widetilde{P}$ coincide $\Rightarrow$ eigenvalues of are **real**.

Also $\mathbf{u}(\lambda) = (\pi_0 Q_0, \pi_1 Q_1, \dots)$ satisfies

$$\mathbf{u}P = \lambda P$$

as

$$\lambda \pi_k Q_n = q_n \pi_n Q_{n-1} + r_n \pi_n Q_n + p_n \pi_n Q_{n+1}$$

$$= p_{n-1} \pi_{n-1} Q_{n-1} + r_n \pi_n Q_n + q_{n+1} \pi_{n+1} Q_{n+1}$$

Diagonalization

If $\lambda_1, \ldots, \lambda_n$ are distinct real EVs of an $n \times n$ matrix A, and if $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are the left and right eigenvectors. Then

$$A^t = \sum_j \frac{\lambda^t v_j^T u_j}{u_j v_j^T} = \int_{\sigma(A)} \lambda^t v^T(\lambda) u(\lambda) d\psi(\lambda) \ ,$$

where $u(\lambda_j) = u_j$, $v(\lambda_j) = v_j$, and

$$\psi(\lambda) = \sum_j \frac{1}{u(\lambda) v^T(\lambda)} \delta_{\lambda_j}(\lambda) = \frac{n}{u(\lambda) v^T(\lambda)} U_{\sigma(A)}(\lambda)$$

Karlin-McGregor refresher

$$P^t = \int \lambda^t \mathbf{q}^T(\lambda)\mathbf{u}(\lambda)d\psi(\lambda)$$

$$= \int \lambda^t \begin{pmatrix} \pi_0 Q_0 Q_0 & \pi_1 Q_0 Q_1 & \cdots \\ \pi_0 Q_1 Q_0 & \pi_1 Q_1 Q_1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} d\psi(\lambda) \ ,$$

where

$$\psi(\lambda) = \lim_{n \to +\infty} \frac{n}{\mathbf{u}(\lambda)\mathbf{q}^T(\lambda)} U_{\sigma(A_n)}(\lambda)$$

- weak limit. A_n is the restriction of P to $< e_0, \ldots, e_{n-1} >$. Observe that the spectrum $\sigma(A_n)$ is the roots of

$$Q_n(\lambda) = 0$$

Example: Chebyshev polynomials.

$$P_{ch} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \ddots \\ 0 & 0 & \frac{1}{2} & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \quad T_0(\lambda) = 1,$$

$T_1(\lambda) = \lambda$ and $T_{k+1}(\lambda) = 2\lambda T_k(\lambda) - T_{k-1}(\lambda)$

Solving rec. relation: $T_k(\lambda) = \cos(k \cos^{-1}(\lambda))$

$\frac{n}{\sum_{k=0}^{n-1} \pi_k T_k^2(\lambda)} U_{\{\cos(n \cos^{-1}(\lambda))=0\}}(\lambda)$

$$(weakly) \quad \to d\psi(\lambda) = \frac{1}{\pi\sqrt{1-\lambda^2}} \chi_{[-1,1]}(\lambda) d\lambda$$

Reversible Markov chains

Theorem.(YK 2008) If P is a reversible M.Ch. Then there exists a spectral prob. meas. $d\psi$ and $Q_j(\lambda)$ - orthogonal polynomials w.r.t. $d\psi$ s.t.

$$P^t = F \left(\quad \int_{-1}^{1} s^t Q_i(s) Q_j(s) d\psi(s) \quad \right) F^T,$$

where $F = \begin{bmatrix} | & | & \\ Q_0(P)e_0^T & Q_1(P)e_0^T & \cdots \\ | & | & \end{bmatrix}$ and $F^T = F^{-1}$

Example: Pentadiagonal Chebyshev operator

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \ddots \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \ddots \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Observe

$$P = s(P_{ch}),$$

where $s(x) = x^2 + \frac{1}{2}x - \frac{1}{2}$

If P is a symmetric random walk reflecting at the origin, that jumps of up to m flights, $\exists$ a polynomial $s(x)$ s.t. $P = s(P_{ch})$ and roots z_j of char. relation in $\lambda \mathbf{c} = P\mathbf{c}$ will lie on $|z| = 1$ and roots $Re(z_j)$ solving $s(x) = \lambda$.

Reason: $\frac{1}{z_j} = \bar{z_j} \Rightarrow \lambda \mathbf{c} = P\mathbf{c}$ can be rewritten as

$$s\left(\frac{1}{2}\left[z + \frac{1}{z}\right]\right) = \lambda \ ,$$

where $\frac{1}{2}\left[z + \frac{1}{z}\right]$ is the Zhukovskiy function.

Pentadiagonal Chebyshev operator

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \ddots \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \ddots \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Pentadiagonal Chebyshev operator

Classical Fourier analysis $\Rightarrow$

$$\Big(e_0, (P - zI)^{-1} e_0\Big) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\frac{1}{2}[\cos(\theta)+\cos(2\theta)]-z}$$

$$= \int_{-\frac{9}{16}}^{1} \frac{d\psi(s)}{s-z} \ ,$$

where

$$d\psi(s) = \frac{1}{2\pi\sqrt{s+\frac{9}{16}}}\left(\frac{\chi_{[-\frac{9}{16},1]}(s)}{\sqrt{1-\left(\sqrt{s+\frac{9}{16}}-\frac{1}{4}\right)^2}} + \frac{\chi_{[-\frac{9}{16},0)}(s)}{\sqrt{1-\left(\sqrt{s+\frac{9}{16}}+\frac{1}{4}\right)^2}}\right) ds$$

Obtaining Jacobi operator

$$P_\triangle = \begin{pmatrix} a_0 & b_0 & 0 & \cdots \\ b_0 & a_1 & b_1 & \ddots \\ 0 & b_1 & a_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \qquad b_j > 0$$

$$(e_0, Pe_0) = (e_0, P_\triangle e_0) = a_0,$$
$$(e_0, P^2 e_0) = (e_0, P_\triangle^2 e_0) = a_0^2 + b_0^2$$
$$(e_0, P^3 e_0) = (e_0, P_\triangle^3 e_0) = (a_0^2 + b_0^2)a_0 + (a_0 + a_1)b_0^2$$

$$(e_0, P^4 e_0) = (e_0, P_\triangle^4 e_0)$$

$$= (a_0^2 + b_0^2)^2 + (a_0 + a_1)^2 b_0^2 + b_0^2 b_1^2$$

Obtaining Jacobi operator

... obtaining a_0, b_0, a_1, $b_1, \ldots$ in

$$P_\triangle = \begin{pmatrix} a_0 & b_0 & 0 & \cdots \\ b_0 & a_1 & b_1 & \ddots \\ 0 & b_1 & a_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \qquad b_j > 0$$

Pentadiagonal Chebyshev operator

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \ddots \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \ddots \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$(e_0, Pe_0) = 0, \;\; (e_0, P^2 e_0) = \frac{1}{4}, \;\; (e_0, P^3 e_0) = \frac{3}{32}$$

$$(e_0, P^4 e_0) = \tfrac{9}{64}, \quad \ldots$$

Pentadiagonal Chebyshev operator

Thus

$$a_0 = 0, \quad b_0 = \frac{1}{2}, \quad a_1 = \frac{3}{8}, \quad b_1 = \frac{\sqrt{11}}{8}, \text{ etc.}$$

and

$$P_\triangle = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \cdots \\ \frac{1}{2} & \frac{3}{8} & \frac{\sqrt{11}}{8} & \ddots \\ 0 & \frac{\sqrt{11}}{8} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

and $Q_0(\lambda) = 1$, $Q_1(\lambda) = 2\lambda$,

$$Q_2(\lambda) = \frac{32}{\sqrt{11}}\lambda^2 - \frac{6}{\sqrt{11}}\lambda - \frac{4}{\sqrt{11}}, \quad \ldots$$

Reversible Markov chains

Theorem.(YK 2008) If P is a reversible M.Ch. Then there exists a spectral prob. meas. $d\psi$ and $Q_j(\lambda)$ - orthogonal polynomials w.r.t. $d\psi$ s.t.

$$P^t = F \left(\quad \int_{-1}^{1} s^t Q_i(s) Q_j(s) d\psi(s) \quad \right) F^T,$$

where $F = \begin{bmatrix} | & | & \\ Q_0(P)e_0^T & Q_1(P)e_0^T & \cdots \\ | & | & \end{bmatrix}$ and

$F^T = F^{-1}$

P.Deift *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach* **(2000)**

Spectral Theorem For every bounded Jacobi operator $\mathcal{A}$, $\exists!$ prob. measure ψ with compact support s.t.

$$G(z) = \left(e_0, (\mathcal{A} - zI)^{-1} e_0\right) = \int_{-\infty}^{+\infty} \frac{d\psi(x)}{x - z}$$

$\mathcal{U} : A \to d\psi$ is **one-to-one onto**, and $\forall f \in L^2(d\psi)$, $(\mathcal{U}\mathcal{A}\mathcal{U}^{-1} f)(s) = s f(s)$ in the following sense

$$(e_0, \mathcal{A} f(\mathcal{A}) e_0) = \int s f(s) d\psi(s)$$

Spectral measure for reversible M.Ch.

We construct a map $\mathcal{M}$ which assigns a prob. meas. $d\psi$ to a reversible P on $\{0, 1, 2, \ldots\}$. W.l.o.g. we assume P is symmetric as

$$\begin{pmatrix} \sqrt{\pi_0} & 0 & \cdots \\ 0 & \sqrt{\pi_1} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} P \begin{pmatrix} \frac{1}{\sqrt{\pi_0}} & 0 & \cdots \\ 0 & \frac{1}{\sqrt{\pi_1}} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}$$

is symmetric, and its is $\sigma(P) \subset [-1, 1]$.

Spectral measure for reversible M.Ch.

$\mathcal{M} : \{\text{reversible P}\} \to \{\psi \text{ on } [-1,1] \text{ compact } supp(\psi)\}$

s.t.

$$(e_0, (P - zI)^{-1} e_0) = \int \frac{d\psi(s)}{s - z}, \quad Im(z) > 0$$

Which implies

$$(e_0, P^k e_0) = \int s^k d\psi(s)$$

Spectral measure for reversible M.Ch.

The method is from P.Deift (2000) (goes back to M.G.Krein and N.I.Akhiezer): for $z \in \mathbb{C} \setminus \mathbb{R}$, let

$$G(z) = (e_0, (P - zI)^{-1} e_0)$$

$$ImG(z) = \frac{1}{2i}\Big[(e_0, (P - zI)^{-1} e_0) - (e_0, (P - \bar{z}I)^{-1} e_0)\Big]$$

$$= (Im(z))|(P - zI)^{-1} e_0|^2$$

$\Rightarrow G(z)$ is a **Herglotz function**, i.e. $G(z)$ is analytic: $\{Im(z) > 0\} \to \{Im(z) > 0\}$

Spectral measure for reversible M.Ch.

$$G(z) = (e_0, (P - zI)^{-1} e_0)$$

is a **Herglotz function** $\Rightarrow$ $G(z)$ can be represented as

$$G(z) = az + b + \int_{-\infty}^{+\infty} \left(\frac{1}{s - z} - \frac{s}{s^2 + 1} \right) d\psi(s), \; Im(z) > 0$$

with $a \geq 0$ and b real constants and $d\psi$, a Borel measure s.t.

$$\int_{-\infty}^{+\infty} \frac{1}{s^2 + 1} d\psi(s) < \infty$$

Spectral measure for reversible M.Ch.

Following P.Deift (2000), we use

$$G(z) = (e_0, (P - zI)^{-1} e_0) = -\frac{1}{z} + O(z^{-2})$$

to show $a = 0$, and

$$b = \int_{-\infty}^{+\infty} \frac{s}{s^2+1} d\psi(s)$$

as well as uniqueness of $d\psi$.

Hence

$$G(z) = \int \frac{d\psi(s)}{s - z}, \quad Im(z) > 0$$

Spectral measure for reversible M.Ch.

$$G(z) = (e_0, (P - zI)^{-1} e_0) = \int \frac{d\psi(s)}{s - z}, \;\; Im(z) > 0$$

implies

$$(e_0, P^k e_0) = \int s^k d\psi(s)$$

We consider

$$\mathcal{U}^{-1}\mathcal{M} : P \to P_\triangle$$

where $P_\triangle$ is a unique Jacobi operator s. t.

$$(e_0, P^k e_0) = \int s^k d\psi(s) = (e_0, P_\triangle^k e_0)$$

Spectral measure for reversible M.Ch.

If $Q_j(\lambda)$ are orthog. poly. w.r.t. $d\psi$ associated with $P_\triangle$, then

$$Q_j(P_\triangle)e_0 = e_j$$

and

$$\delta_{i,j} = (e_i, e_j) = (Q_i(P_\triangle)e_0, Q_j(P_\triangle)e_0)$$

$$= (Q_i(P)e_0, Q_j(P)e_0)$$

$\Rightarrow$ If P is irreducible, then $f_j = Q_j(P)e_0$ is an orthonormal basis.

Spectral measure for reversible M.Ch.

$f_j = Q_j(P)e_0$ is an orthonormal basis. Let

$$F = \begin{bmatrix} | & | & \\ f_0^T & f_1^T & \cdots \\ | & | & \end{bmatrix}, \text{ then}$$

$$P^t = \left(\quad (P^t e_i, e_j) \quad \right) = F \left(\quad \int_{-1}^{1} s^t Q_i(s) Q_j(s) d\psi(s) \quad \right) F^T,$$

as F translates from $(P^t e_i, e_j)$ to $(P_\triangle^t e_i, e_j)$

$$= (P_\triangle^t Q_i(P_\triangle)e_0, Q_j(P_\triangle)e_0) = (P^t Q_i(P)e_0, Q_j(P)e_0)$$

Observe: $F^T = F^{-1}$.

Obtaining Jacobi operator

$$P_\triangle = \begin{pmatrix} a_0 & b_0 & 0 & \cdots \\ b_0 & a_1 & b_1 & \ddots \\ 0 & b_1 & a_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \qquad b_j > 0$$

$$(e_0, Pe_0) = (e_0, P_\triangle e_0) = a_0,$$
$$(e_0, P^2 e_0) = (e_0, P_\triangle^2 e_0) = a_0^2 + b_0^2$$
$$(e_0, P^3 e_0) = (e_0, P_\triangle^3 e_0) = (a_0^2 + b_0^2)a_0 + (a_0 + a_1)b_0^2$$

$$(e_0, P^4 e_0) = (e_0, P_\triangle^4 e_0)$$

$$= (a_0^2 + b_0^2)^2 + (a_0 + a_1)^2 b_0^2 + b_0^2 b_1^2$$

Another approach: Riemann surfaces

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \ddots \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \ddots \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$P = s(P_{ch})$, where $s(x) = \left(x + \frac{1}{4}\right)^2 - \frac{9}{16}$

Another approach: Riemann surfaces

$$\mathbf{q}^T(\lambda) = \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{pmatrix}, \text{ where } Q_0(\lambda) = 1 \text{ and}$$

$Q_1(\lambda) = \mu(\lambda)$ **Question:** $\mu(\lambda) =?$

$P = s(P_{ch})$, where $s(x) = \left(x + \frac{1}{4}\right)^2 - \frac{9}{16}$

Two candidates: $\mu_+(\lambda) = \frac{-1+\sqrt{9+16\lambda}}{4}$ and $\mu_-(\lambda) = \frac{-1-\sqrt{9+16\lambda}}{4}$ over $[-\frac{9}{16}, 0)$

$\mu(\lambda) = \mu_+(\lambda)$ over $[0, 1]$.

Another approach: Riemann surfaces

Two candidates: $\mu_+(\lambda) = \frac{-1+\sqrt{9+16\lambda}}{4}$ and $\mu_-(\lambda) = \frac{-1-\sqrt{9+16\lambda}}{4}$ over $[-\frac{9}{16}, 0)$

$\mu(\lambda) = \mu_+(\lambda)$ over $[0, 1]$.

Plemelj formula:

$$\mu(z) = -\frac{1}{4} + z^{\frac{1}{2}} \exp\left\{ \frac{1}{2} \int_{-\frac{9}{16}}^{0} \frac{ds}{s-z} \right\} ,$$

where $\mu_+(\lambda) = \lim_{z\to\lambda,\ Im(z)>0} \mu_1(z)$ and $\mu_-(\lambda) = \lim_{z\to\lambda,\ Im(z)<0} \mu_1(z)$.

Another approach: Riemann surfaces

Plemelj formula:

$$\mu(z) = -\frac{1}{4} + z^{\frac{1}{2}} \exp\left\{\frac{1}{2}\int_{-\frac{9}{16}}^{0} \frac{ds}{s-z}\right\}$$

Then

$$P^t = \int_{\left[-\frac{9}{16},0\right)_- \cup \left[-\frac{9}{16},0\right)_+ \cup [0,1]} \lambda^t \mathbf{q}^T(\lambda)\mathbf{u}(\lambda)d\psi(\lambda),$$

where

$$d\psi(\lambda) = \left(\frac{\chi_{[-\frac{9}{16},0)_-}(\lambda)}{\sqrt{1-\left(\sqrt{\lambda+\frac{9}{16}}+\frac{1}{4}\right)^2}} + \frac{\chi_{[-\frac{9}{16},0)_+}(\lambda)+\chi_{[0,1]}(\lambda)}{\sqrt{1-\left(\sqrt{\lambda+\frac{9}{16}}-\frac{1}{4}\right)^2}}\right)\frac{d\lambda}{2\pi\sqrt{\lambda+\frac{9}{16}}}$$

Applications: mixing rates

$$t_{mix}(\epsilon)=\text{min}\left\{t:\left\|e_i F\left(\int_{(-1,1)} s^t Q_j(s) Q_k(s) d\psi(s)\right) F^T\right\|_{\ell_1}\leq 2\epsilon\right\},$$

where $\mu_0 = i$

Applications: the generator

The generator

$$G(z) = \left(\quad \sum_{t=0}^{+\infty} z^{-t} p_t(i,j) \quad \right)$$

$$= F \left(\quad -z \int_{-1}^{1} \frac{Q_i(\lambda) Q_j(\lambda)}{\lambda - z} d\psi(\lambda) \quad \right) F^T$$

Applications: Riemann-Hilbert problems and RWRE

- Use the Fokas, Its and Kitaev results, and benefit from the connection between orthogonal polynomials and Riemann-Hilbert problems.

- Interpret random walks in random environment as a random spectral measure.

Other applications

- Martingales

- Spectral geometry and manifold learning