Occupation times and modified Bessel functions

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Joint work with N.Meredith

Occupation time. Consier n-state Markov process with generator

$$Q = \begin{pmatrix} -\sum_{j\neq 0}\lambda_{0,j} & \lambda_{0,1} & \dots \\ \lambda_{1,0} & -\sum_{j\neq 1}\lambda_{1,j} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

For the site 0, let $f_k(t,x)$ be the occupation time density, given that initially, the process is at site k.

Integral equations.

Integral equations relating $\{f_k(t,x)\}_{k=0,1,\ldots,n-1}$ via conditioning:

$$\begin{aligned}
f_0(t,x) &= e^{-(\sum_{m\neq 1}\lambda_{0,m})t}\delta_t(x) \\
&+ \sum_{k\neq 0}\int_0^t f_k(t-y,x-y)\lambda_{0,k}e^{-(\sum_{m\neq 0}\lambda_{0,m})y}dy \\
f_j(t,x) &= e^{-(\sum_{m\neq j}\lambda_{j,m})t}\delta_0(x) \\
&+ \sum_{k\neq j}\int_0^t f_k(t-y,x)\lambda_{j,k}e^{-(\sum_{m\neq j}\lambda_{j,m})y}dy
\end{aligned}$$

for j=1,2,. . . n-1 .

Integral equations.

Substituting $\psi = t - y$:

$$
\begin{aligned}
f_0(t,x) &= e^{-(\sum_{m\neq 1}\lambda_{0,m})t}\delta_t(x) \\
&+ \sum_{k\neq 0}\int_0^t f_k(\psi, x-t+\psi)\lambda_{0,k}e^{-(\sum_{m\neq 0}\lambda_{0,m})(t-\psi)}d\psi \\
f_j(t,x) &= e^{-(\sum_{m\neq j}\lambda_{j,m})t}\delta_0(x) \\
&+ \sum_{k\neq j}\int_0^t f_k(\psi, x)\lambda_{j,k}e^{-(\sum_{m\neq j}\lambda_{j,m})(t-\psi)}d\psi
\end{aligned}
$$

for j=1,2,. . . n-1 .

Fourier transform.

Taking the Fourier transform with respect to x:

$$\hat{f}_0(t,s_2) = e^{-(\sum_{m\neq 0}\lambda_{0,m} - is_2)t} + \sum_{k\neq 0}\int_{-\infty}^{\infty}\int_0^t f_k(\psi, x-t+\psi)\lambda_{0,k} e^{-(\sum_{m\neq 0}\lambda_{0,m})(t-\psi)} d\psi e^{is_2 x} dx$$

$$\hat{f}_j(t,s_2) = e^{-(\sum_{m:\ m\neq j}\lambda_{j,m})t} + \sum_{k\neq j}\int_{-\infty}^{\infty}\int_0^t f_k(\psi, x)\lambda_{j,k} e^{-(\sum_{m\neq j}\lambda_{j,m})(t-\psi)} d\psi e^{is_2 x} dx$$

for j=1,2,. . . n-1 .

Fourier transform.

Simplifying:

$$e^{(\sum_{m\neq 0}\lambda_{0,m}-is_2)t}\hat{f}_0(t,s_2)$$
$$= 1+\sum_{k\neq 0}\int_0^t \hat{f}_k(\psi,s_2)\lambda_{0,k}e^{(\sum_{m\neq 0}\lambda_{0,m}-is_2)\psi}d\psi$$

$$e^{(\sum_{m\neq j}\lambda_{j,m})t}\hat{f}_j(t,s_2)$$
$$= 1+\sum_{k\neq j}\int_0^t \hat{f}_k(\psi,s_2)\lambda_{j,k}e^{(\sum_{m\neq j}\lambda_{j,m})\psi}d\psi$$

for j=1,2,. . . n-1 .

Fourier transform.

Differentiating w.r.t. variable t:

$$\left(\sum_{m\neq 0}\lambda_{0,m} - is_2\right)\hat{f}_0(t,s_2)+\frac{\partial}{\partial t}\hat{f}_0(t,s_2) = \sum_{k:\ k\neq 0}\lambda_{0,k}\hat{f}_k(t,s_2)$$

$$\left(\sum_{m\neq j}\lambda_{j,m}\right)\hat{f}_j(t,s_2) + \frac{\partial}{\partial t}\hat{f}_j(t,s_2) = \sum_{k:\ k\neq j}\lambda_{j,k}\hat{f}_k(t,s_2)$$

for j=1,2,. . . n-1 .

Laplace-Fourier transform.

Observe: $\widehat{f}_j(0, s_2) = 1$ for all j. Taking Laplace transform w.r.t. variable t:

$$\left(\sum_{m \neq 0} \lambda_{0,m} + s_1 - i s_2\right) \mathcal{L}_{\widehat{f}_0}(s_1, s_2) = 1 + \sum_{k \neq 0} \lambda_{0,k} \mathcal{L}_{\widehat{f}_k}(s_1, s_2)$$

$$\left(\sum_{m \neq j} \lambda_{j,m} + s_1\right) \mathcal{L}_{\widehat{f}_j}(s_1, s_2) = 1 + \sum_{k \neq j} \lambda_{j,k} \mathcal{L}_{\widehat{f}_k}(s_1, s_2)$$

for j=1,2,. . . n-1 .

Spectral representation.

Let $\text{Ⱡ}_{\widehat{f}}(s_1, s_2) = \begin{bmatrix} \text{Ⱡ}_{\widehat{f}_0}(s_1, s_2) \\ \text{Ⱡ}_{\widehat{f}_1}(s_1, s_2) \\ \vdots \end{bmatrix}$ and $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix}$.

We proved

$$(Q - s_1 I)\text{Ⱡ}_{\widehat{f}}(s_1, s_2) = -\mathbf{1} - is_2\text{Ⱡ}_{\widehat{f}_0}(s_1, s_2)e_0$$

Thus

$$\text{Ⱡ}_{\widehat{f}}(s_1, s_2) = -(Q - s_1 I)^{-1}\mathbf{1} - is_2\text{Ⱡ}_{\widehat{f}_0}(s_1, s_2)(Q - s_1 I)^{-1}e_0$$

Spectral representation.

Therefore

$$\mathcal{L}_{\hat{f}_0}(s_1, s_2) = \frac{-\Big((Q - s_1 I)^{-1}\mathbf{1}, e_0\Big)}{1 + is_2\Big((Q - s_1 I)^{-1} e_0, e_0\Big)}$$

Now $(Q - s_1 I)\mathbf{1} = -s_1\mathbf{1}$ implies

$$\mathcal{L}_{\hat{f}_0}(s_1, s_2) = \frac{1/s_1}{1 - is_2 h(s_1)},$$

where $h(s) = -\Big((Q - sI)^{-1} e_0, e_0\Big) = \Big((\int_0^\infty e^{-st} e^{Qt} dt) e_0, e_0\Big)$
$= \int_0^\infty e^{-st} p_t(0,0) dt.$

Laplace transform w.r.t. t.

The Fourier transform can be inverted via complex integration:

$$\mathcal{L}_{f_0}(s_1, x) = \frac{1}{s_1 h(s_1)} \exp\left\{-\frac{x}{h(s_1)}\right\},$$

where $h(s) = -\Big((Q - sI)^{-1} e_0, e_0\Big)$.

Modified Bessel functions.

In many cases the inverse Laplace transform of

$$\text{Ł}_{f_0}(s_1, x) = \frac{1}{s_1 h(s_1)} \exp\left\{ -\frac{x}{h(s_1)} \right\},$$

can be represented via modified Bessel functions

$$I_m(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{m+2k}}{k!(k+m)!}$$

that solve

$$z^2 u''(z) + zu(z) - (z^2 + m^2)u(z) = 0$$

Modified Bessel functions.

$$\int_0^\infty I_0(2\sqrt{at})e^{-st}dt = \frac{1}{s}e^{\frac{a}{s}}$$

and therefore

$$e^{-sx}\frac{1}{s}e^{\frac{a}{s}} = \int_x^\infty I_0(2\sqrt{a(t-x)})e^{-st}dt$$

Also

$$\int_0^\infty \left(\frac{t}{a}\right)^{\frac{m}{2}} I_m(2\sqrt{at})e^{-st}dt = s^{-(m+1)}e^{\frac{a}{s}}$$

Example: Two-state Markov processes.

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$(Q - s_1 I)^{-1} = \frac{-1}{s_1^2 + (\lambda + \mu)s_1} \begin{pmatrix} \mu + s_1 & \lambda \\ \mu & \lambda + s_1 \end{pmatrix}$$

$$\text{Ł}(s_1, x) = e^{-x(s_1+\lambda)} e^{\frac{\lambda\mu x}{s_1+\mu}} + \frac{\lambda}{s_1 + \mu} e^{-x(s_1+\lambda)} e^{\frac{\lambda\mu x}{s_1+\mu}}$$

$$f_0(t, x) = e^{-\lambda t}\delta_t(x) + \lambda e^{-\lambda x} e^{-\mu(t-x)} I_0(2\sqrt{\lambda\mu x(t-x)}) + \sqrt{\tfrac{\lambda\mu x}{t-x}} I_1(2\sqrt{\lambda\mu x(t-x)}) e^{-\lambda x} e^{-\mu(t-x)}$$

Birth-and-death process. Karlin-McGregor polynomials $P_0(s) \equiv 1$, $P_1(s)$, $P_2(s), \ldots$

$$P[s] = \begin{bmatrix} P_0(s) \\ P_1(s) \\ P_2(s) \\ \vdots \end{bmatrix}$$

satisfying

$$(Q - sI)P[s] = 0 \ .$$

$\{P_k\}$ are orthogonal w.r.t. probability measure μ, i.e.

$$\int_{(-\infty,0]} P_k(s)P_m(s)d\mu(s) = \frac{\delta_{k,m}}{\pi_k} \ ,$$

where $\pi_k = \frac{\lambda_0\lambda_1 \ldots \lambda_{k-1}}{\mu_1\mu_2 \ldots \mu_k}$

Birth-and-death process. (Karlin-McGregor)

$$(Q - sI)P[s] = 0$$

$$\int_{(-\infty,0]} P_k(s)P_m(s)d\mu(s) = \frac{\delta_{k,m}}{\pi_k} \ .$$

Here

$$h(s) = -\int_{(-\infty,0]} \frac{d\mu(x)}{x - s}$$

- Cauch integral w.r.t. μ

Example: Birth-and-death process with equal rates.

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & \ldots \\ 0 & 1 & -2 & 1 & \ldots \\ 0 & 0 & 1 & -2 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

$$\int_{(-\infty,0]} \frac{d\mu(x)}{x-s} = \frac{-2}{s+\sqrt{s^2+4s}}$$

$$f_0(x,t) = e^{-t}\delta_0(t-x) + e^{x-2t} I_0\left(2\sqrt{t(t-x)}\right) + \sqrt{\tfrac{t}{t-x}}\, e^{x-2t} I_1\left(2\sqrt{t(t-x)}\right)$$

Example: Simple random walk on $\mathbb{Z}$.

$$Q = \begin{pmatrix} \ddots & \ddots & 0 & 0 & \cdots \\ \ddots & -2 & 1 & 0 & \cdots \\ 0 & 1 & -2 & 1 & \cdots \\ 0 & 0 & 1 & -2 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Simple Fourier analysis:

$$h(s) = \frac{1}{2\pi}\int_0^{2\pi} \frac{dx}{2\cos x - (2+s)} = -\int_{-2}^{0} \frac{d\mu(y)}{y-s} \ ,$$

where $d\mu(y) = \frac{2}{\pi\sqrt{y(y-4)}} dy$.

Why modified Bessel functions? Moments of μ:

$$m_j = \int_{(-\infty,0]} (-x)^j d\mu(x) \ .$$

Assume μ has compact support.

$$\frac{1}{h(z)} = z + m_1 - (m_2 - m_1^2)z^{-1} + \phi(z^{-1})z^{-2} \ .$$

If $\mathcal{L}_{F_0}(s_1, x) = \frac{1}{s_1} \exp\left\{-\frac{x}{h(s_1)}\right\}$, then

$$F_0(t,x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{z} \exp\left\{zt - \frac{x}{h(z)}\right\} dz$$

$$= \frac{e^{-m_1 x}}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{z} \exp\left\{z(t-x) + (m_2 - m_1^2)xz^{-1} - x\phi(z^{-1})z^{-2}\right\} dz$$

Why modified Bessel functions? Writing

$$e^{-x\phi(z^{-1})z^{-2}} = 1 + \sum_{k=2}^{\infty} v_k(x) z^{-k},$$

get

$$F_0(t,x) = e^{-m_1 x} I_0\left(2\sqrt{(m_2 - m_1^2)(t-x)x}\right)$$
$$+e^{-m_1 x} \sum_{k=2}^{\infty} v_k(x) \left(\frac{t-x}{(m_2-m_1^2)x}\right)^{\frac{k}{2}} I_k\left(2\sqrt{(m_2 - m_1^2)(t-x)x}\right),$$

where $f_0(t,x) = -\frac{\partial}{\partial x} F_0(t,x)$.