Multi-particle processes with reinforcements.

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Abstract

We consider a multi-particle generalization of linear edge-reinforced random walk (ERRW). We observe that in absence of exchangeability, new techniques are needed in order to study the multi-particle model. We describe an unusual coupling construction associated with the two-point edge-reinforced process on $\mathbb{Z}$ and prove a form of recurrence: the two particles meet infinitely often a.s. [1]

1 Introduction

The edge-reinforced random walk was first introduced in [2] and [5]. In this paper we will study linear multi-particle edge-reinforced processes on $\mathbb{Z}$. In the original edge-reinforced random walk model, each edge of a locally finite non-directed graph is initially assigned weight $a > 0$. With each step, the particle jumps to a nearest-neighbor vertex. The probability of the jump equals to the fraction of the weight attached to the traversed edge in the total sum of the weights of the edges coming out of the vertex where the particle is located prior to the jump. Each time an edge is traversed, its weight is increased by 1. In other words, the linear edge-reinforced random walk is a random walk on a weighted graph, where the weight of an edge is increased by one each time it is being traversed.

One of the most important open problems in the theory of reinforced random walks is that of checking if the linear edge-reinforced random walk is recurrent on $\mathbb{Z}^d$ for dimensions $d \geq 2$. Linear edge-reinforced random walk is exchangeable making the model an important example for applying the theorem of de Finetti (see [7]) and its generalizations (see [4], [5]). The history and some of the most important results in reinforced processes can be found in [5], [3], [10], [8], [1], [12], [11] and references therein.

*This research was supported in part by NSF VIGRE Grant DMS 9983726 at UCLA
[1]Keywords: edge-reinforced processes, random walk, random environment, urn model, coupling method, recurrence
AMS Subject Classification: 60C05, 60G07, 60G09, 60K37

In this paper we will consider a multi-particle modification of the edge-reinforced random walk model similar to some of the reinforced processes studied in [11]. We let the walker (or particle) in the edge-reinforced random walk model wait an independent exponential time with rate one between the jumps. So, the walker jumps from a site to a near by site with rates equal to the corresponding ratios. Now we are ready to define an n-point process $\eta_t = \{\eta_1(t), \ldots, \eta_n(t)\}$, where all n particles travel along the edges of a graph G, jumping from a site to a neighboring site in S, the set of all sites. Now, let $W_t(e_1), ..., W_t(e_k)$ be the weights assigned to all k edges $e_1, ..., e_k$ coming out of a given site $v \in S$ at time t. Once again, the initial weights are all assigned to be equal to $a > 0$, i.e. $W_0(e_1) = W_0(e_2) = \cdots = W_0(e_k) = a$. If one of the particles, say η_j, is at site v at t when its exponential clock rings, then the particle traverses e_i $(1 \leq i \leq k)$ with the rate $= \frac{W_{t-}(e_i)}{W_{t-}(e_1)+\cdots+W_{t-}(e_k)}$. In which case the corresponding edge weight increases by 1, i.e. $W_t(e_i) = W_{t-}(e_i) + 1$. The recurrence/transience questions arising in this more general model are as important as the corresponding questions in the theory of one-article edge-reinforced random walks.

The edge-reinforced process on $\mathbb{Z}$ with drift $\Delta > 0$ can be defined in the following way: if a particle is at site $v \in \mathbb{Z}$ at the jump time t, then the probability of the particle jumping to $v+1$ is

$$\frac{W_{t-}(v, v+1) + \Delta}{W_{t-}(v-1, v) + W_{t-}(v, v+1) + \Delta}$$

while the probability of it jumping to $v-1$ is

$$\frac{W_{t-}(v-1, v)}{W_{t-}(v-1, v) + W_{t-}(v, v+1) + \Delta}.$$

On trees, the edge-reinforced process with a toward-the-root drift $\Delta > 0$ can be stated accordingly.

We will concentrate on the most basic case of multi-particle reinforced processes: the two point reinforced process $\eta_t = \{\eta_1(t), \eta_2(t)\}$ on $\mathbb{Z}$ with drift $\Delta \geq 0$. We will describe an unusual coupling construction associated with the process and as a consequence prove the recurrence of $(\eta_2(t) - \eta_1(t))$ whenever $0 \leq \Delta < 1$. In the case of a two-particle process on $\mathbb{Z}$, one of the two particles is located to the left of another, except for the times when both particles are at the same site. We will denote by l_t the location of the left particle, and by r_t the location of the right particle at time t. When $r_t = l_t$ there is no need to distinguish between the "left" and the "right" particles. The difference becomes apparent only when one of the particles leaves the site. So, $\eta_t = \{l_t, r_t\}$, and here is the main result of this paper:

Theorem 1. *For all $0 \leq \Delta < 1$ and $a > 0$, $(r_t - l_t)$ is recurrent.*

Let us begin by reviewing the Polya's urn model. The urn initially contains R_0 marbles of red color, and B_0 marbles of blue color. We fix a positive integer number D. A marble is randomly and uniformly drawn from the urn, returned, and D marbles of the same color are added. Let R_n and B_n be respectively the number of red and blue marbles in the urn after n drawings, and let $\rho_n = \frac{R_n}{R_n+B_n}$ be the fraction of the red marbles in the urn after n drawings. It is easy to show that ρ_n is a martingale, and therefore, by martingale convergence theorem, converges to a random variable. That random variable ρ_∞ is in turn shown to be a beta random variable with parameters $\frac{R_0}{D}$ and $\frac{B_0}{D}$, i.e. one with beta density function

$$\frac{1}{\beta(\frac{R_0}{D}, \frac{B_0}{D})} x^{\frac{R_0}{D}-1}(1-x)^{\frac{B_0}{D}-1}, \tag{1}$$

where $\beta(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$. One can check that the urn model is *exchangeable* (see [7]), that is if one permutes the results of m consecutive drawings, the probability of the outcome does not change. By de Finetti's theorem, conditioned on ρ_∞, the results of the drawings are independent Bernoulli trials, where each time a red marble is selected with probability ρ_∞ and a blue marble is selected with probability $1 - \rho_\infty$.

The model trivially extends to the case when R_0, B_0 and D are positive *real* numbers, as well as when there are more than two different types of marbles. For instance, consider the case when there are three types of marbles, red, blue and green, in the urn. If we start with the amounts R_0, B_0 and G_0 of respectively red, blue and green marbles, then the limiting fractions vector will be a Dirichlet distributed random vector, i.e. the cumulative density function for the limiting fractions of red and blue marbles will be

$$f(x,y) = \frac{\Gamma\left(\frac{R_0+B_0+G_0}{D}\right)}{\Gamma\left(\frac{R_0}{D}\right)\Gamma\left(\frac{B_0}{D}\right)\Gamma\left(\frac{G_0}{D}\right)} x^{\frac{R_0}{D}-1} y^{\frac{B_0}{D}-1}(1-x-y)^{\frac{G_0}{D}-1} \text{ if } x>0, y>0 \text{ and } x+y<1,$$

where the above density is derived by applying (1) twice.

See [7] for basic facts on exchangeability, the Polya's urn model and a simple version of de Finetti's theorem. A simple proof of the convergence to beta distribution can be found in [15].

Polya's urns were used to study linear edge-reinforced random walks (see [12]). There, if the walk lives on an acyclic graph, say $\mathbb{Z}$, we can assign a Polya's urn for each site. When the walker is at site v, we do a drawing from the urn associated with v, where the number of the red (respectively blue) marbles in the urn is equal to the weight attached to the edge $[v-1, v]$ (respectively $[v, v+1]$) at the time. If a red marble is drawn, the walker jumps to $v-1$ and we add $D = 2$ red marbles to the urn associated with v. Similarly, if a blue marble is drawn, the walker jumps to $v+1$ and we add $D = 2$ blue marbles. We do so because the graph is acyclic: if the particle ever returns to the vertex, it will be from the same direction it took when it left the vertex. For example, if the red marble is drawn, the walker will traverse the edge $[v-1, v]$ twice before returning to v, thus increasing the weight of the edge exactly by $D = 2$. The initial conditions $[R_0(v), B_0(v)]$ at an urn associated with site v should be set equal to the weights attached to the edges $[v-1, v]$ and $[v, v+1]$ respectively at the time of the first arrival to v. As an example, consider the edge-reinforced random walk on $\mathbb{Z}$ that begins at site 0. There the correct initial conditions for a Polya's urn assigned to site $v \in \mathbb{Z}$ should be set equal to

$$[R_0(v), B_0(v)] = \begin{cases} [a, a+1] \text{ if } v < 0, \\ [a, a] \text{ if } v = 0, \\ [a+1, a] \text{ if } v > 0. \end{cases}$$

What follows is that one can do an infinite number of drawings independently for each of the Polya's urns associated with the vertices of an acyclic graph before the walk begins, thus completely predetermining the trajectory of the walker. Now, the exchangeability property of Polya's urns and de Finetti's theorem mentioned above allows one to restate the edge-reinforced random walk as a random walk in random environment (RWRE), where the environment is distributed as the limiting beta random variables obtained for Polya's urn processes associated with each vertex of the acyclic graph. After that, other techniques such as large deviations are of use in answering the corresponding recurrence/transience questions for the RWRE model (see [12]).

Does the same approach work for the two point process $\eta_t = \{l_t, r_t\}$ on $\mathbb{Z}$? The answer is "no". Consider the case when the drift $\Delta = 0$. Suppose there is an urn at each vertex of $\mathbb{Z}$. Suppose a vertex v is visited by the right particle r_t, and the drawing was done from the urn associated with v and a blue marble was selected, so that the right particle r_t jumps to $v+1$. We *cannot* add $D = 2$ blue marbles into the urn, as it could happen that the left particle l_t arrives to the urn from $(-\infty, v-1]$ before the right particle r_t returns to v from $[v+1, \infty)$. In the latter case, there will be more blue marbles than the weight amount attached to the edge $[v, v+1]$ on the right and the rates will not agree. In other words, the representation with Polya's urns and similar approaches will not work because the two-point linear edge-reinforced process is nonexchangeable. The non-exchangeability of the process was the main obstacle for studying it as well as for proving Theorem 1.

2 The Polya's urn modified

Although the representation with classical Polya's urns fails for the two-point process $\eta_t = \{l_t, r_t\}$, there is a way to modify it. Suppose that at each vertex of $\mathbb{Z}$, the associated urn contains not only the red and blue marbles, but also a special marble, called *magic marble*, such that when the left particle l_t arrives to the site, the magic marble becomes red, while when the right particle r_t arrives to the site, the magic marble becomes blue. Each urn will contain exactly one magic marble in addition to red and blue marbles. When magic marble is selected, two marbles of the color assumed by the magic marble will be added into the urn. Once again the particles move according to the colors of marbles selected from the urns. In other words, if the magic marble is selected when it is red, two more red marbles will be added to the urn and the particle will jump left. Similarly, if the magic marble is selected when it is blue, two more blue marbles will be added to the urn and the particle will jump right.

Let $R_t(v)$ and $B_t(v)$ denote respectively the number of red and the number of blue marbles inside an urn associated with site $v \in \mathbb{Z}$, at time t. The initial number of red and blue marbles, $R_0(v)$ and $B_0(v)$, together with the magic marble must represent the corresponding weights assigned to edges $[v-1, v]$ and $[v, v+1]$ at the time of the first arrival to v by any of the two particles. The left particle is the first to visit the sites to the left of l_0, i.e. all $v < l_0$, and the right particle is first to visit the sites to the right of r_0. Hence, for all $a > 0$, the following must be the initial configuration of red and blue marbles assigned to the urns associated with sites in $\mathbb{Z}$:

$$[R_0(v), B_0(v)] = \begin{cases} [a-1, 1+a+\Delta] \text{ if } v < l_0, \\ [a-1, a+\Delta] \text{ if } v = l_0, \\ [a, a+\Delta] \text{ if } l_0 < v < r_0, \\ [a, a-1+\Delta] \text{ if } v = r_0, \\ [a+1, a-1+\Delta] \text{ if } r_0 < v \end{cases} \tag{2}$$

plus a magic marble in every urn. Here we can allow $a - 1 < 0$ since there is also a magic marble in the urn, which is red when the left particle is at the site, and blue when the right particle is at the site.

We now explain the reason why the magic marble was introduced. First we check that the above urn representation produces correct rates up until the first recurrence time $\tau_1 := min\{t : l_t = r_t\}$. We consider the case when the right particle departs from site v at jump time $t < \tau_1$. Suppose that the next arrival to v happens before τ_1, then there are three possible scenarios.

Case I: the right particle jumps to the left, and returns to v before the left particle arrives. So $r_{t-} = v$, $r_t = v - 1$, $l_t < v$, and we need to add two red marbles into the urn, i.e. $R_t(v) = R_{t-}(v) + 2$. The magic marble stays blue, and as it was the case with one particle ERRW model, the rates agree.

Case II: the right particle jumps to the right, but returns to v before the left particle arrives. That is $r_{t-} = v$, $r_t = v + 1$, $l_t < v$ and we need to add two blue marbles into the urn, i.e. $B_t(v) = B_{t-}(v) + 2$. The rates agree since the right particle returns to v before the next visit to v by the left particle. Again, the magic marble stays blue, and as it was the case with one particle ERRW model, the rates agree.

Case III: the right particle jumps to the right, and the left particle arrives to v before the right particle returns from $[v + 1, +\infty)$. This is the case where the chameleon property of the magic marble is used. Once again $r_{t-} = v$, $r_t = v + 1$, $l_t < v$ and we need to add two blue marbles into the urn, i.e. $B_t(v) = B_{t-}(v) + 2$. Before the departure of the right particle from site v at time t,

$$\text{the weight assigned to } [v-1, v] \text{ was } = R_{t-}(v)$$

and

$$\text{the weight assigned to } [v, v+1] \text{ was } = B_{t-}(v) + 1$$

as the magic marble was blue in the presence of the right particle. When the left particle jumps to v from $v - 1$ at time $t_1 \in (t, \tau_1)$ before the return of the right particle, the magic marble re-colors into red, and

$$\text{the weight assigned to } [v-1, v] \;= R_{t_1}(v) + 1 = R_{t-}(v) + 1$$

and

$$\text{the weight assigned to } [v, v+1] \;= B_{t_1}(v) = B_t(v) = B_{t-}(v) + 2.$$

One can see that the weights are correct since each edge $[v-1, v]$ and $[v, v+1]$ was traversed exactly once.

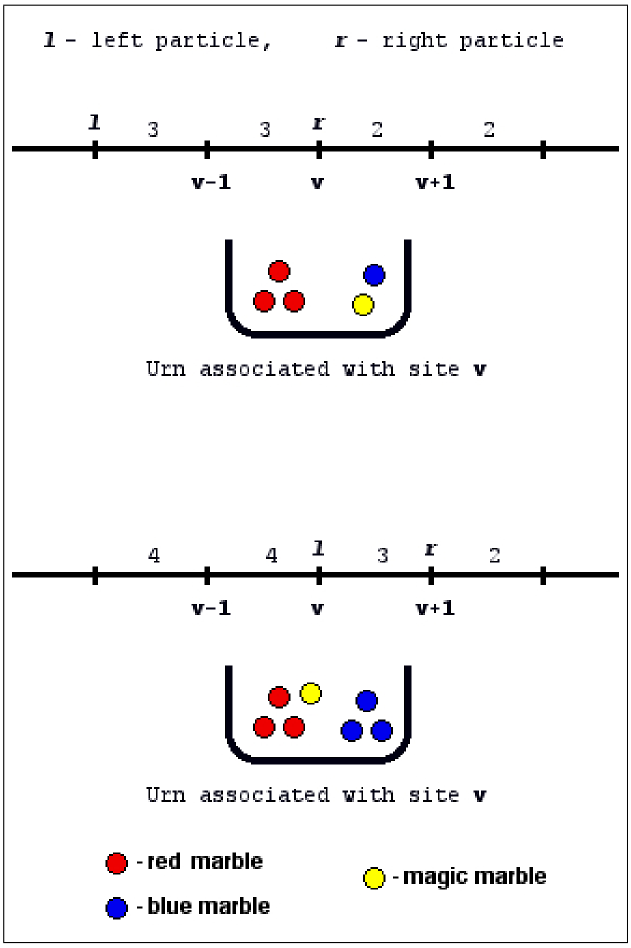

Figure 1: *Above:* $\Delta = 0$*, the right particle is at site* v*,* $W_t(v-1, v) = 3$ *and* $W_t(v, v+1) = 2$*. The urn associated with* v *contains three red, one blue and one magic marble. The magic marble is temporarily colored in blue as the right particle is present. Below: with probability* $\frac{2}{5}$*, either blue or magic marble was selected, the right particle jumps to* $v+1$ *and two blue marbles are added into the urn. The left particle arrives to* v *from the left while the right particle is still at* $v+1$*. The weight that corresponds to* $[v-1, v]$ *is equal to* 4*, while the weight of* $[v, v+1]$ *is equal to* 3*. There are three red, one magic and three blue marbles in the urn. However, the magic marble is temporarily red as the left particle is present at site* v*.*

That explains why adding magic marble works. The case when the left particle is at site v can be checked by the analogy with the case above. See Figure 1 for a visual example.

Observe that the above coupling of the urn process with η_t works **only** up until time $\tau_1 = min\{t : \ l_t = r_t\}$, the first time that the particles meet. Now, we need to show that the particles meet at least once. Therefore before proving Theorem 1, we will need to prove the following one-time recurrence result:

Theorem 2. *For all* $0 \leq \Delta < 1$ *and all* $a > 0$*,* $\tau_1 < \infty$*.*

Since the above urn process is coupled with the above described urn process until the decoupling time τ_1, it suffices to prove Theorem 2 for the urn process. Later it will be shown that the construction will also imply the full recurrence, i.e. Theorem 1.

3 Recurrence via coupling.

The urn construction construction defined in the preceding section determines $\eta_t = (l_t, r_t)$ for $0 \leq t \leq \tau_1$. Here we let $q_t^l(v)$ and $p_t^l(v)$ be respectively the left and the right jump rates for the *left* particle at site v. We also denote by $q_t^r(v)$ and $p_t^r(v)$ respectively the left and the right jump rates for the *right* particle at site v. By construction,

$$q_t^l(v) = \frac{R_t(v)+1}{R_t(v)+B_t(v)+1} \quad \text{and} \quad p_t^l(v) = \frac{B_t(v)}{R_t(v)+B_t(v)+1},$$

and similarly,

$$q_t^r(v) = \frac{R_t(v)}{R_t(v)+B_t(v)+1} \quad \text{and} \quad p_t^r(v) = \frac{B_t(v)+1}{R_t(v)+B_t(v)+1}.$$

We recall that in the case of edge-reinforced random walks on $\mathbb{Z}$, the drawings predetermined the outcome of the whole process. There the results of all drawings from the urns associated with all the vertices of the graph determined uniquely the trajectory of the walker. In the one-particle case one determines the limiting fractions of blue marbles for all sites in the form of respective independent beta random variables. Then one interprets the walk as a birth and death chain with these rates. We want to implement a similar trick for the two-point edge-reinforced process. We will embed the urn process defined in the preceding section into a three-color Polya's urn process.

3.1 Magic family

For each site v and the urn associated with v, we define the *magic family* as all the marbles that were added as the result of selecting the magic marble, plus the magic marble itself. Here is how the magic family is constructed: at the beginning the magic family consists of only the magic marble itself. When the magic marble is selected from the urn for the first time, and two marbles (of one of the two colors) are added to the urn, we include the two into the magic family. Each time a marble from the magic family is selected and two new marbles are added, we let the two marbles into the magic family regardless of their color.

All the red marbles that are not in the magic family will be called **pure** red, and all the blue marbles that are not in the magic family will be called **pure** blue. Observe that for each site v, the urn associated with v is a Polya's urn with respect to three types of marbles: pure red, pure blue and the marbles in the magic family. Each time when a pure red marble is selected, two more pure red marbles are added to the urn. Same is true for pure blue marbles. See Figure 2 for a visual example.

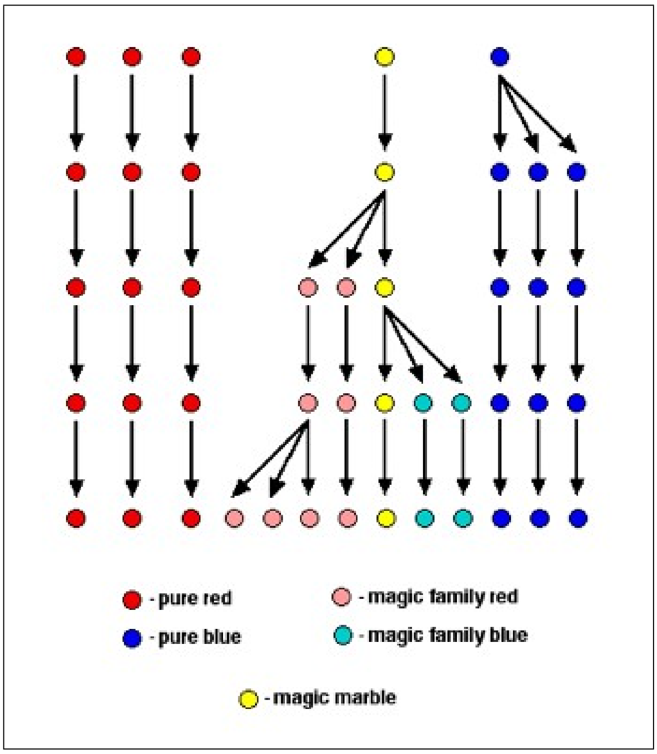

Figure 2: *Suppose $a = 2$, $\Delta = 0$ and $v > r_0$. We begin with three red, one blue and one magic marble inside the urn associated with site v. The right particle was at v at the time of the first drawing. After waiting for the jump time with rate one, a pure blue marble is selected, two more pure blue marbles are added to the urn and the right particle jumps to $v+1$. Next, the left particle arrives from $v-1$, and the magic marble assumes red color. The magic marble is selected in the second drawing, two magic family red marbles are added to the urn and the left particle jumps back to $v-1$. The right particle arrives from the right for the third drawing, the magic marble is selected with probability $\frac{1}{9}$, in which case two magic family blue marbles are added and the right particle jumps to site $v+1$. The left particle arrives from $v-1$, a magic family red marble is selected, two more magic family red marbles are added, and the left particle jumps back to $v-1$.*

For each site v, we let $\bar{R}_n(v)$, $\bar{B}_n(v)$ and $\bar{M}_n(v)$ denote respectively the number of pure red marbles, the number of pure blue marbles and the number of marbles in the magic family inside the urn associated with site v after n drawings. The proportion vector of pure red marbles, marbles in the magic family and pure blue marbles

$$\Big[\bar{R}_n(v), \bar{M}_n(v), \bar{B}_n(v)\Big]/(\bar{R}_n(v)+\bar{M}_n(v)+\bar{B}_n(v))$$

converges to a Dirichlet random vector with parameters $\frac{R_0(v)}{2}$, $\frac{1}{2}$ and $\frac{B_0(v)}{2}$.

We observe that after n drawings, $\bar{M}_n(v)-1$ marbles in the magic family are of either red or blue color. Therefore $\bar{R}_n(v) \le R_n(v)$ and $\bar{B}_n(v) \le B_n(v)$. Let $\mathcal{B}(\alpha,\beta)$ denote the beta distribution with parameters $\alpha > 0$ and $\beta > 0$. If for each v we define $p^l_{Polya}(v)$ as the limiting fraction of pure blue marbles, then $p^l_{Polya}(v)$ will be a beta random variable with parameters $\frac{B_0(v)}{2}$ and $\frac{R_0(v)+1}{2}$. Looking back at (2), one can write down the corresponding $\mathcal{B}\left(\frac{B_0(v)}{2}, \frac{R_0(v)+1}{2}\right)$ distribution of $p^l_{Polya}(v)$ for each $v \in \mathbb{Z}$. For $a \le 1-\Delta$,

$$p^l_{Polya}(v) \text{ is } \begin{cases} \mathcal{B}(\frac{a+1+\Delta}{2}, \frac{a}{2}) \text{ if } v < l_0, \\ \mathcal{B}(\frac{a+\Delta}{2}, \frac{a}{2}) \text{ if } v = l_0, \\ \mathcal{B}(\frac{a+\Delta}{2}, \frac{a+1}{2}) \text{ if } l_0 < v < r_0, \\ 0 \text{ if } r_0 \le v \end{cases}$$

and for $a > 1-\Delta$,

$$p^l_{Polya}(v) \text{ is } \begin{cases} \mathcal{B}(\frac{a+1+\Delta}{2}, \frac{a}{2}) \text{ if } v < l_0, \\ \mathcal{B}(\frac{a+\Delta}{2}, \frac{a}{2}) \text{ if } v = l_0, \\ \mathcal{B}(\frac{a+\Delta}{2}, \frac{a+1}{2}) \text{ if } l_0 < v < r_0, \\ \mathcal{B}(\frac{a-1+\Delta}{2}, \frac{a+1}{2}) \text{ if } v = r_0, \\ \mathcal{B}(\frac{a-1+\Delta}{2}, 1+\frac{a}{2}) \text{ if } r_0 < v. \end{cases}$$

Similarly, $q^r_{Polya}(v)$ defined as the limiting fraction of pure red marbles in the urn will be a beta random variable with parameters $\frac{R_0(v)}{2}$ and $\frac{B_0(v)+1}{2}$. So for $a \le 1$,

$$q^r_{Polya}(v) \text{ is } \begin{cases} 0 \text{ if } v \le l_0, \\ \mathcal{B}(\frac{a}{2}, \frac{a+1+\Delta}{2}) \text{ if } l_0 < v < r_0, \\ \mathcal{B}(\frac{a}{2}, \frac{a+\Delta}{2}) \text{ if } v = r_0, \\ \mathcal{B}(\frac{a+1}{2}, \frac{a+\Delta}{2}) \text{ if } r_0 < v. \end{cases}$$

and for $a > 1$,

$$q^r_{Polya}(v) \text{ is } \begin{cases} \mathcal{B}(\frac{a-1}{2}, 1+\frac{a+\Delta}{2}) \text{ if } v < l_0, \\ \mathcal{B}(\frac{a-1}{2}, \frac{a+1+\Delta}{2}) \text{ if } v = l_0, \\ \mathcal{B}(\frac{a}{2}, \frac{a+1+\Delta}{2}) \text{ if } l_0 < v < r_0, \\ \mathcal{B}(\frac{a}{2}, \frac{a+\Delta}{2}) \text{ if } v = r_0, \\ \mathcal{B}(\frac{a+1}{2}, \frac{a+\Delta}{2}) \text{ if } r_0 < v. \end{cases}$$

Let $q^l_{Polya} := 1 - p^l_{Polya}$ and $p^r_{Polya} := 1 - q^r_{Polya}$. Then the pairs $(q^l_{Polya}(v), p^l_{Polya}(v))_{v\in\mathbb{Z}}$ and $(q^r_{Polya}(v), p^r_{Polya}(v))_{v\in\mathbb{Z}}$ can be viewed as two dependent random environments. We define

l_t^{Polya} as a random walk in the random environment $(q^l_{Polya}(v), p^l_{Polya}(v))_{v\in\mathbb{Z}}$ that starts at $l_0^{Polya} = l_0$ and jumps from v to $v+1$ with rate $p^l_{Polya}(v)$ or to $v-1$ with rate $q^l_{Polya}(v)$. Similarly, we define r_t^{Polya} as a random walk in the random environment $(q^r_{Polya}(v), p^r_{Polya}(v))_{v\in\mathbb{Z}}$ that starts at $r_0^{Polya} = r_0$ and jumps from v to $v+1$ with rate $p^r_{Polya}(v)$ or to $v-1$ with rate $q^r_{Polya}(v)$.

We observe that conditioned on the two dependent environments, the two random walks, l_t^{Polya} and r_t^{Polya}, can coexist as two independent birth and death chains. In the next subsection we will couple $\{l_t^{Polya}, r_t^{Polya}\}$ with $\{l_t, r_t\}$ so that

$$l_t^{Polya} \le l_t \le r_t \le r_t^{Polya} \quad \text{for } 0 \le t \le \tau_1 \ .$$

Then showing the recurrence of $(r_t^{Polya} - l_t^{Polya})$ will prove Theorem 2. Here is the heuristic explanation for the coupling construction to follow. If in the original urn model $\{l_t, r_t\}$, we substitute the magic marble with a red marble in every urn, then the left particle process l_t will be distributed as the random walk in random environment l_t^{Polya}. If in turn we substitute the magic marble with a blue marble in every urn, then the right particle process will have the same distribution as r_t^{Polya}. Observe that this heuristics can be generalized to work in the case of more than two particles, e.g. three-particle linear edge-reinforced processes on $\mathbb{Z}$.

3.2 Coupling with RWRE

We notice that the process $\{l_t, r_t\}$ can be predetermined by the results of all drawings from the Polya's urns with three types of marbles: pure red, pure blue and magic family. We can first do the drawings from the urns associated with all the sites in $\mathbb{Z}$, determining $[\bar{R}_n(v), \bar{M}_n(v), \bar{B}_n(v)]_{n=0,1,2,\dots}$ for all v and the limiting fractions $\{q^r_{Polya}(v), p^l_{Polya}(v)\}_{v\in\mathbb{Z}}$ of pure red and pure blue marbles. By de Finetti's theorem, conditioned on $q^r_{Polya}(v), p^l_{Polya}(v)$, $[\bar{R}_n(v), \bar{M}_n(v), \bar{B}_n(v)]_{n=0,1,2,\dots}$ are determined as independent trials with probabilities $q^r_{Polya}(v)$ and $p^r_{Polya}(v)$ for pure red and pure blue marbles respectively. When one of the two particles visits site v, the results $[\bar{R}_n(v), \bar{M}_n(v), \bar{B}_n(v)]$ of the n-th drawing (if it is the time of n-th departure from the site) determine the destination site and the coloring of the marbles in the magic family.

We recall that in the Polya's urn model the limiting fraction of marbles of one color is a particular beta random variable and the density function $f_{\{q^r_{Polya}(v), p^l_{Polya}(v)\}}(x,y)$ for the pair of limiting fractions $q^r_{Polya}(v)$ and $p^l_{Polya}(v)$ is Dirichlet

$$f_{\{q^r_{Polya}(v), p^l_{Polya}(v)\}}(x,y) = \frac{\Gamma\left(\frac{R_0(v)+B_0(v)+1}{2}\right)}{\Gamma\left(\frac{R_0(v)}{2}\right)\Gamma\left(\frac{B_0(v)}{2}\right)\Gamma\left(\frac{1}{2}\right)} x^{\frac{R_0(v)}{2}-1} y^{\frac{B_0(v)}{2}-1} (1-x-y)^{-\frac{1}{2}}$$

if $x > 0, y > 0$ and $x + y \le 1$.

Now we construct the coupled process $\{l_t^{Polya}, l_t, r_t, r_t^{Polya}\}$:

- We condition on the Dirichlet variables $\{q^r_{Polya}(v), p^l_{Polya}(v)\}_{v\in\mathbb{Z}}$.

- We begin with $l_0^{Polya} = l_0 < r_0 = r_0^{Polya}$.
- The trajectories of l_t and r_t are determined by drawings from the urns.
- l_t^{Polya} and r_t^{Polya} are independent birth and death chains with corresponding probabilities $(q^l_{Polya}(v), p^l_{Polya}(v))_{v\in\mathbb{Z}}$ and $(q^r_{Polya}(v), p^r_{Polya}(v))_{v\in\mathbb{Z}}$ that move independently of each other and of l_t and r_t except for the times when $l_t^{Polya} = l_t$ or $r_t^{Polya} = r_t$.
- When the left particles l and l^{Polya} happen to be at the same site v, they wait for the departure time with rate one. Then the results of the next drawing from the corresponding urn are studied. If the selected marble is red or magic marble, both particles jump to $v-1$. However, if the selected marble is blue, the left particle l jumps to $v+1$, while l^{Polya} jumps to $v+1$ only if the marble is pure blue, and jumps to $v-1$ otherwise. The probability that l^{Polya} jumps to $v+1$ is equal to $p^l_{Polya}(v)$, and each such drawing is independent of all the other drawings. l_t^{Polya} will still be a random walk in random environment $(q^l_{Polya}(v), p^l_{Polya}(v))_{v\in\mathbb{Z}}$, while $l_t^{Polya} \leq l_t$ is preserved.
- Similarly, when the right particles r and r^{Polya} happen to be at the same site v, they wait for the departure time with rate one. Then the results of the next drawing are studied. If the selected marble is a blue or a magic marble, both particles jump to $v+1$. If the marble is red, the right particle r jumps to $v-1$, while r^{Polya} jumps to $v-1$ only if the marble is pure red, and jumps to $v+1$ otherwise. The probability that r^{Polya} jumps to $v-1$ is equal to $q^r_{Polya}(v)$, and each drawing is independent of the others. r_t^{Polya} will still be a random walk in random environment $(q^r_{Polya}(v), p^r_{Polya}(v))_{v\in\mathbb{Z}}$, while $r_t \leq r_t^{Polya}$ is preserved.

In the above coupled process, conditioned on the environments, the independence of l_t^{Polya} and r_t^{Polya} is preserved and

$$l_t^{Polya} \leq l_t \leq r_t \leq r_t^{Polya} .$$

Thus showing the recurrence of $(r_t^{Polya} - l_t^{Polya})$ is enough to prove Theorem 2.

Some theory of RWREs: a RWRE on $\mathbb{Z}$ with the right jump probability $p(v)$ chosen to be $\mathcal{B}(\frac{a+1}{2}, \frac{a+\Delta}{2})$ distributed at all sites v is transient to the right whenever $0 \leq \Delta < 1$, explaining the bound on the drift Δ in Theorem 1. In general, RWRE on $\mathbb{Z}$ is a.s. transient to the right if and only if $E[\log\left(\frac{p(v)}{1-p(v)}\right)] > 0$ (see [14] for the proof, and [13], [9] and references therein for more on the subject). Not surprisingly the RWRE with the environment $\{p(v)\}_{v\in\mathbb{Z}}$

independently $\mathcal{B}(\alpha_1, \alpha_2)$ distributed with $\alpha_1 > \alpha_2$ is a.s. transient to the right:

$$\begin{aligned}
E[\log\Big(\frac{p(v)}{1-p(v)}\Big)] &= \frac{1}{\beta(\alpha_1,\alpha_2)}\int_0^1 \log\Big(\frac{x}{1-x}\Big)x^{\alpha_1-1}(1-x)^{\alpha_2-1}dx \\
&= \frac{4}{\beta(\alpha_1,\alpha_2)}\int_{-\infty}^{\infty} se^{2\alpha_1 s}(1+e^{2s})^{-(\alpha_1+\alpha_2)}ds \\
&= \frac{2^{3-\alpha_1-\alpha_2}}{\beta(\alpha_1,\alpha_2)}\int_0^{\infty} s\sinh((\alpha_1-\alpha_2)s)(\cosh s)^{-(\alpha_1+\alpha_2)}ds \\
&> 0,
\end{aligned}$$

where we substitute $\frac{x}{1-x} = e^{2s}$. In the above general case, $E\Big[\frac{1-p(v)}{p(v)}\Big] = \frac{\beta(\alpha_1-1,\alpha_2+1)}{\beta(\alpha_1,\alpha_2)} = \frac{\alpha_2}{\alpha_1-1}$ if $\alpha_1 > 1$, and $E\Big[\frac{1-p(v)}{p(v)}\Big] = +\infty$ if $\alpha_1 \le 1$.

The expected time of return is a.s. finite only when $E\Big[\frac{1-p(v)}{p(v)}\Big] < 1$ (see [14]). That is only when $\alpha_1 > 1 + \alpha_2$. Now, if one considers a RWRE on $\mathbb{Z}$ with independent right jump probabilities $\{p(v)\}_{v\in\mathbb{Z}}$, each $\mathcal{B}(\frac{a+1}{2}, \frac{a+\Delta}{2})$ distributed, the recurrence time is infinite since $\frac{a+1}{2} < 1 + \frac{a+\Delta}{2}$ for $\Delta > 0$.

The above implies that the expected time of return to site v for r^{Polya} is infinite if $v < r_t^{Polya}$ and is finite if $r_t^{Polya} < v$. Similarly, the expected time of return to site v for l^{Polya} is infinite if $l_t^{Polya} < v$ and is finite if $v < l_t^{Polya}$.

3.3 Proof of Theorem 2.

In this subsection we will prove Theorem 2.

Proof of Theorem 2: Recall that the environments $\{(p^l_{Polya}(v), q^l_{Polya}(v)\}_{v\in\mathbb{Z}}$ and $\{(p^r_{Polya}(v), q^r_{Polya}(v)\}_{v\in\mathbb{Z}}$ of l_t^{Polya} and r_t^{Polya} are dependent, but conditioned on the environments, the walks l_t^{Polya} and r_t^{Polya} are independently. We claim that, even though the expected time of return of $(r_t^{Polya} - l_t^{Polya})$ to zero is infinite, $(r_t^{Polya} - l_t^{Polya})$ is **recurrent**. The problem can be summarized by the following more general lemma.

Lemma 1. *Let $p_1(1), p_1(2), ...$ be i.i.d. random variables defined on $(0,1)$ with*

$$\mu_1 = E\left[\log\left(\frac{1-p_1(i)}{p_1(i)}\right)\right] > 0,$$

e.g. $p_1(i) \sim \mathcal{B}(a_1, b_1)$ for $0 < a_1 < b_1$, and let $p_2(1), p_2(2), ...$ be i.i.d. random variables defined on $(0,1)$ with

$$\mu_2 = E\left[\log\left(\frac{1-p_2(i)}{p_2(i)}\right)\right] > 0,$$

e.g. $p_2(i) \sim \mathcal{B}(a_2, b_2)$ for $0 < a_2 < b_2$. Also let $p_1(0) = p_2(0) = 1$.

If $p_1(0), p_1(1), p_1(2), ...$ are the forward rates for the birth-and-death chain Z_t^r, and $p_2(0), p_2(1), p_2(2), ...$ are the forward rates for the birth-and-death chain Z_t^r, then the two dimensional RWRE $X_t = (Z_t^l, Z_t^r)$ returns to zero infinitely often.

The following is equivalent statement that we can apply in our case: Suppose Z_t^r is a RWRE on $\mathbb{Z}_+$ such that for any $i \geq 0$, Z_t^r jumps from site i to $i+1$ with rate $p_1(i)$ and to $i-1$ with rate $1-p_1(i)$, and suppose Z_t^l is a RWRE on $\mathbb{Z}_-$ such that for any $i \geq 0$, Z_t^l jumps from site $-i$ to $-(i+1)$ with rate $p_2(i)$ and to $-(i-1)$ with rate $1-p_2(i)$. If conditioned on the environments $\{p_1(i)\}_i$ and $\{p_2(j)\}_j$, Z_t^r and Z_t^l are independent birth and death chains, then $Z_t^r - Z_t^l$ is recurrent.

The proof of the above lemma can be thought of as an exercise on use of harmonic functions in stochastic processes. It can be done with Lyapunov functions (see [6]), or alternatively with conductivities. $\square$

3.4 Proof of Theorem 1

It was essential for proving Theorem 2 that $\eta_t = (l_t, r_t)$ was defined via urns and magic marbles for $t \leq \tau_1$. There are many ways to complete the proof of Theorem 1, one is to notice that when the particles separate after the first meeting time τ_1, we can do the whole coupling construction (that lead us to the proof of Theorem 2) anew, starting from scratch. The only thing different will be the initial marble configuration for $\eta_t = (l_t, r_t)$, and the environments of l_t^{Polya} and r_t^{Polya}, but only at finitely many sites, thus establishing the finiteness of the second meeting time τ_2. The finiteness of $\tau_3, \tau_4, \ldots$ follows by induction. $\square$

This proof followed from the unusual coupling construction with the magic marbles and the domination by two RWREs, one on the right and one on the left. As we already mentioned, the above domination can be constructed without using the magic marble approach, and generalized to work for more particles than two. However the approach taken in this paper allows us better understand the dynamics behind the two-point processes, and is valuable as an innovative coupling technique.

Acknowledgment

The author wishes to thank all members of UCLA probability group for their support in the three years that the author worked there as a postdoc, and Silke Rolles who worked at UCLA at that time for useful discussions. While working on this version of the paper, the author received a lot of encouragement from members of the probability group at Oregon State University. The author wishes to thank Robert Burton and Mina Ossiander for sharing thoughts on nonexchangeable processes.

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