# The Brownian Bridge Asymptotics in the Subcritical Phase of Bernoulli Bond Percolation Model.

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**Abstract**

For a given point $\vec{\mathbf{a}}$ in $\mathbb{Z}^d$, we prove that a cluster in the $d$-dimensional subcritical Bernoulli bond percolation model conditioned on connecting points $(0, ..., 0)$ and $n\vec{\mathbf{a}}$ if scaled by $\frac{1}{n\|\vec{\mathbf{a}}\|}$ along $\vec{\mathbf{a}}$ and by $\frac{1}{\sqrt{n}}$ in the orthogonal directions converges asymptotically to Time $\times$ $(d-1)$-dimensional Brownian bridge. [1]

## 1 Introduction.

In this paper we describe the limiting structure of a bond percolation cluster in subcritical phase, conditioned on reaching a faraway point on square lattice. We will start with a brief description of the Bernoulli bond percolation model based on the material rigorously presented in [9] and [11]. We will also recall the definition of Brownian bridge. The result of this research establishes a link between the geometrical behavior of a large percolation cluster in subcritical phase and that of Brownian bridge, and will be presented in light of recent developments in the field.

**Percolation:** For each edge of the $d$-dimensional square lattice $\mathbb{Z}^d$ in turn, we declare the edge *open* with probability $p$ and *closed* with probability $1-p$, independently of all other edges. If we delete the closed edges, we are left with a random subgraph of $\mathbb{Z}^d$. A connected component of the subgraph is called a "cluster", and the number of edges in a cluster is the "size" of the cluster. The probability $\theta(p)$ that the point $(0, 0)$ belongs to a cluster of an infinite size is zero if $p = 0$, and one if $p = 1$. However, there exists a critical probability $0 < p_c < 1$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$. In the first case, we say that we are in the *subcritical* phase of Bernoulli bond percolation model and in the second case we say that we are in the *supercritical* phase of Bernoulli bond percolation model. We say that two points in $\mathbb{Z}^d$ are connected to each other whenever they belong to the same cluster.

**Brownian bridge:** Given that $B_t$ is the d-dimensional Brownian motion, a sample-continuous Gaussian process $B_t^o \equiv B_t - tB_1$ $(0 \leq t \leq 1)$ is called the Brownian bridge. We observe that $\mathbf{E}[B_{i,s}^o B_{j,t}^o] = \delta_{i,j} s(1-t)$ for all $0 \leq s \leq t \leq 1$ and all $1 \leq i, j \leq d$, where $B_t^o = (B_{1,t}^o, B_{2,t}^o, ..., B_{d,t}^o)$ and $\delta_{i,j}$ is the Kronecker coefficient. In fact the d-dimensional Brownian

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bridge can be also defined as a sample-continuous Gaussian process $B_t^o = (B_{1,t}^o, B_{2,t}^o, ..., B_{d,t}^o)$ on $[0,1]$ with mean zero and covariance $\mathbf{E}[B_{i,s}^o B_{j,t}^o] = \delta_{i,j} s(1-t)$ for all $0 \le s \le t \le 1$ and all $1 \le i,j \le d$. Notice that, given either of the two equivalent definitions, $B_t^o$ is a Gaussian process such that $B_0^o = B_1^o = 0$ a.s. For more details see [2], [7] or [8]. The process $B_t^{o,\vec{\mathbf{a}}} \equiv B_t^o + t\vec{\mathbf{a}}$ is the Brownian bridge that connects points zero and $\vec{\mathbf{a}}$.

We will introduce the reader to the problem: Consider a point $\vec{\mathbf{a}}$ in $\mathbb{Z}^d$ and the $d$-dimensional model of subcritical bond percolation ($p < p_c$) conditioned on the event of zero being connected to $n\vec{\mathbf{a}}$. We first show that a specifically chosen path connecting points zero and $n\vec{\mathbf{a}}$ and going through some appropriately defined points on the cluster (regeneration points), if scaled $\frac{1}{n\|\vec{\mathbf{a}}\|}$ times along $\vec{\mathbf{a}}$ and $\frac{1}{\sqrt{n}}$ times in the direction orthogonal to $\vec{\mathbf{a}}$, converges to Time $\times$ $(d-1)$-dimensional Brownian bridge as $n \to +\infty$, where the scaled interval connecting points zero and $n\vec{\mathbf{a}}$ serves as a $[0,1]$ time interval. In other words, we prove that a scaled "skeleton" going through the regeneration points of the cluster converges to Time $\times$ $(d-1)$-dimensional Brownian bridge. In a subsequent step, we show that after scaling the hitting area of the orthogonal hyper-planes shrinks, implying that for $n$ large enough, all the points of the scaled cluster are within an $\varepsilon$-neighborhood of the points in the "skeleton". One of the major tools used in this research was the renewal technique developed in [1], [4], [5], [6] and [10] as part of the derivation of the Ornstein-Zernike estimate for the subcritical bond percolation model and other processes. A major result related to the study is that for $\vec{\mathbf{a}} = (1,0,...,0)$, the hitting distribution of the cluster in the intermediate planes, $x_1 = tn\vec{\mathbf{a}}$, $0 < t < 1$ obeys a multidimensional local limit theorem (see [4]). Dealing with all non-axis $\vec{\mathbf{a}}$ became possible only after the corresponding technique analyzing the so called Wulff shape and further mastering the regeneration structures and equi-decay profiles was developed in [5] and [10]. This technique played a central role in obtaining the research results of the paper.

**Invariance principle.** The result of section 3 (see subsection 3.2) that we cite below is to play an important role in proving the Brownian bridge asymptotics for subcritical bond percolation in section 2. It can be also interpreted on its own: establishing a Brownian bridge limiting behavior for a scaled "directed" random walk, conditioned on arriving to a faraway point $n\vec{\mathbf{a}}$, where by a directed random walk we mean a random walk in which the steps $\{\zeta_i\}$ are i.i.d. and the probability $P[\zeta_i \cdot \vec{\mathbf{a}} > 0] = 1$.

We let $X_1, X_2, ...$ to be i.i.d. random variables (vectors) on $\mathbb{Z}^d$ with the span of the lattice distribution equal to one (see [8], section 2.5), and let there be a $\lambda > 0$ such that the moment-generating function

$$\mathbf{E}[e^{\theta \cdot X_1}] < \infty$$

for all $\|\theta\| < \lambda$. Now, suppose there is a vector $\vec{\mathbf{a}} \in \mathbb{Z}^d$ such that $P[\vec{\mathbf{a}} \cdot X_i] > 0] = 1$. Let $\vec{g}_1, ..., \vec{g}_d$ denote the new orthonormal basis such that $\vec{g}_1 \| \vec{\mathbf{a}}$, and lets write $[\cdot,\cdot]_g \in \mathbb{R} \times \mathbb{R}^{d-1}$ for the first and the last $d-1$ coordinate of a point in the new basis. Then we can define $t_i \in \mathbb{R}$ and $Y_i \in \mathbb{R}^{d-1}$ to be such that $X_1 + ... + X_i = [t_i, Y_i]_g \in \mathbb{Z}^d$ when written in the new orthonormal basis. Observe that $\vec{\mathbf{a}} = [\|\vec{\mathbf{a}}\|, 0]_g$ .

Now, let $\mathbf{M}$ and $\mathbf{M}_a$ denote respectively the covariance matrix of $X_1$ and the ($d$-1)-dimensional linear transformation corresponding to the last $d$-1 coordinates of $\mathbf{M}$ in the new basis (the covariance matrix of the last $d$-1 coordinates of $X_1$). Then there is a ($d$-1)-dimensional linear transformation $\mathbf{A}_a$ such that $\mathbf{M}_a = \mathbf{A}_a \mathbf{A}_a^T$ (see [3], p.384). We will also denote by $\mu_a$ the projection of $\mathbf{E}[X_1]$ on $< \vec{\mathbf{a}} >$, e.g. $\mu_a = (\mathbf{E}[X_1] \cdot \vec{g}_1)\vec{g}_1$.

We define the process $[t, Y^*_{n,k}(t)]_g$ to be the interpolation of 0 and $[\frac{1}{n\|\vec{\mathbf{a}}\|} t_i, \frac{1}{\sqrt{n}} Y_i]_g^{i=0,1,\ldots,k}$, in subsection 3.2 we will show that

**Technical Theorem.** *The process*

$$\{Y^*_{n,k} \text{ for some } k \text{ such that } [t_k, Y_k]_g = n\vec{\mathbf{a}}\}$$

*conditioned on the existence of such $k$ converges weakly to $\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}} \mathbf{A}_a B^o$, where $B^o = \{B^o_t\}$ is the (d-1)-dimensional Brownian bridge.*

# 2 The Main Result in Subcritical Percolation.

In this section we will work only with subcritical percolation probabilities $p < p_c$.

## 2.1 Preliminaries

Here we briefly go over the definitions that one can find in Section 4 of [5].
We start with the inverse correlation length $\xi_p(\vec{x})$:

$$\xi_p(\vec{x}) \equiv - \lim_{n\to\infty} \frac{1}{n} \log P_p(0 \leftrightarrow [n\vec{x}]),$$

where the limit is always defined due to the FKG property of the Bernoulli bond percolation (see [9]). Now, $\xi_p(\vec{x})$ is the support function of the compact convex set

$$\mathbf{K}^p \equiv \bigcap_{\vec{n}\in\mathbb{S}^{d-1}} \{\vec{r} \in \mathbb{R}^d : \vec{r} \cdot \vec{n} \leq \xi_p(\vec{n})\},$$

with non-empty interior int$\{\mathbf{K}^p\}$ containing point zero.
Let $\vec{\mathbf{r}} \in \partial\mathbf{K}^p$, and let $\vec{e}$ be a basis vector such that $\vec{e} \cdot \vec{\mathbf{r}}$ is maximal. For $\vec{x}, \vec{y} \in \mathbb{Z}^d$ define

$$S^r_{\vec{x},\vec{y}} \equiv \{\vec{z} \in \mathbb{R}^d | \vec{\mathbf{r}} \cdot \vec{x} \leq \vec{\mathbf{r}} \cdot \vec{z} \leq \vec{\mathbf{r}} \cdot \vec{y}\}.$$

Note that $S^r_{\vec{x},\vec{y}} = \emptyset$ if $\vec{\mathbf{r}} \cdot \vec{y} < \vec{\mathbf{r}} \cdot \vec{x}$.
Let $\mathbf{C}^r_{\vec{x},\vec{y}}$ denote the corresponding common open cluster of $x$ and $y$ when we run the percolation process on $S^r_{\vec{x},\vec{y}} \bigcap \mathbb{Z}^d$. Let also $\Delta_r$ be the set of all basis vectors orthogonal to $\vec{\mathbf{r}}$, and their negatives. For the simplicity of notations (to avoid writing $(1-p)^{|\Delta_r|}$ coefficient) in the future, we restrict ourself to the case when vector $\vec{\mathbf{r}}$ has all non-zero coefficients (e.g. $|\Delta_r| = 0$).

**Definition 1.** For $\vec{x}, \vec{y} \in \mathbb{Z}^d$ lets define $h_r$-connectivity $\{\vec{x} \overset{h_r}{\longleftrightarrow} \vec{y}\}$ of $\vec{x}$ and $\vec{y}$ to be the event that
1. $\vec{x}$ and $\vec{y}$ are connected in the restriction of the percolation configuration to the slab $S^r_{\vec{x},\vec{y}}$.
2. If $\vec{x} \neq \vec{y}$, then $\mathbf{C}^r_{\vec{x},\vec{y}} \bigcap S^r_{\vec{x},\vec{x}+\vec{e}} = \{\vec{x}, \vec{x}+\vec{e}\}$ and $\mathbf{C}^r_{\vec{x},\vec{y}} \bigcap S^r_{\vec{y}-\vec{e},\vec{y}} = \{\vec{y}-\vec{e}, \vec{y}\}$.

Set

$$h_r(\vec{x}) \equiv P_p[0 \longleftarrow^{h_r}\longrightarrow \vec{x}]$$

and $h_r(0) = 1$.

**Definition 2.** For $\vec{x}, \vec{y} \in \mathbb{Z}^d$ lets define $f_r$-connectivity $\{\vec{x} \longleftarrow^{f_r}\longrightarrow \vec{y}\}$ of $\vec{x}$ and $\vec{y}$ to be the event that
1. $\vec{x} \neq \vec{y}$
2. $\vec{x} \longleftarrow^{h_r}\longrightarrow \vec{y}$ .
3. For no $\vec{z} \in \mathbb{Z}^d \setminus \{\vec{x}, \vec{y}\}$ both $\{\vec{x} \longleftarrow^{h_r}\longrightarrow \vec{z}\}$ and $\{\vec{z} \longleftarrow^{h_r}\longrightarrow \vec{y}\}$ take place.

Set

$$f_r(\vec{x}) \equiv P_p[0 \longleftarrow^{f_r}\longrightarrow \vec{x}]$$

and $f_r(0) = 0$.

**Definition 3.** Suppose $0 \longleftarrow^{h_r}\longrightarrow \vec{x}$, we say that $\vec{z} \in \mathbb{Z}^d$ is **a regeneration point** of $\mathbf{C}^r_{0,\vec{x}}$ if
1. $\vec{\mathbf{r}} \cdot \vec{e} \leq \vec{\mathbf{r}} \cdot \vec{z} \leq \vec{\mathbf{r}} \cdot (\vec{x} - \vec{e})$
2. $S^r_{\vec{z}-\vec{e},\vec{z}+\vec{e}} \bigcap \mathbf{C}^r_{0,\vec{x}}$ contains exactly three points: $\vec{z} - \vec{e}$, $\vec{z}$ and $\vec{z} + \vec{e}$, where $\vec{e}$ is defined as before.

The following is the Ornstein-Zernike equality in subcritical phase of Bernoulli bond percolation.

**Theorem.** $\exists$ *positive functions* $A(\cdot, \cdot)$ *on* $(0, p_c) \times \mathbf{S}^{d-1}$ *s. t.*

$$P_p[0 \leftrightarrow \vec{x}] = \frac{A(p, n(\vec{x}))}{\|\vec{x}\|^{\frac{d-1}{2}}} e^{-\xi_p(\vec{x})}(1 + o(1)) \tag{1}$$

*uniformly in* $\vec{x} \in \mathbb{Z}^d$*, where* $n(\vec{x}) \equiv \frac{\vec{x}}{\|\vec{x}\|}$.

We refer the reader to [5] for the proof of the above theorem.

## 2.2 Probability measure $Q^r_{r_0}(x)$

It had been proved in section 4 of [5] that for a given $\vec{\mathbf{r}}_0 \in \partial \mathbf{K}^p$ there exists $\bar{\lambda} > 0$ such that

$$F_{r_0}(\vec{\mathbf{r}}) = \sum_{x \in \mathbb{Z}^d} f_{\vec{\mathbf{r}}_0}(x) e^{\vec{\mathbf{r}} \cdot \vec{x}} = 1 \text{ whenever } \vec{\mathbf{r}} \in \mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0) \bigcap \partial \mathbf{K}^p,$$

where $\mathcal{B}_{\bar{\lambda}}(\cdot)$ denotes the Euclidean ball of radius $\bar{\lambda}$ around the given point in parenthesis, and therefore

$$Q^r_{r_0}(\vec{x}) \equiv f_{r_0}(\vec{x}) e^{\vec{\mathbf{r}} \cdot \vec{x}} \text{ is a probability measure on } \mathbb{Z}^d.$$

Also, it was shown that

$$\mu = \mu_{r_0}(\vec{\mathbf{r}}) \equiv \mathbf{E}^r_{r_0} X = \sum_{\vec{x} \in \mathbb{Z}^d} \vec{x} Q^r_{r_0}(\vec{x}) = \nabla_r log F_{r_0}(\vec{\mathbf{r}}) \neq 0$$

and

$$F_{r_0}(\vec{\mathbf{r}}) < \infty \text{ for all } \vec{\mathbf{r}} \text{ in } \mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0).$$

The later implies

$$F_{r_0}(\vec{\mathbf{r}}) = \sum_{\vec{x}\in\mathbb{Z}^d} f_{r_0}(\vec{x}) e^{\vec{\mathbf{r}}\cdot\vec{x}} = \sum_{\vec{x}\in\mathbb{Z}^d} Q_{r_0}^{r_0}(\vec{x}) e^{\theta\cdot\vec{x}} < \infty$$

for $\theta = \vec{\mathbf{r}} - \vec{\mathbf{r}}_0 \in \mathcal{B}_{\bar{\lambda}}(0)$, e.g. the moment generating function $\mathbf{E}_{r_0}^{r_0}(e^{\theta\cdot X_1})$ of the law $Q_{r_0}^{r_0}$ is finite for all $\theta \in \mathcal{B}_{\bar{\lambda}}(0)$.

Now, there is a renewal relation (see section 1 and section 4 of [5]),

$$h_{r_0}(\vec{x}) = \sum_{\vec{z}\in\mathbb{Z}^d} f_{r_0}(\vec{z}) h_{r_0}(\vec{x}-\vec{z}) \text{ with } h_{r_0}(0) = 1$$

and therefore

$$h_{r_0}([N\mu]) = e^{-r\cdot[N\mu]} \sum_k \bigotimes_1^k Q_{r_0}^r(X_1 + ... + X_k = [N\mu]) \text{ for } N > 0,$$

where $X_1, X_2, ...$ is a sequence of i.i.d. random variables distributed according to $Q_{r_0}^r$, as $h_{r_0}$-connection is a chain of $f_{r_0}$-connections with junctions at the regeneration points of $\mathbf{C}_{0,x}^{r_0}$.

## 2.3 Important Observation

We would like the reader to notice a certain relationship between the notions of the regeneration points and that of $f_{r_0}$-connectivity as they were defined in section 2.1. That is for a given vector $\vec{x} \in \mathbb{Z}^d$, the event of

- $\{0 \overset{h_{r_0}}{\longleftrightarrow} \vec{x}$ with exactly one regeneration point $\vec{x}_1$ $\}$

is equivalent to the union of two independent events:

- $\{\ 0 \overset{f_{r_0}}{\longleftrightarrow} \vec{x}_1\ \}$,
- $\{\ \vec{x}_1 \overset{f_{r_0}}{\longleftrightarrow} \vec{x}\ \}$.

Thus the probability of the above event is equal to

$$f_{r_0}(\vec{x}_1) f_{r_0}(\vec{x} - \vec{x}_1).$$

More generally, the probability $P_X$ that $0 \overset{h_{r_0}}{\longleftrightarrow} \vec{x}$ with exactly $k$-1 regeneration points $\vec{x}_1, \vec{x}_1 + \vec{x}_2, ..., \sum_{i=1}^{k-1} \vec{x}_i$ (where $\sum_{i=1}^{k} \vec{x}_i = \vec{x}$) can be factored as following

$$\begin{aligned} P_X &\equiv P[0 \overset{h_{r_0}}{\longleftrightarrow} \vec{x}\ ;\ \text{regeneration points: } \vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^{k-1}\vec{x}_i] \\ &= P[0 \overset{f_{r_0}}{\longleftrightarrow} \vec{x}_1] P[\vec{x}_1 \overset{f_{r_0}}{\longleftrightarrow} \vec{x}_1 + \vec{x}_2] ... P[\sum_{i=1}^{k-1}\vec{x}_i \overset{f_{r_0}}{\longleftrightarrow} \sum_{i=1}^{k}\vec{x}_i = \vec{x}] \\ &= f_{r_0}(\vec{x}_1) f_{r_0}(\vec{x}_2) ... f_{r_0}(\vec{x}_k). \end{aligned} \quad (2)$$

## 2.4 The Result

In this section we fix $\vec{\mathbf{a}} \in \mathbb{Z}^d$, we let $\vec{\mathbf{r}}_0 = \vec{\mathbf{a}}\mathbb{R}^+ \bigcap \partial\mathbf{K}^p$ and $\vec{\mathbf{r}} \in \mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0) \bigcap \partial\mathbf{K}^p$ (say $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0$). Then we recall that

$$\mathbf{E}^r_{r_0}[e^{\theta \cdot X_1}] < \infty$$

for all $\|\theta\| < \bar{\lambda}(0)$. We also denote $h(x) \equiv h_{r_0}(x)$ and $f(x) \equiv f_{r_0}(x)$.
First, we introduce the new orthonormal basis $\{\vec{g_1}, \vec{g_2}, ..., \vec{g_d}\}$, where $\vec{g_1} = \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|}$. We use $[\cdot,\cdot]_g \in \mathbb{R} \times \mathbb{R}^{d-1}$ to denote the coordinates of a vector with respect to the new basis. Obviously $\vec{\mathbf{a}} = [\|\vec{\mathbf{a}}\|, 0]_g$. We want to prove that the process corresponding to the last $d-1$ coordinates in the new basis of the scaled ($\frac{1}{n\|\vec{\mathbf{a}}\|}$ times along $\vec{\mathbf{a}}$ and $\frac{1}{\sqrt{n}}$ times in the orthogonal d-1 dimensions) interpolation of regeneration points of $\mathbf{C}^{r_0}_{0,n\vec{\mathbf{a}}}$ conditioned on $0 \overset{h}{\longleftrightarrow} n\vec{\mathbf{a}}$ converges weakly to a linear transformation of the ($d$-1)-dimensional Brownian bridge $B^o(t)$, where $t$ represents the scaled first coordinate in the new basis.

Let $\mathbf{M}_{Q,r_0,r}$ denote the covariance matrix of a random variable distributed according to $Q^r_{r_0}$ and let $\bar{\mathbf{M}}_{r_0,r}$ denote the ($d$-1)-dimensional linear transformation corresponding to the last $d$-1 coordinates of $\mathbf{M}_{Q,r_0,r}$ in the new basis (the covariance matrix of the last $d-1$ coordinates of a random variable distributed according to $Q^r_{r_0}$). We also recall from [5] that the covariance matrix $\mathbf{M}_{Q,r_0,r} = \mathbf{Hess}\big(\log F_{r_0}(\vec{\mathbf{r}})\big)$ is uniformly non-degenerate for $\vec{\mathbf{r}} \in \mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0) \bigcap \partial\mathbf{K}^p$. There is a ($d$-1)-dimensional linear transformation $\mathbf{A}_{r_0,r}$ such that $\bar{\mathbf{M}}_{r_0,r} = \mathbf{A}_{r_0,r}\mathbf{A}^T_{r_0,r}$. We recall that $\mu_{r_0}(\vec{\mathbf{r}})$ denotes the mean of a $Q^r_{r_0}$-distributed random variable and we let $\mu_a$ denote the projection of $\mu_{r_0}(\vec{\mathbf{r}})$ on $< \vec{\mathbf{a}} >$, e.g. $\mu_a = (\mu_{r_0}(\vec{\mathbf{r}}) \cdot \vec{g}_1)\vec{g}_1$.

Let $X_1, X_2, ...$ be i.i.d. random variables distributed according to $Q^r_{r_0}$ law. We interpolate $0, X_1, (X_1 + X_2), ..., (X_1 + ... + X_k)$ and scale by $\frac{1}{n\|\vec{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}}$ along $< \vec{\mathbf{a}} > \times < \vec{\mathbf{a}} >^{\perp}$ to get the process $[t, Y^*_{n,k}(t)]_g$. The technical theorem (see section 1 or 3.2) implies the following

**Theorem 1.** *The process*

$$\{Y^*_{n,k} \textit{ for some } k \textit{ such that } X_1 + ... + X_k = n\vec{\mathbf{a}}\}$$

*conditioned on the existence of such $k$ converges weakly to $\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\mathbf{A}_{r_0,r}B^o$, where $B^o = \{B^o_t\}$ is the (d-1)-dimensional Brownian bridge.*

Now, let for $\vec{y}_1, ..., \vec{y}_k \in \mathbb{Z}^d$ with positive increasing first coordinates $\gamma(\vec{y}_1, ..., \vec{y}_k)$ be the last $(d-1)$ coordinates in the new basis of the scaled ($\frac{1}{n\|\vec{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}}$) interpolation of points $0, \vec{y}_1, ..., \vec{y}_k$ (where the first coordinate is time). To be precise we write $\vec{y}_i = [y_{i,x}, y_{i,y}]_g$, where $y_{i,x} \in \mathbb{R}$ and $y_{i,y} \in \mathbb{R}^{d-1}$. If we also let $[y_{0,x}, y_{0,y}]_g = 0$, then $y_{0,x} < y_{1,x} < y_{2,x} < ... < y_{k,x}$. Now we can explicitly define $\gamma(\vec{y}_1, ..., \vec{y}_k)$ as

$$\gamma(\vec{y}_1, ..., \vec{y}_k)[t] = \frac{1}{\sqrt{n}} y_{i,y} + \frac{1}{\sqrt{n}}\Big(\frac{n\|\vec{\mathbf{a}}\|t - y_{i,x}}{y_{i+1,x} - y_{i,x}}\Big)(y_{i+1,y} - y_{i,y})$$

whenever $\frac{y_{i,x}}{n\|\vec{\mathbf{a}}\|} \leq t \leq \frac{y_{i+1,x}}{n\|\vec{\mathbf{a}}\|}$ $(i = 0, 1, ..., k-1)$. Notice that $\gamma(\vec{y}_1, ..., \vec{y}_k) \in C_o[0,1]^{d-1}$ as a function of scaled first coordinate whenever $\vec{y}_k = n\vec{\mathbf{a}}$. By the important observation (2) that

we have made before, for any function $\Psi(\cdot)$ on $C[0,1]^{d-1}$,
$\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} \Psi(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i))$

$$\times P[0 \longleftarrow^{h_{r_0}} \longrightarrow n\vec{\mathbf{a}} \text{ ; regeneration points: } \vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^{k-1} \vec{x}_i]$$

$$= \sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} \Psi(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) f(\vec{x}_1)...f(\vec{x}_k)$$

$$= e^{-r\cdot n\vec{\mathbf{a}}} \sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} \Psi(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) Q^r_{r_0}(\vec{x}_1)...Q^r_{r_0}(\vec{x}_k).$$

Therefore, for any $E \subset C[0,1]^{d-1}$ and corresponding indicator function $I_E(\cdot)$ on $C[0,1]^{d-1}$,

$P_p[\gamma(\text{regeneration points}, n\vec{\mathbf{a}}) \in E \mid 0 \longleftarrow^{h} \longrightarrow n\vec{\mathbf{a}}]$

$$= \frac{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} I_E(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) f(\vec{x}_1)...f(\vec{x}_k)}{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} f(\vec{x}_1)...f(\vec{x}_k)}$$

$$= \frac{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} I_E(\gamma(\vec{x}_1, \vec{x}_1+\vec{x}_2, ..., \sum_{i=1}^k \vec{x}_i)) Q^r_{r_0}(\vec{x}_1)...Q^r_{r_0}(\vec{x}_k)}{\sum_k \sum_{\vec{x}_1+...+\vec{x}_k=n\vec{\mathbf{a}}} Q^r_{r_0}(\vec{x}_1)...Q^r_{r_0}(\vec{x}_k)}$$

$= P[Y^*_{n,k} \in E$ for the $k$ such that $X_1+...+X_k = n\vec{\mathbf{a}} \mid \exists k$ such that $X_1+...+X_k = n\vec{\mathbf{a}}]$.

Hence, we have proved the following

**Corollary.** *The process corresponding to the last* $d-1$ *coordinates (in the new basis* $\{\vec{g}_1, ..., \vec{g}_d\}$*) of the scaled (*$\frac{1}{n\|\vec{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}}$*) interpolation of regeneration points of* $\mathbf{C}^{r_0}_{0,n\vec{\mathbf{a}}}$ *(where the first coordinate is time) conditioned on* $0 \longleftarrow^{h} \longrightarrow n\vec{\mathbf{a}}$ *converges weakly to* $\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\mathbf{A}_{r_0,r}B^o$*, where* $B^o = \{B^o_t\}$ *is the (d-1)-dimensional Brownian bridge.*

Observe that if $\vec{\mathbf{a}} \| \vec{e}_1$, then we do not need to change the basis, e.g. $\vec{\mathbf{a}}$ lies on the first axis and we let $\vec{g}_i = \vec{e}_i$ for all $i = 1, 2, ..., d$. We also let $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0$. In this case, if $\vec{X} = (\mathcal{X}_1, ..., \mathcal{X}_d)$ is a random vector distributed according to $Q^r_{r_0}$, then, due to the lattice symmetry,

$$Cov(\mathcal{X}_i, \mathcal{X}_j) = \delta_{i,j}\sigma^2_{\mathcal{X}} \text{ whenever } 2 \le i, j \le d,$$

where $\delta_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$ and $\sigma^2_{\mathcal{X}} = Var(\mathcal{X}_i)$ for all $i = 2, ..., d$. Hence $\mathbf{A}_{r_0,r} = \sigma_{\mathcal{X}} I$ and

$$\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\mathbf{A}_{r_0,r}B^o = \sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\sigma_{\mathcal{X}}B^o.$$

Thus the enhanced version of the above corollary, stated at the end of this section, will once again confirm the multidimensional local limit theorem proved in [4].

## 2.5 Shrinking of the Cluster and Main Theorem

Here for $\vec{\mathbf{a}} \in \mathbb{Z}^d$ we let $\mathbf{r}_0 = \vec{\mathbf{a}}\mathbb{R}^+ \bigcap \partial\mathbf{K}^p$ again. Before we proceed with the proof that the scaled percolation cluster $\mathbf{C}^{r_0}_{0,n\vec{\mathbf{a}}}$ shrinks to the scaled interpolation skeleton of regeneration points, we need to observe the following easy consequence of $\xi_p(\cdot)$ being the support function of $\mathbf{K}^p$ and strict convexity of $\partial\mathbf{K}^p$ (see [5]): if $\vec{\mathbf{r}} = \nabla\xi_p(\vec{\mathbf{r}}_0)$ then $Q^r_{r_0}$ is a probability measure.

With the help of the above statement we shall show that the consequent regeneration points are situated relatively close to each other:

**Lemma.**

$$P_p[\max_i |x_i - x_{i-1}| > n^{1/3},\ x_i\text{- } reg.\ points \mid 0 \overset{h}{\longleftrightarrow} n\vec{\mathbf{a}}] < \frac{1}{n}$$

*for $n$ large enough.*

*Proof.* Let $\vec{\mathbf{r}} \equiv \nabla\xi_p(\vec{\mathbf{r}}_0) = \nabla\xi_p(\vec{\mathbf{a}})$. Since $\xi_p(x)$ is strictly convex (see section 4 in [5]),

$$\frac{\xi_p(\vec{\mathbf{a}}) - \xi_p(\vec{\mathbf{a}} - \frac{\vec{x}}{n})}{(\frac{\|\vec{x}\|}{n})} < \frac{\vec{x}}{\|\vec{x}\|} \cdot \nabla\xi_p(\vec{\mathbf{a}})$$

for $\vec{x} \in \mathbb{Z}^d$ $(\vec{x} \neq 0)$, and therefore

$$\xi_p(n\vec{\mathbf{a}}) - \xi_p(n\vec{\mathbf{a}} - \vec{x}) = \|\vec{x}\| \frac{\xi_p(\vec{\mathbf{a}}) - \xi_p(\vec{\mathbf{a}} - \frac{\vec{x}}{n})}{(\frac{\|\vec{x}\|}{n})} < \vec{x} \cdot \nabla\xi_p(\vec{\mathbf{a}}) = \vec{\mathbf{r}} \cdot \vec{x}.$$

Thus, since $Q^r_{r_0}(x)$ decays exponentially and therefore

$$e^{\xi_p(n\vec{\mathbf{a}}) - \xi_p(n\vec{\mathbf{a}} - x)} < Q^r_{r_0}(x)$$

and also decays exponentially. Hence by Ornstein-Zernike result (1),

$$P_p[n^{1/3} < |x|,\ x\text{-first reg. point } |0 \overset{h}{\longleftrightarrow} n\vec{\mathbf{a}}] = \sum_{n^{1/3} < |x|} f(x) \frac{h(n\vec{\mathbf{a}} - x)}{h(n\vec{\mathbf{a}})} < \frac{1}{n^2}$$

for $n$ large enough. So, since the number of the regeneration points is no greater than $n$,

$$P_p[\max_i |x_i - x_{i-1}| > n^{1/3},\ x_i\text{- reg. points} \mid 0 \overset{h}{\longleftrightarrow} n\vec{\mathbf{a}}] < \frac{1}{n}$$

for $n$ large enough.

□

Now, it is really easy to check that there is a constant $\lambda_g > 0$ such that

$$f(\vec{x}) > e^{-\lambda_g \|\vec{x}\|}$$

for all $\vec{x}$ such that $f(\vec{x}) \neq 0$ (here we only need to connect points $\vec{e}$ and $\vec{x} - \vec{e}$ with two non-intersecting open paths surrounded by the closed edges), and there exists a constant $\lambda_u > 0$ such that

$$P_p[\text{ percolation cluster } \mathbf{C}(0) \not\subset [\mathbb{R}; \mathcal{B}^{d-1}_R(0)]_g] < e^{-\lambda_u R}$$

for $R$ large enough due to the exponential decay of the radius distribution for subcritical probabilities (see [9]). Here $\mathcal{B}_R^{d-1}$ denotes the ($d$-1)-dimensional Euclidean ball of radius $R$. Hence, for a given $\epsilon > 0$

$$P_p[\text{ cluster } \mathbf{C}_{0,\vec{x}}^{r_0} \not\subset [\mathbb{R}, \mathcal{B}_{\epsilon\sqrt{n}}^{d-1}(0)]_g \mid 0 \longleftarrow^{f} \longrightarrow x] < e^{\lambda_g \|\vec{x}\| - \lambda_u \epsilon \sqrt{n}},$$

and therefore, summing over the regeneration points, we get

$$P_p[\text{ scaled cluster } \mathbf{C}_{0,n\vec{\mathbf{a}}}^{r_0} \not\subset \epsilon\text{-neighbd. of } [0,1] \times \gamma(\text{ reg. points }) \mid 0 \longleftarrow^{g} \longrightarrow n\vec{\mathbf{a}}]$$

$$< \frac{1}{n} + n e^{\lambda_g n^{1/3} - \lambda_u \epsilon \sqrt{n}}$$

for $n$ large enough.

We can now state the main result of this paper:

**Main Theorem.** *The process corresponding to the last $d-1$ coordinates (in the new basis $\{\vec{g}_1, \vec{g}_2, ..., \vec{g}_d\}$) of the scaled ($\frac{1}{n\|\vec{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}}$) interpolation of regeneration points of $\mathbf{C}_{0,n\vec{\mathbf{a}}}^{r_0}$ (where the first coordinate is time) conditioned on $0 \longleftarrow^{h} \longrightarrow n\vec{\mathbf{a}}$ converges weakly to $\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}} \mathbf{A}_{r_0,r} B^o$, where $B^o = \{B_t^o\}$ is the (d-1)-dimensional Brownian bridge.*

*Also for a given $\epsilon > 0$*

$$P_p[\text{ scaled cluster } \mathbf{C}_{0,n\vec{\mathbf{a}}}^{r_0} \not\subset \epsilon\text{-neighbd. of } [0,1] \times \gamma(\text{ reg. points, } n\vec{\mathbf{a}}) \mid 0 \longleftarrow^{h} \longrightarrow n\vec{\mathbf{a}}] \to 0$$

*as $n \to \infty$.*

## 2.6 From $S_{0,n\vec{\mathbf{a}}}^{r} \cap \mathbb{Z}^d$ to all of $\mathbb{Z}^d$.

Here we are going to transform the preceding theorem that deals only with the restriction $\mathbf{C}_{0,n\vec{\mathbf{a}}}^{r_0}$ of the percolation cluster to the strip $S_{0,n\vec{\mathbf{a}}}^{r_0}$ to the statement about the entire percolation cluster $\mathbf{C}_{0,n\vec{\mathbf{a}}}$, where $\mathbf{C}_{\vec{x},\vec{y}}$ denotes the cluster connecting points $\vec{x}$ and $\vec{y}$. For this we will need to employ the techniques originally developed in [6] (pages 228-229) , [4], [5] (pages 347-349) and [10] (pages 674-675).

**Definition 4.** For $\vec{x}, \vec{y} \in \mathbb{Z}^d$ and $\vec{\mathbf{r}} \in \partial\mathbf{K}^p$, we say that $x$ and $y$ are $d_r$-connected and write $\{\vec{x} \longleftarrow^{d_r} \longrightarrow \vec{y}\}$ whenever the following two conditions hold:
1. $\vec{x}$ and $\vec{y}$ are connected (both points belong to the common percolation cluster $\mathbf{C}_{\vec{x},\vec{y}}$).
2. There is no $\vec{z} \in S_{\vec{x}+\vec{e},\vec{y}-\vec{e}}^{r}$ such that $\mathbf{C}_{\vec{x},\vec{y}} \bigcap S_{\vec{z}-\vec{e},\vec{z}+\vec{e}}^{r} = \{\vec{z}-\vec{e}, \vec{z}, \vec{z}+\vec{e}\}$, where, as before, $\vec{e}$ is the basis vector that maximizes $\vec{\mathbf{r}} \cdot \vec{e}$.

The points $\vec{z}$ from part 2 of the above definition are once again called **the regeneration points** in the direction of $\vec{\mathbf{r}}$. Only this time we need the endpoints to be connected on the entire lattice, without restricting it to $S_{0,n\vec{\mathbf{a}}}^{r}$. The $d_r$-connectivity defined as above is the connectivity with no regeneration points in the given direction. We set $d_r(\vec{x}) \equiv P_p[0 \longleftarrow^{d_r} \longrightarrow \vec{x}]$ and $d_r(0) = 0$.

Now, we define the connectivity with exactly one regeneration point:

**Definition 5.** For $\vec{x}, \vec{y} \in \mathbb{Z}^d$ and $\mathbf{\vec{r}} \in \partial \mathbf{K}^p$, we say that $x$ and $y$ are $u_r$-connected and write $\{\vec{x} \leftarrow^{u_r} \rightarrow \vec{y}\}$ whenever the following three conditions hold:
1. $\vec{x}$ and $\vec{y}$ are connected (both points belong to the common percolation cluster $\mathbf{C}_{\vec{x},\vec{y}}$).
2. If $\vec{x} \neq \vec{y}$, then $\mathbf{C}_{\vec{x},\vec{y}} \bigcap S^r_{\vec{y}-\vec{e},\vec{y}} = \{\vec{y}-\vec{e}, \vec{y}\}$, where $\vec{e}$ is the basis vector that maximizes $\mathbf{\vec{r}} \cdot \vec{e}$.
3. There is **exactly one** point $\vec{z} \in S^r_{\vec{x}+\vec{e},\vec{y}-\vec{e}}$ such that $\mathbf{C}_{\vec{x},\vec{y}} \bigcap S^r_{\vec{z}-\vec{e},\vec{z}+\vec{e}} = \{\vec{z}-\vec{e}, \vec{z}, \vec{z}+\vec{e}\}$

We set $u_r(\vec{x}) \equiv P_p[0 \leftarrow^{u_r} \rightarrow \vec{x}]$ and $u_r(0) = 0$.

We observe that for a given direction $\mathbf{\vec{r}}$, the rightmost and the leftmost regeneration points connecting zero to $\vec{x}$ are $h_r$-connected, while if glued together, the remaining parts of the cluster compile into a $u_r$-connected component:

$$P[0 \leftrightarrow \vec{x}] = d_r(\vec{x}) + \sum_{\vec{x}_1+\vec{x}_2=\vec{x}} u_r(\vec{x}_1) h_r(\vec{x}_2). \tag{3}$$

According to the construction in [5], for a fixed $\mathbf{\vec{r}} \in \partial \mathbf{K}^p$, there exist constants $\nu_1 > 0$ and $\nu_2 > 0$ such that

$$d_r(\vec{x}) \leq e^{-\nu_1 \|\vec{x}\|} P_p[0 \leftrightarrow \vec{x}]$$

and

$$u_r(\vec{x}) \leq e^{-\nu_2 \|\vec{x}\|} P_p[0 \leftrightarrow \vec{x}].$$

Thus, for a given $\mathbf{\vec{a}} \in \mathbb{Z}^d$, $\mathbf{\vec{r}}_0 = \mathbf{\vec{a}}\mathbb{R}^+ \bigcap \partial \mathbf{K}^p$ and $n$ large enough, the probability $P_p[0 \leftarrow^{d} \rightarrow n\mathbf{\vec{a}} \mid 0 \leftrightarrow n\mathbf{\vec{a}}]$ is negligibly small. Also, if we denote by $\vec{x}_g$ and $\vec{x}_L$ the first and the last regeneration points whenever $0 \leftrightarrow n\mathbf{\vec{a}}$, then there exist $\nu_3 \in (0, \nu_2)$ such that

$$\begin{aligned} P_p[\|n\mathbf{\vec{a}} - \vec{x}_L + \vec{x}_g\| \geq n^{\frac{1}{3}} \mid 0 \leftrightarrow n\mathbf{\vec{a}}] &= \frac{1}{P_p[0 \leftrightarrow n\mathbf{\vec{a}}]} \sum_{\vec{x} \in \mathbb{Z}^d : \|\vec{x}\| \geq n^{\frac{1}{3}}} u(\vec{x}) h(n\mathbf{\vec{a}} - \vec{x}) \\ &\leq \frac{e^{-\nu_3 n^{\frac{1}{3}}}}{P_p[0 \leftrightarrow n\mathbf{\vec{a}}]} \sum_{\vec{x} \in \mathbb{Z}^d : \|\vec{x}\| \geq n^{\frac{1}{3}}} h(\vec{x}) h(n\mathbf{\vec{a}} - \vec{x}) \\ &\rightarrow 0 \end{aligned} \tag{4}$$

as

$$\begin{aligned} \frac{1}{h(n\mathbf{\vec{a}})} \sum_{\vec{x} \in \mathbb{Z}^d} h(\vec{x}) h(n\mathbf{\vec{a}} - \vec{x}) &= \sum_{\vec{x} \in \mathbb{Z}^d} P_p[\vec{x} \text{ is a regeneration point} \mid 0 \leftrightarrow n\mathbf{\vec{a}}] \\ &= E[\#\text{regeneration points} \mid 0 \leftrightarrow n\mathbf{\vec{a}}] \\ &< n\|\mathbf{\vec{a}}\|. \end{aligned}$$

The above formulas (3) and (4), together with the Main theorem of the previous subsection imply the more general form of the result valid for the percolation process conditioned on connecting zero to $n\mathbf{\vec{a}}$ on the entire lattice without restrictions. Here the direction $\mathbf{\vec{r}}_0$, the new basis and the interpolation skeleton $\gamma$ of the regeneration points and boundary points are defined in exactly the same way as in the preceding subsection of this manuscript.

**Main Theorem on $\mathbb{Z}^d$.** *The process corresponding to the last $d-1$ coordinates (in the new basis $\{\vec{g}_1, \vec{g}_2, ..., \vec{g}_d\}$) of the scaled $(\frac{1}{n\|\vec{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}})$ interpolation of regeneration points of $\mathbf{C}_{0,n\vec{\mathbf{a}}}$ (where the first coordinate is time) conditioned on $0 \leftrightarrow n\vec{\mathbf{a}}$ converges weakly to $\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\mathbf{A}_{r_0,r}B^o$, where $B^o = \{B^o_t\}$ is the (d-1)-dimensional Brownian bridge.*

*Also for a given $\epsilon > 0$*

$$P_p[\text{ scaled cluster } \mathbf{C}_{0,n\vec{\mathbf{a}}} \not\subset \epsilon\text{-neighbd. of } [0,1] \times \gamma(\text{ reg. points, } n\vec{\mathbf{a}}) \mid 0 \leftrightarrow n\vec{\mathbf{a}}] \to 0$$

*as $n \to \infty$.*

The last part of the above theorem states that the rescaled percolation cluster shrinks to the interpolation skeleton of regeneration and boundary points. The proof of it is almost identical to that presented in subsection 2.5. Once again $\vec{\mathbf{r}}$ is a point in $\mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0) \bigcap \partial\mathbf{K}^p$ The covariance matrices $\mathbf{M}_{Q,r_0,r}$ and $\bar{\mathbf{M}}_{r_0,r}$ as well as $\mathbf{A}_{r_0,r}$ and the projection of the mean $\mu_a$ are defined as before.

# 3 Convergence to Brownian Bridge.

As it was mentioned in the introduction, this section is entirely dedicated to proving the Technical theorem. Recall that we already used the theorem to prove the main result of the preceding section.

Various forms of conditional limit theorems and conditional invariance principles constitute a traditional subject in the probability theory, and were thoroughly studied by Liggett, Durrett, Resnick and others. The conditional invariance principle that we will cite in the first subsection of this section is based upon the results proved in [13], [14] and [15] where weak convergence of conditioned sums of independent random variables was studied in full generality for many different cases. We will use one of the results from [15] to prove the Technical theorem. Notice that the Technical theorem of this section is a specific version of the conditional invariance principle, useful in its own way. The reader is referred to [2] for more on weak convergence, and to [12] for more on weak convergence in this particular case.

## 3.1 An invariance principle.

We let $Z_1, Z_2, ...$ be i.i.d. random variables (vectors) on lattice $\mathbb{L}$ ( e.g. $P[Z_1 \in \mathbb{L}] = 1$) with the span of the $d$-dimensional lattice distribution equal to one (see [8], section 2.5) with finite mean $\mu = \mathbf{E}Z_1$ and covariance matrix $\mathbf{M}_Z$. Also we require the origin to be a point in the interior of the closed convex hull of $\{z : P[Z_1 = z] > 0\}$.

Consider a $d$-dimensional walk $X_j$ that starts with $X_0 = 0$ and for a given $X_j$, the (j+1)-st step to be $X_{j+1} = X_j + Z_{j+1}$. After interpolation we get

$$X(t) = X_{[t]} + (t - [t])(X_{[t]+1} - X_{[t]})$$

for $0 \leq t < \infty$. Now, define $X_n(t) \equiv \frac{X(nt)}{\sqrt{n}}$ for $0 \leq t \leq 1$. Observe that $X_n(t) \in C^d_0[0,1]$, where $C^d_0[0,1]$ is the space of all $d$-dimensional continuous functions on $[0,1]$ with the corresponding topology defined by the distance function $\rho(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$ for any two functions $f$ and $g$ in $C^d_0[0,1]$.

Since $\mathbf{M}_Z$ is symmetric nonnegative definite, there is a linear transformation $\mathbf{A}_Z : \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathbf{M}_Z = \mathbf{A}_Z \mathbf{A}_Z^T$. One of the implications of [15] (p. 202) is

**Theorem 2.** *$X_n(t)$ conditioned on $X_n(1) = 0$ converges weakly to $\mathbf{A}_Z B_d^o$, where $B_d^o$ is the $d$-dimensional Brownian Bridge.*

and for $\mathbf{a} \in \mathbb{R}^d$,

**Theorem 3.** *If $\mathbf{a}(n) \in \mathbb{L}$ is such that $\frac{\mathbf{a}(n)}{\sqrt{n}} = \mathbf{a} + o(1)$, then the process $\{X_n(t) - tX_n(1)\}$, conditioned on $X_n = \mathbf{a}(n)$, converges weakly to $\mathbf{A}_Z B_d^o$.*

Also the convergence is **uniform** in the following sense: here and in the next section, when we say that convergence is *uniform*, what we really mean is that if we fix a compact set $L \in \mathbb{R}^d$, then for a given Borel set $E \in C_0^d[0,1]$ and any small $\epsilon > 0$, there is an integer $N_L > 0$ such that for all $n > N_L$

$$\left| P[\frac{1}{\sigma}\{X_n(t) - tX_n(1)\} \in E \;|X_n \in L] - P[\mathbf{A}_Z B_d^o \in E] \right| < \epsilon. \tag{5}$$

Observe that the local CLT in this case will imply that

$$\left| P[\frac{1}{\sigma}\{X_n(t) - tX_n(1)\} \in E \;|X_n = \mathbf{a}(n)] - P[\mathbf{A}_Z B_d^o \in E] \right| < \epsilon.$$

for all $n$ large enough and all $\mathbf{a}(n)$ such that $\frac{\mathbf{a}(n)}{\sqrt{n}} \in L$ .

## 3.2 General Case.

We begin with the settings needed for the invariance principle that will be proved in this subsection. For a given non-zero vector $\vec{\mathbf{a}} \in \mathbb{Z}^d$, we let $X_1, X_2, \ldots$ be i.i.d. random variables (vectors) in $\mathbb{Z}^d$ with the span of the lattice distribution equal to one (see [8]) such that the probability $P[\vec{\mathbf{a}} \cdot X_1 > 0] = 1$ with the finite mean $\mu = \mathbf{E}X_1 \in \mathbb{Z}^d$ and there is a constant $\lambda > 0$ such that the moment-generating function

$$\mathbf{E}[e^{\theta \cdot X_1}] < \infty$$

for all $\theta$ inside the Euclidean ball of radius $\lambda$ around zero. Also we let $\mathbf{P}_{\vec{a}}$ denote the projection map on $< \vec{\mathbf{a}} >$ and $\mathbf{P}_{\vec{a}}^{\perp}$ denote the orthogonal projection on $< \vec{\mathbf{a}} >^{\perp}$. Now we can decompose the mean $\mu = \mu_a + \mu_{or}$, where $\mu_a \equiv \mathbf{P}_{\vec{a}}\mu$ and $\mu_{or} \equiv \mathbf{P}_{\vec{a}}^{\perp}\mu$.

Now, borrowing the notations from the introduction section, we let $\mathbf{M}$ and $\mathbf{M}_a$ denote respectively the covariance matrix of $X_1$ and the ($d$-1)-dimensional linear transformation corresponding to the last $d$-1 coordinates of $\mathbf{M}$ in the new basis (the covariance matrix of the last $d$-1 coordinates of $X_1$). Then, as we already mentioned, there is a $d$-dimensional linear transformation $\mathbf{A}$ such that $\mathbf{M} = \mathbf{A}\mathbf{A}^T$ (see [3], p.384). Similarly there is a ($d$-1)-dimensional linear transformation $\mathbf{A}_a$ such that $\mathbf{M}_a = \mathbf{A}_a \mathbf{A}_a^T$.

As we did in the preceding sections of this manuscript, we introduce the new orthonormal basis $\{\vec{g_1}, \vec{g_2}, \ldots, \vec{g_d}\}$, where $\vec{g_1} = \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|}$. We again use $[\cdot,\cdot]_g \in \mathbb{R} \times \mathbb{R}^{d-1}$ to denote the coordinates of a vector with respect to the new basis. We let $X_1+...+X_i=[t_i,Y_i]_g \in \mathbb{Z}\times\mathbb{Z}^{d-1}$. Note: $t_i$ and $Y_i$ don't have to be independent. Since the first coordinate $T_i$ is positive with probability one, the first step will be to interpolate $[t_i,Y_i]_g$, and prove that if scaled and conditioned on $[t_n,Y_n]_g = X_1+...+X_n = [n\|\vec{\mathbf{a}}\|,0]_g = n\vec{\mathbf{a}}$, the new process will converge weakly to the Brownian bridge (with the first coordinate being the time axis).

We first let $\bar{X}_i \equiv X_i-\mu_a$, then $\mathbf{E}\bar{X}_i=\mu_{or}$ and $\mathbf{M}$ is the covariance matrix for each $\bar{X}_i$. From here on we let $S_j \equiv [t_j,Y_j]_g = X_1+...+X_j$ and $\bar{S}_j \equiv \bar{X}_1+...+\bar{X}_j = S_j - j\mu_a$ for any positive integer $j$. We again interpolate:

$$\bar{X}(t) = \bar{S}_{[t]} + (t-[t])\bar{X}_{[t]+1}$$

for $0\le t\le\infty$, and scale down by introducing $\bar{X}_k(t)\equiv \frac{\bar{X}(kt)}{\sqrt{k}}$. The statement below follows immediately from the last theorem. $[\cdot]$ will denote the integer rounding for points in $\mathbb{R}^d$.

**Corollary.** *For $k=k(n)=[\frac{n\|\vec{\mathbf{a}}\|}{\|\mu_a\|}+k_0\sqrt{n}]$, $\{\bar{X}_k(t)-t\bar{X}_k(1)\}$ conditioned on $\bar{S}_k = n\vec{\mathbf{a}}-k\mu_a$ (e.g.$[t_k,Y_k]_g=n\vec{\mathbf{a}}$) converges weakly to $\{\mathbf{A}B^o_d\}$, where $B^o_d$ is the d-dimensional Brownian bridge.*

Here we boldly used the theorem of the preceding subsection: namely observe that

$$\bar{S}_k = n\vec{\mathbf{a}}-k\mu_a = -k_0\sqrt{n}\mu_a + O(1).$$

If we let $\alpha(k)=\frac{n\vec{\mathbf{a}}-k\mu_a}{\sqrt{k}}$, then the process $\{\bar{X}_k(t)-t\bar{X}_k(1)\}$ is being conditioned on $\bar{X}_k(1)\equiv \frac{\bar{S}_k}{\sqrt{k}}=\alpha(k)$, where $\alpha(k)=-k_0\sqrt{\frac{\|\mu_a\|}{\|\vec{\mathbf{a}}\|}}\mu_a+o(1)$.

Notice that we didn't use the fact that the moment generating function is finite on some interval around zero. We only needed the first two moments of $X_1$ to be finite. One can also notice that since $\alpha(k)=-k_0\sqrt{\frac{\|\mu_a\|}{\|\vec{\mathbf{a}}\|}}\mu_a+o(1)$, the convergence must be uniform (in sense of (5) ) for all $k_0$ in a compact set . Now, if we consider only the last $d-1$ coordinates of $\bar{X}_k(t)$, w.r.t. the new basis the last Corollary will imply the following:

**Lemma 1.** *For $k=k(n)=[\frac{n\|\vec{\mathbf{a}}\|}{\|\mu_a\|}+k_0\sqrt{n}]$, $Y_k(t)$ conditioned on $t_k=n\|\vec{\mathbf{a}}\|$ and $Y_k=0$ converges weakly to $\{\mathbf{A}_aB^o\}$, where $B^o$ is the $(d-1)$-dimensional Brownian bridge.*

Notice that the convergence is again "uniform" for $k_0$ in a compact set. The Lemma above above can be interpreted in the following way: the interpolation of $[\frac{i}{k},\frac{1}{\sqrt{k}}Y_i]_g$ conditioned on $[t_k,Y_k]_g=n\vec{\mathbf{a}}$ converges to Time$\times\{\mathbf{A}_aB^o\}$, where $B^o$ for the rest of the section will denote the $(d-1)$-dimensional Brownian bridge. Now, if we define the process $[t,Y^*_{n,k}(t)]_g$ to be the interpolation of $[\frac{1}{n\|\vec{\mathbf{a}}\|}t_i,\frac{1}{\sqrt{n}}Y_i]_g^{i=0,1,...,k}$, then

**Theorem 4.** *For $k=k(n)=[\frac{n\|\vec{\mathbf{a}}\|}{\|\mu_a\|}+k_0\sqrt{n}]$, $\sqrt{\frac{n}{k}}Y^*_{n,k}(t)$ conditioned on $t_k=n\|\vec{\mathbf{a}}\|$ and $Y_k=0$ converges weakly to $\{\mathbf{A}_aB^o\}$.*

*Proof:* Here we observe that the mean $\mathbf{E}[\frac{t_i}{n\|\vec{\mathbf{a}}\|}-\frac{t_{i-1}}{n\|\vec{\mathbf{a}}\|}]$ is actually equal to $\frac{\|\mu_a\|}{n\|\vec{\mathbf{a}}\|}=\frac{1}{k-k_0\sqrt{n}}+o(\frac{1}{n})$, and that for a given $\epsilon>0$, the probability of the $\|[\frac{1}{n\|\vec{\mathbf{a}}\|}t_i,\frac{1}{\sqrt{n}}Y_i]_g-[\frac{i}{k},\frac{1}{\sqrt{k}}Y_i]_g\| = |\frac{t_j}{n\|\vec{\mathbf{a}}\|}-\frac{j}{k}|$ exceeding $\epsilon$ for some $j\le k$,

$$P[\max_{0\le j\le k}|t_j - \frac{n\|\vec{\mathbf{a}}\|}{k}j| \ge n\epsilon \mid S_n = n\vec{\mathbf{a}}] \le P[\max_{0\le j\le k}\|S_j - \frac{n\|\vec{\mathbf{a}}\|j}{k}\mu_a\| \ge n\epsilon \mid S_k = n\vec{\mathbf{a}}]$$
$$\le P[\max_{0\le j\le k}|\bar{S}_j| \ge n\frac{\epsilon}{2} \mid \bar{S}_k = [n\|\vec{\mathbf{a}}\| - k\|\mu_a\|, 0]_g]$$
$$\to 0$$

as $n \to +\infty$ since $n\|\vec{\mathbf{a}}\| - k\|\mu_a\| = -\|\mu_a\|k_0\sqrt{n} + o(\sqrt{n})$. □

Now, the next step is to prove that the process

$$\{Y^*_{n,k} \text{ for some } k \text{ such that } [t_k, Y_k]_g = n\vec{\mathbf{a}}\}$$

conditioned on the existence of such $k$ converges weakly to the $\{\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\mathbf{A}_a B^o\}$

First we notice that the last theorem easily implies the following

**Lemma 2.** *For given* $k = k(n) = [\frac{n\|\vec{\mathbf{a}}\|}{\|\mu_a\|} + k_0\sqrt{n}]$, $Y^*_{n,k}(t)$ *conditioned on* $t_k = n\|\vec{\mathbf{a}}\|$ *and* $Y_k = 0$ *converges weakly to* $\{\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\mathbf{A}_a B^o\}$

For a fixed $M > 0$, convergence in the lemma above is also uniform for $k \in [\frac{n\|\vec{\mathbf{a}}\|}{\|\mu_a\|} - M\sqrt{n}, \frac{n\|\vec{\mathbf{a}}\|}{\|\mu_a\|} + M\sqrt{n}]$ (e.g. $k_0 \in [-M, M]$). For the future purposes we denote $\kappa \equiv \frac{\|\mu_a\|}{\|\vec{\mathbf{a}}\|}$ and $I_M \equiv [\frac{n}{\kappa} - M\sqrt{n}, \frac{n}{\kappa} + M\sqrt{n}] \bigcap \mathbb{Z}$.

Finally, we want to prove the following technical result, in which we use the truncation argument as $M \to +\infty$ to show the weak convergence of $Y^*_{n,k}$ to $\{\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\mathbf{A}_a B^o\}$ in case when we condition only on the existence of such $k$.

**Technical Theorem.** *The process*

$$\{Y^*_{n,k} \textit{ for some } k \textit{ such that } [t_k, Y_k]_g = n\vec{\mathbf{a}}\}$$

*conditioned on the existence of such* $k$ *converges weakly to* $\{\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\mathbf{A}_a B^o\}$

*Proof:* Take $M$ large. We notice that due to the uniformity of convergence for all $k = k(n) \in I_M$ established following the previous theorem, for given $E \subset C^{d-1}[0,1]$,

$$\max_{k\in I_M}|P[Y^*_k \in E \mid [t_k, Y_k]_g = n\vec{\mathbf{a}}] - P[\{\frac{1}{\sqrt{\kappa}}\mathbf{A}_a B^o\} \in E]| = o(1).$$

Hence,

$$lim_{n\to+\infty}\frac{\sum_{k\in I_M} P[S_k = n\vec{\mathbf{a}}]P[Y^*_{n,k} \in E|S_k = n\vec{\mathbf{a}}]}{\sum_{k\in I_M} P[S_k = n\vec{\mathbf{a}}]} = P[\{\frac{1}{\sqrt{\kappa}}\mathbf{A}_a B^o\} \in E].$$

Therefore we are only left to prove that truncation works as $M \to +\infty$. Now, for any $\epsilon > 0$ there exists $M > 0$ such that

$$(1+\epsilon)\sum_{k\in I_M} P[S_k = n\vec{\mathbf{a}}] \le \sum_k P[S_k = n\vec{\mathbf{a}}] \le (1+2\epsilon)\sum_{k\in I_M} P[S_k = n\vec{\mathbf{a}}]$$

for $n$ large enough, as by the large deviation upper bound, there is a constant $\bar{C}_{LD} > 0$ such that

$$P[S_k = n\vec{\mathbf{a}}] \le e^{-\bar{C}_{LD}\frac{(n-k\kappa)^2}{k} \wedge |n-k\kappa|},$$

and therefore $\exists C_{LD} > 0$ such that

$$\sum_{|n-k\kappa|>n^{2/3}} P[S_k = n\vec{\mathbf{a}}] < e^{-C_{LD}n^{1/3}}.$$

Also, by the local CLT,

$$P[S_k = n\vec{\mathbf{a}}] = P[\bar{S}_k = (n-k\kappa)\vec{\mathbf{a}}] = k^{-\frac{d}{2}}\big((2\pi)^d \det \mathbf{M}\big)^{-\frac{1}{2}} e^{-\frac{(n-k\kappa)^2}{2k}\vec{\mathbf{a}}^T\mathbf{M}^{-1}\vec{\mathbf{a}}} + o(\frac{1}{k^{d/2}})$$

implying (if one writes the corresponding Riemann sum) that

$$\sum_{|n-k\kappa|\le n^{2/3}} P[S_k = n\vec{\mathbf{a}}] = \left(\frac{\kappa}{n}\right)^{\frac{d-1}{2}} [\big((2\pi)^d \det \mathbf{M}\big)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}\vec{\mathbf{a}}^T\mathbf{M}^{-1}\vec{\mathbf{a}}}dx + o(1)]$$

where

$$\sum_{k\in I_M} P[S_k = n\vec{\mathbf{a}}] = \left(\frac{\kappa}{n}\right)^{\frac{d-1}{2}} [\big((2\pi)^d \det \mathbf{M}\big)^{-\frac{1}{2}} \int_{-M\sqrt{\kappa}}^{M\sqrt{\kappa}} e^{-\frac{x^2}{2}\vec{\mathbf{a}}^T\mathbf{M}^{-1}\vec{\mathbf{a}}}dx + o(1)].$$

Therefore

$$\frac{1}{1+2\epsilon}\frac{\sum_{k\in I_M} P[S_k = n\vec{\mathbf{a}}]P[Y^*_{n,k}\in E|S_k = n\vec{\mathbf{a}}]}{\sum_{k\in I_M} P[S_k = n\vec{\mathbf{a}}]} \le \frac{\sum_k P[S_k = n\vec{\mathbf{a}}]P[Y^*_{n,k}\in E|S_k = n\vec{\mathbf{a}}]}{\sum_k P[S_k = n\vec{\mathbf{a}}]}$$

$$\le \frac{1}{1+\epsilon}\frac{\sum_{k\in I_M} P[S_k = n\vec{\mathbf{a}}]P[Y^*_{n,k}\in E|S_k = n\vec{\mathbf{a}}]}{\sum_{k\in I_M} P[S_k = n\vec{\mathbf{a}}]}$$

for all open $E \subset C^{d-1}[0,1]$. Taking the lim inf and lim sup of the fraction in the middle completes the proof.

□

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