

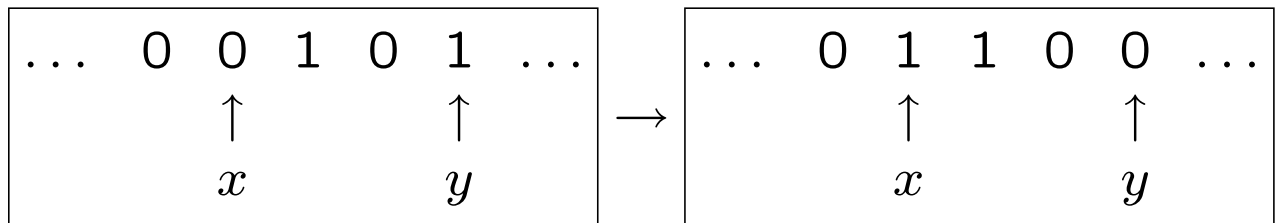
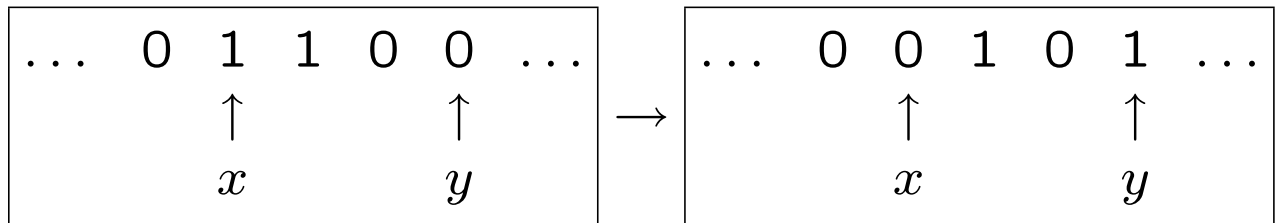
Generalized Symmetric Exclusion Processes

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October 22, 2005

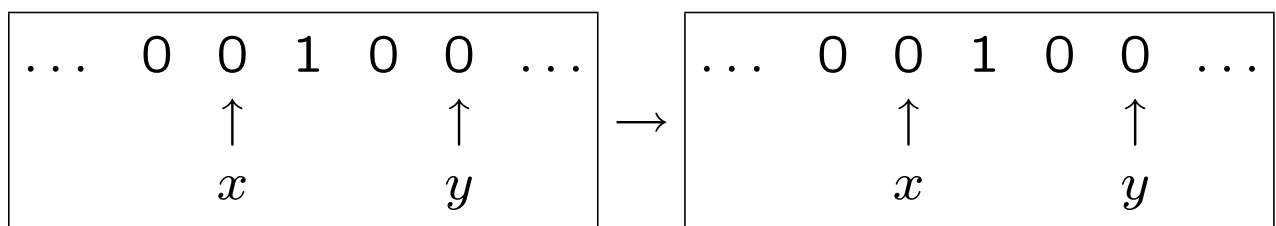
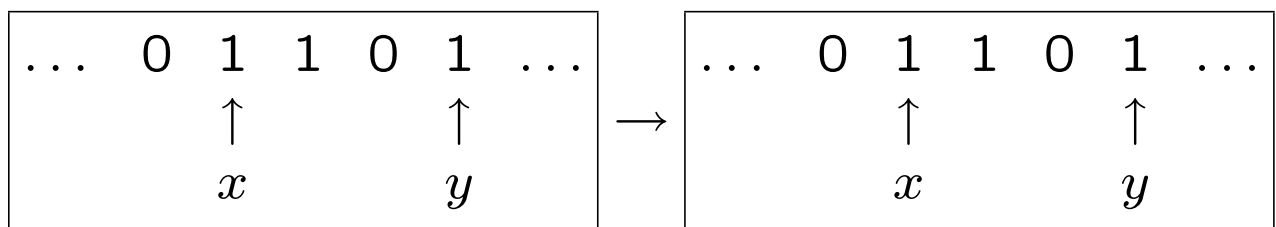
Symmetric Exclusion Processes

$$p(x, y) = p(y, x)$$



each time with rate $p(x, y)$.

But also, we let



with rate $p(x, y)$, if no particle is tagged.

For each transposition $\tau_{x,y}$ ($x, y \in S, x \neq y$) we assign rate

$$q(\tau_{x,y}) = p(y, x) = p(x, y)$$

at which the transposition occurs:

$$\eta \rightarrow \tau_{x,y}(\eta) = \eta_{x,y} \quad \text{at rate} \quad q(\tau_{x,y}).$$

Natural generalization of symmetric exclusion suggested by T.Liggett: We can assign the rates not to the particles inhabiting the space S , but to various permutations of finitely many points of S .

For each $\eta \in \{0, 1\}^S$, let $\sigma(\eta)$ be the new configuration of particles after the permutation σ was applied to η , i.e.

$$\sigma(\eta)(x) := \eta(\sigma^{-1}(x)) \quad \text{for all} \quad x \in S.$$

We construct a continuous time Feller process, where rates $q(\sigma)$ are assigned so that

$$\eta \rightarrow \sigma(\eta) \quad \text{at rate} \quad q(\sigma).$$

Sufficient conditions for the process to be well defined Feller process?

• **Exclusion processes:** $\sup_{y \in S} \sum_x p(x, y) < \infty$

• **“Permutation processes:”**

$$M_I := \sup_{x \in S} \sum_{\sigma: x \in \text{Range}(\sigma)} q(\sigma) < \infty,$$

where $\text{Range}(\sigma) = \{x \in S : \sigma(x) \neq x\}$.

\Rightarrow the semigroup Ω_t of the permutation process η_t , generated by

$$\Omega f(\eta) := \sum_{\sigma} q(\sigma) [f(\sigma(\eta)) - f(\eta)]$$

(for cylinder functions f), is well defined.

Why was symmetric exclusion so successfully studied?

Duality: $P^\eta[\eta_t \equiv 1 \text{ on } A] = P^A[\eta \equiv 1 \text{ on } A_t]$

$\eta : \dots$	1	0	1	1	0	1	0	0	\dots
$A : \dots$	—	●	●	●	—	—	●	—	\dots



$\eta_t : \dots$	0	1	1	1	0	0	1	0	\dots
$A : \dots$	—	●	●	●	—	—	●	—	\dots

has the same probability as

$\eta : \dots$	1	0	1	1	0	1	0	0	\dots
$A : \dots$	—	●	●	●	—	—	●	—	\dots



$\eta : \dots$	1	0	1	1	0	1	0	0	\dots
$A_t : \dots$	●	—	●	●	—	●	—	—	\dots

What are the conditions for permutation processes to be self-dual?

For $\eta \in \{0, 1\}^S$ and a set $A \subset S$,

$$H(\eta, A) = \prod_{x \in A} \eta(x).$$

$$\begin{aligned} \Omega H(\cdot, A)(\eta) &= \sum_{\sigma} q(\sigma) [H(\sigma(\eta), A) - H(\eta, A)] \\ &= \sum_{\sigma} q(\sigma) [H(\eta, \sigma^{-1}(A)) - H(\eta, A)] \\ &= \sum_{\sigma} q(\sigma) [H(\eta, \sigma(A)) - H(\eta, A)], \end{aligned}$$

where the last line is true whenever

$$\boxed{q(\sigma) = q(\sigma^{-1})}$$

- Permutation process is said to be *symmetric* whenever the above condition is satisfied.

Then $\Omega H(\cdot, A)(\eta) = \Omega H(\eta, \cdot)(A)$, and

$$\boxed{P^\eta[\eta_t \equiv 1 \text{ on } A] = P^A[\eta \equiv 1 \text{ on } A_t]}.$$

$\mathcal{I} = \{\text{class of stationary distributions}\},$

$\mathcal{I}_e = \{\text{set of all the extreme points of } \mathcal{I}\},$

$\mathcal{S} = \{\text{class of shift invariant prob. measures}\}$

ν_ρ - homogeneous product measure on $\{0, 1\}^S$
with marginal probability ρ , e.g.

$$\nu_\rho\{\eta : \eta := 1 \text{ on } A\} = \rho^{|A|} \text{ for any } A \subset S.$$

We want to generalize two theorems: Let $S = \mathbb{Z}^d$ with shift-invariant random walk rates $p(x, y) = p(0, y - x)$.

Theorem. (F.Spitzer - recurrent case,
T.Liggett - transient case) For the symmetric
exclusion process, $\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$.

Theorem(R.Holley - special case,
T.Liggett - general case) For the general ex-
clusion process, $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$.

Possible conditions: We know two.

- **Condition I (Existence):**

$$M_I := \sup_{x \in S} \sum_{\sigma: x \in \text{Range}(\sigma)} q(\sigma) < \infty,$$

where $\text{Range}(\sigma) = \{x \in S : \sigma(x) \neq x\}$.

- **Condition II (Symmetry):**

$$q(\sigma) = q(\sigma^{-1})$$

Consider three more.

- **Condition III (Irreducibility):** $\forall x, y \in S$,
 \exists permutations $\sigma_1, \dots, \sigma_k$ with **positive** rates

$$\text{s.t.} \quad \boxed{\sigma_k \circ \dots \circ \sigma_1(x) = y}.$$

- **Condition IV (Finiteness):**

$$\boxed{M_{IV} := \sup_{\{\sigma: q(\sigma) > 0\}} |Range(\sigma)| < \infty},$$

where $Range(\sigma) = \{x \in S : \sigma(x) \neq x\}$ and $|\cdot|$ denotes cardinality.

- **Condition V (Ellipticity):**

If $Range(\sigma_1) = Range(\sigma_2)$ and $q(\sigma_1) > 0$, then $q(\sigma_2) > 0$. Also

$$\boxed{M_V := \sup_{\sigma_1, \sigma_2: Range(\sigma_1) = Range(\sigma_2)} \left| \frac{q(\sigma_1)}{q(\sigma_2)} \right| < \infty}$$

↑

sup taken over $q(\sigma_1), q(\sigma_2) > 0$.

Main Results:

Consider irreducible permutation processes on $S = \mathbb{Z}^d$ with translation invariant rates. Suppose conditions

I (Existence), **III** (Irreducibility), **IV** (Finiteness) and **V** (Ellipticity) are satisfied.

Theorem 1. (K. 2004) For the symmetric permutation processes, $\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$.

i.e. **II** (Symmetry) $\Rightarrow \mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$

Theorem 2. (K. 2004) For the general permutation process (conditions **I,III, IV** and **V** only), $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$.

Definition. A bounded function $f : S \rightarrow \mathbb{R}$ is *harmonic* if

$$f(\eta) = \sum_{\zeta \in \{0,1\}^S} P^\eta[\eta_t = \zeta] f(\zeta).$$

We will need the following

Proposition 1. If f is a bounded harmonic function for the well defined finite permutation process A_t , then f is constant on $\{A : |A| = n\}$ for each given integer $n \geq 1$.

Now, Proposition 1 \Rightarrow Theorem 1

Reason: self-duality.

Definition. A probability measure μ on $\{0, 1\}^S$ is called *exchangeable* if for any finite $A \subset S$, $\mu\{\eta : \eta \equiv 1 \text{ on } A\}$ is a function of $|A|$.

Proof (Proposition 1 \Rightarrow Theorem 1).

Ω_t - semigroup of the permutation process.

Self-duality \Rightarrow for a prob. measure μ on $\{0, 1\}^S$,

$$\begin{aligned} \mu\Omega_t\{\eta : \eta \equiv 1 \text{ on } A\} &= \int P^\eta[\eta_t \equiv 1 \text{ on } A]d\mu \\ &= \int P^A[\eta \equiv 1 \text{ on } A_t]d\mu \\ &= \sum_B P^A[A_t = B]\mu\{\eta : \eta \equiv 1 \text{ on } B\} \end{aligned}$$

Consider $f(A) = \mu\{\eta : \eta \equiv 1 \text{ on } A\}$.

$$\boxed{\mu \in \mathcal{I}, \text{ i.e. } \mu\Omega_t = \mu, t \geq 0} \Leftrightarrow \boxed{f(A) \text{ is harmonic}}$$

$$(\text{RHS} = f(A)) \uparrow \qquad (\text{Proposition 1}) \downarrow$$

$$\boxed{f(A) \text{ is a function of } |A|, \text{ i.e. } \mu \text{ is exchangeable}}$$

$$\Updownarrow \qquad \Updownarrow (\text{de Finetti's thm.}) \Updownarrow \qquad \Updownarrow$$

$$\boxed{\mu \text{ is mixture of homogeneous product measures } \nu_\rho}$$

Proof of Proposition 1 in recurrent case.

Proposition 1. If f is a bounded harmonic function for the well defined finite permutation process A_t , then f is constant on $\{A : |A| = n\}$ for each given integer $n \geq 1$.

Enough: show $f(A_0) = f(B_0) \quad \forall A_0, B_0 \subset S$

s.t. $\boxed{|A_0| = |B_0| = |A_0 \cap B_0| + 1}$

Need: construct a *successful* coupling of two copies A_t and B_t of the process with initial states A_0 and B_0 .

Successful coupling:

$P[A_t = B_t \text{ for all } t \text{ beyond some time}] = 1.$

Then

$$\begin{aligned} |f(A_0) - f(B_0)| &= |Ef(A_t) - Ef(B_t)| \\ &\leq E|f(A_t) - f(B_t)| \leq \|f\|P[A_t \neq B_t] \rightarrow 0 \end{aligned}$$

as $t \rightarrow \infty$.

Reconstructing Spitzer's coupling.

d_t^+ denotes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ discrepancy

d_t^- denotes $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ discrepancy

$A_t :$...	1	0	1	1	0	1	0	0	...
$B_t :$...	1	0	0	1	0	1	1	0	...
				↑				↑		
				d_t^+				d_t^-		

Challenge: canceling the original two discrepancies, not creating new discrepancies and still constructing a successful coupling.

Recurrence

 \Rightarrow
 $\{d_t^+ \text{ and } d_t^- \in \text{Range}(\sigma), q(\sigma) > 0\}$
 will happen infinitely often

Coupling. When d_t^+ and $d_t^- \in \text{Range}(\sigma_0) = R$, where $q(\sigma_0) > 0$, we pick a **cycle** σ_R

s.t. $\boxed{q(\sigma_R) > 0 \text{ and } \sigma_R(A_t) = B_t}$

Rates: Let $m(R) = \min_{\sigma: \text{Range}(\sigma)=R} \{q(\sigma)\}$.

Then $\begin{pmatrix} A_t \\ B_t \end{pmatrix}$ transforms into

$$\left\{ \begin{array}{l} \begin{pmatrix} \sigma_R^2(A_t) \\ \sigma_R(B_t) \end{pmatrix} = \begin{pmatrix} \sigma_R(B_t) \\ \sigma_R(B_t) \end{pmatrix} \text{ with rate } m(R), \\ \\ \begin{pmatrix} \sigma_R^3(A_t) \\ \sigma_R^2(B_t) \end{pmatrix} = \begin{pmatrix} \sigma_R^2(B_t) \\ \sigma_R^2(B_t) \end{pmatrix} \text{ with rate } m(R), \\ \\ \dots \\ \begin{pmatrix} \sigma_R^{|R|-1}(A_t) \\ \sigma_R^{|R|-2}(B_t) \end{pmatrix} = \begin{pmatrix} \sigma_R^{|R|-2}(B_t) \\ \sigma_R^{|R|-2}(B_t) \end{pmatrix} \text{ with rate } m(R), \\ \\ \begin{pmatrix} \sigma_R(A_t) \\ \sigma_R^{|R|-1}(B_t) \end{pmatrix} = \begin{pmatrix} B_t \\ A_t \end{pmatrix} \text{ with rate } m(R), \end{array} \right.$$

Rates(Continued):

$\begin{pmatrix} A_t \\ B_t \end{pmatrix}$ transforms into

$$\left\{ \begin{array}{l} \begin{pmatrix} \sigma_R(A_t) \\ \sigma_R(B_t) \end{pmatrix} \text{ with rate } q(\sigma_R) - m(R), \\ \\ \begin{pmatrix} \sigma_R^2(A_t) \\ \sigma_R^2(B_t) \end{pmatrix} \text{ with rate } q(\sigma_R^2) - m(R), \\ \\ \dots \\ \begin{pmatrix} \sigma_R^{|R|-1}(A_t) \\ \sigma_R^{|R|-1}(B_t) \end{pmatrix} \text{ with rate } q(\sigma_R^{|R|-1}) - m(R), \\ \\ \begin{pmatrix} \sigma(A_t) \\ \sigma(B_t) \end{pmatrix} \text{ with rate } q(\sigma) \text{ if } \text{Range}(\sigma) = R \\ \text{and } \sigma \neq \sigma_R^i, \text{ all } i. \end{array} \right.$$

The coupling is successful.

At the holding time, the discrepancies will cancel with probability

$$\begin{aligned} &\geq \sum_{R: d_t^-, d_t^+ \in R} \frac{m(R)}{z_d(t)} \\ &\geq \sum_{R: d_t^-, d_t^+ \in R} \frac{Z(R)}{\mathcal{P}(M_{IV}) M_V z_d(t)} = \frac{1}{\mathcal{P}(M_{IV}) M_V} \end{aligned}$$

where

$$Z(R) = \sum_{\sigma: \text{Range}(\sigma) = R} q(\sigma),$$

$$z_d(t) := \sum_{\substack{\text{range sets } R : \\ d_t^-, d_t^+ \in R}} Z(R) \leq M_I$$

and $\mathcal{P}(n) = \sum_{k=0}^n \binom{n}{k} (-1)^k (n-k)!$ denotes the number of permutations of n distinct objects such that each object is displaced.

Example. Let $S = \mathbb{Z}$. Each permutation in $\bigcup_{x \in \mathbb{Z}} \{ \sigma_x := \overline{(x, x+1, x+2)}, \sigma_x^2 = \sigma_x^{-1} \}$ has positive rate q , i.e.

$$\boxed{q(\sigma_x) = q(\sigma_x^{-1}) = q > 0}.$$

Range sets: $R_x = \{x, x+1, x+2\}$ for $x \in \mathbb{Z}$.

Then $M_I = 6q$, $M_{IV} = 3$ and $M_V = 1$.

There $m(R_x) = q$ and $Z(R_x) = 2q$ since $R_x = \text{Range}(\sigma_x) = \text{Range}(\sigma_x^{-1})$.

Suppose

$$\begin{array}{cccccccccc} A_t : & \dots & 1 & 1 & 1 & 0 & 0 & 1 & 0 & \dots \\ B_t : & \dots & 1 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ & & & \uparrow & \uparrow & \uparrow & & & & \\ & & & y-1 & y & y+1 & & & & \end{array}$$

$$\boxed{R = R_{y-1} \Rightarrow \sigma_R = \sigma_{y-1}^{-1}},$$

and

$$\boxed{R = R_y \Rightarrow \sigma_R = \sigma_y}.$$

$$\begin{array}{l}
\left(\begin{array}{c} A_t \\ B_t \end{array} \right) \rightarrow \left\{ \begin{array}{l}
\begin{array}{l}
\left(\begin{array}{c} \sigma_{y-1}(A_t) \\ \sigma_{y-1}^{-1}(B_t) \end{array} \right) = \dots \quad \begin{array}{cccc} 0 & 1 & 1 & 0 \end{array} \quad \dots \\
\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ y-1 & y & y+1 \end{array} \\
\end{array} \\
\begin{array}{l}
\left(\begin{array}{c} \sigma_{y-1}^{-1}(A_t) \\ \sigma_{y-1}(B_t) \end{array} \right) = \dots \quad \begin{array}{cccc} 1 & 0 & 1 & 0 \end{array} \quad \dots \\
\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ y-1 & y & y+1 \end{array} \\
\end{array} \\
\begin{array}{l}
\left(\begin{array}{c} \sigma_y^{-1}(A_t) \\ \sigma_y(B_t) \end{array} \right) = \dots \quad \begin{array}{cccc} 1 & 0 & 0 & 1 \end{array} \quad \dots \\
\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ y-1 & y & y+1 \end{array} \\
\end{array} \\
\begin{array}{l}
\left(\begin{array}{c} \sigma_y(A_t) \\ \sigma_y^{-1}(B_t) \end{array} \right) = \dots \quad \begin{array}{cccc} 1 & 0 & 1 & 0 \end{array} \quad \dots \\
\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ y-1 & y & y+1 \end{array} \\
\end{array}
\end{array} \right.
\end{array}$$

with rate q each time.

Example(Continued). Here

$$z_d(t) = Z(R_y) + Z(R_{y-1}) = 4q.$$

Discrepancies cancel with probability

$$= \frac{2q}{z_d(t)} = \frac{1}{2} = \frac{1}{\mathcal{P}(M_{IV})M_V}.$$

Proof of Proposition 1: transient case.

$f(A)$ bounded harmonic (i.e. $\Omega_t f = f$) and

$$g_n(A) := P^A \left\{ \begin{array}{l} \exists t \in (0, \infty) : A_i(t) \neq A_i(t-) \text{ and} \\ A_j(t) \neq A_j(t-) \text{ some } i \neq j \end{array} \right\},$$

where $A_t = (A_1(t), \dots, A_n(t))$ is the n -point permutation process.

We prove

$$\boxed{|f(A) - C| = |\Omega_t f(A) - C| \leq c \Omega_t g_n(A), \quad A \in T_n}$$

for some constants C and c , where

$$T_n = \{x = (x_1, \dots, x_n) \in S^n : x_i \neq x_j \text{ for all } i \neq j\}.$$

We show (by transience)

$$\lim_{x \rightarrow \infty} \Omega_t g_n(A) = 0$$

for all $A \in T_n$.

Question: How far can we extend these methods? Can we prove some of the same results under weaker conditions?