Generalized Symmetric Exclusion Processes

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Exclusion Processes

S - general countable set,
$p(x,y)$ - transition probabilities for a Markov chain on S

η_t - continuous time Feller process with values in $\{0,1\}^S$

Transition rates:

$\eta \to \eta_{x,y}$ at rate $p(x,y)$ if $\eta(x)=1, \eta(y)=0$

where $\eta_{x,y}(u) \equiv \begin{cases} \eta(u) \text{ when } u \notin \{x,y\}, \\ \eta(y) \text{ when } u = x \\ \eta(x) \text{ when } u = y. \end{cases}$

$$\begin{array}{ccccccccccc} \ldots & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots \\ & & & \uparrow & & & & \uparrow & & \\ & & & x & & & & y & & \end{array}$$

Symmetric Exclusion Processes

$$p(x,y) = p(y,x)$$

$$\boxed{\begin{array}{ccccccc} \ldots & 0 & 1 & 1 & 0 & 0 & \ldots \\ & & \uparrow & & & \uparrow & \\ & & x & & & y & \end{array}} \rightarrow \boxed{\begin{array}{ccccccc} \ldots & 0 & 0 & 1 & 0 & 1 & \ldots \\ & & \uparrow & & & \uparrow & \\ & & x & & & y & \end{array}}$$

$$\boxed{\begin{array}{ccccccc} \ldots & 0 & 0 & 1 & 0 & 1 & \ldots \\ & & \uparrow & & & \uparrow & \\ & & x & & & y & \end{array}} \rightarrow \boxed{\begin{array}{ccccccc} \ldots & 0 & 1 & 1 & 0 & 0 & \ldots \\ & & \uparrow & & & \uparrow & \\ & & x & & & y & \end{array}}$$

each time with rate $p(x,y)$.

But also, we let

$$\boxed{\begin{array}{ccccccc} \ldots & 0 & 1 & 1 & 0 & 1 & \ldots \\ & & \uparrow & & & \uparrow & \\ & & x & & & y & \end{array}} \rightarrow \boxed{\begin{array}{ccccccc} \ldots & 0 & 1 & 1 & 0 & 1 & \ldots \\ & & \uparrow & & & \uparrow & \\ & & x & & & y & \end{array}}$$

$$\boxed{\begin{array}{ccccccc} \ldots & 0 & 0 & 1 & 0 & 0 & \ldots \\ & & \uparrow & & & \uparrow & \\ & & x & & & y & \end{array}} \rightarrow \boxed{\begin{array}{ccccccc} \ldots & 0 & 0 & 1 & 0 & 0 & \ldots \\ & & \uparrow & & & \uparrow & \\ & & x & & & y & \end{array}}$$

with rate $p(x,y)$, if no particle is tagged.

For each transposition $\tau_{x,y}$ ($x, y \in S$, $x \neq y$) we assign rate

$$q(\tau_{x,y}) = p(y,x) = p(x,y)$$

at which the transposition occurs:

$$\eta \to \tau_{x,y}(\eta) = \eta_{x,y} \quad \text{at rate} \quad q(\tau_{x,y}).$$

Natural generalization of symmetric exclusion suggested by T.Liggett: We can assign the rates not to the particles inhabiting the space S, but to various permutations of finitely many points of S.

For each $\eta \in \{0,1\}^S$, let $\sigma(\eta)$ be the new configuration of particles after the permutation σ was applied to η, i.e.

$$\sigma(\eta)(x) := \eta(\sigma^{-1}(x)) \quad \text{for all} \quad x \in S.$$

We construct a continuous time Feller process, where rates $q(\sigma)$ are assigned so that

$$\eta \to \sigma(\eta) \quad \text{at rate} \quad q(\sigma).$$

Sufficient conditions for the process to be well defined Feller process?

- **Exclusion processes:** $\boxed{\sup_{y\in S}\sum_x p(x,y) < \infty}$

- **"Permutation processes:"**

$\boxed{M_I := \sup_{x\in S}\sum_{\sigma: x\in Range(\sigma)} q(\sigma) < \infty}$,

where $Range(\sigma) = \{x \in S : \quad \sigma(x) \neq x\}$.

$\Rightarrow$ the semigroup Ω_t of the permutation process η_t, generated by

$$\Omega f(\eta) := \sum_\sigma q(\sigma)[f(\sigma(\eta)) - f(\eta)]$$

(for cylinder functions f), is well defined.

Why was symmetric exclusion so successfully studied?

Duality: $P^{\eta}[\eta_t \equiv 1 \text{ on } A] = P^{A}[\eta \equiv 1 \text{ on } A_t]$

$\eta : \ldots$	1	0	1	1	0	1	0	0	$\ldots$
$A : \ldots$	–	•	•	•	–	–	•	–	$\ldots$

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

$\eta_t : \ldots$	0	1	1	1	0	0	1	0	$\ldots$
$A : \ldots$	–	•	•	•	–	–	•	–	$\ldots$

has the same probability as

$\eta : \ldots$	1	0	1	1	0	1	0	0	$\ldots$
$A : \ldots$	–	•	•	•	–	–	•	–	$\ldots$

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

$\eta : \ldots$	1	0	1	1	0	1	0	0	$\ldots$
$A_t : \ldots$	•	–	•	•	–	•	–	–	$\ldots$

What are the conditions for permutation processes to be self-dual?

For $\eta \in \{0,1\}^S$ and a set $A \subset S$,

$$H(\eta, A) = \prod_{x \in A} \eta(x).$$

$$\begin{aligned}
\Omega H(\cdot, A)(\eta) &= \sum_\sigma q(\sigma)[H(\sigma(\eta), A) - H(\eta, A)] \\
&= \sum_\sigma q(\sigma)[H(\eta, \sigma^{-1}(A)) - H(\eta, A)] \\
&= \sum_\sigma q(\sigma)[H(\eta, \sigma(A)) - H(\eta, A)],
\end{aligned}$$

where the last line is true whenever

$$\boxed{q(\sigma) = q(\sigma^{-1})}$$

- Permutation process is said to be *symmetric* whenever the above condition is satisfied.

Then $\Omega H(\cdot, A)(\eta) = \Omega H(\eta, \cdot)(A)$, and

$$\boxed{P^\eta[\eta_t \equiv 1 \text{ on } A] = P^A[\eta \equiv 1 \text{ on } A_t]}.$$

$\mathcal{I} = \{$class of stationary distributions$\}$,
$\mathcal{I}_e = \{$set of all the extreme points of $\mathcal{I}\}$,
$\mathcal{S} = \{$class of shift invariant prob. measures$\}$
ν_ρ - homogeneous product measure on $\{0,1\}^S$ with marginal probability ρ, e.g.

$$\boxed{\nu_\rho\{\eta : \eta := 1 \text{ on } A\} = \rho^{|A|} \text{ for any } A \subset S}.$$

We want to generalize two theorems: Let $S = \mathbb{Z}^d$ with shift-invariant random walk rates $p(x,y) = p(0, y-x)$.

Theorem. (F.Spitzer - recurrent case, T.Liggett - transient case) For the symmetric exclusion process, $\boxed{\mathcal{I}_e = \{\nu_\rho : 0 \le \rho \le 1\}}$.

Theorem(R.Holley - special case, T.Liggett - general case) For the general exclusion process, $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho : 0 \le \rho \le 1\}$.

Possible conditions: We know two.

- **Condition I (Existence):**

$$\boxed{M_I := \sup_{x\in S} \sum_{\sigma : x \in Range(\sigma)} q(\sigma) < \infty},$$

where $Range(\sigma) = \{x \in S : \quad \sigma(x) \neq x\}$.

- **Condition II (Symmetry):**

$$\boxed{q(\sigma) = q(\sigma^{-1})}$$

Consider three more.

- **Condition III (Irreducibility):** $\forall x, y \in S$,
$\exists$ permutations $\sigma_1, \ldots, \sigma_k$ with **positive** rates

$$\text{s.t.} \qquad \boxed{\sigma_k \circ ... \circ \sigma_1(x) = y}.$$

- **Condition IV (Finiteness):**

$$\boxed{M_{IV} := \sup_{\{\sigma : q(\sigma) > 0\}} \left| Range(\sigma) \right| < \infty},$$

where $Range(\sigma) = \{x \in S : \quad \sigma(x) \neq x\}$ and
$|\cdot|$ denotes cardinality.

- **Condition V (Ellipticity):**
If $Range(\sigma_1) = Range(\sigma_2)$ and $q(\sigma_1) > 0$, then
$q(\sigma_2) > 0$. Also

$$\boxed{M_V := \sup_{\sigma_1, \sigma_2 : Range(\sigma_1) = Range(\sigma_2)} \left| \frac{q(\sigma_1)}{q(\sigma_2)} \right| < \infty}$$

$\uparrow$

sup taken over $q(\sigma_1), q(\sigma_2) > 0$.

Main Results:

Consider irreducible permutation processes on $S = \mathbb{Z}^d$ *with translation invariant rates.* Suppose conditions

$\boxed{\textbf{I} \text{ (Existence)}, \textbf{III} \text{ (Irreducibility)}, \textbf{IV} \text{ (Finiteness)}}$ and $\boxed{\textbf{V} \text{ (Ellipticity)}}$ are satisfied.

Theorem 1. (K. 2004) For the symmetric permutation processes, $\boxed{\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}}$.

i.e. $\boxed{\textbf{II} \text{ (Symmetry)}} \Rightarrow \boxed{\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}}$

Theorem 2. (K. 2004) For the general permutation process (conditions **I,III, IV** and **V** only), $\boxed{(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho : 0 \leq \rho \leq 1\}}$.

Definition. A bounded function $f : S \to \mathbb{R}$ is *harmonic* if

$$\boxed{f(\eta) = \sum_{\zeta \in \{0,1\}^S} P^\eta[\eta_t = \zeta] f(\zeta)}.$$

We will need the following

Proposition 1. If f is a bounded harmonic function for the well defined finite permutation process A_t, then f is constant on $\{A : |A| = n\}$ for each given integer $n \geq 1$.

Now, $\boxed{\text{Proposition 1}} \Rightarrow \boxed{\text{Theorem 1}}$

Reason: self-duality.

Definition. A probability measure μ on $\{0,1\}^S$ is called *exchangeable* if for any finite $A \subset S$, $\mu\{\eta : \eta \equiv 1 \text{ on } A\}$ is a function of $|A|$.

Proof (Proposition 1 $\Rightarrow$ Theorem 1**).**

Ω_t - semigroup of the permutation process.

Self-duality $\Rightarrow$ for a prob. measure μ on $\{0,1\}^S$,

$$\begin{aligned}\mu\Omega_t\{\eta : \eta \equiv 1 \text{ on } A\} &= \int P^\eta[\eta_t \equiv 1 \text{ on } A]d\mu \\ &= \int P^A[\eta \equiv 1 \text{ on } A_t]d\mu \\ &= \textstyle\sum_B P^A[A_t = B]\mu\{\eta : \eta \equiv 1 \text{ on } B\}\end{aligned}$$

Consider $f(A) = \mu\{\eta : \eta \equiv 1 \text{ on } A\}$.

$\boxed{\mu \in \mathcal{I}, \text{ i.e. } \mu\Omega_t = \mu,\ t \geq 0}$ $\Leftrightarrow$ $\boxed{f(A) \text{ is harmonic}}$

(RHS= $f(A)$) $\Uparrow$ (Proposition 1) $\Downarrow$

$\boxed{f(A) \text{ is a function of } |A|, \text{ i.e. } \mu \text{ is exchangeable}}$

$\Updownarrow$ $\Updownarrow$ (de Finetti's thm.) $\Updownarrow$ $\Updownarrow$

$\boxed{\mu \text{ is mixture of homogeneous product measures } \nu_\rho}$

Proof of Proposition 1 in recurrent case.

Proposition 1. If f is a bounded harmonic function for the well defined finite permutation process A_t, then f is constant on $\{A : |A| = n\}$ for each given integer $n \geq 1$.

Enough: show $f(A_0) = f(B_0) \quad \forall A_0, B_0 \subset S$

s.t. $\boxed{|A_0| = |B_0| = |A_0 \cap B_0| + 1}$

Need: construct a *successful* coupling of two copies A_t and B_t of the process with initial states A_0 and B_0.

Successful coupling:
$P[A_t = B_t$ for all t beyond some time $] = 1$.

Then

$$|f(A_0) - f(B_0)| = |Ef(A_t) - Ef(B_t)|$$

$$\leq E|f(A_t) - f(B_t)| \leq \|f\| P[A_t \neq B_t] \to 0$$

as $t \to \infty$.

Reconstructing Spitzer's coupling.

d_t^+ denotes $\binom{1}{0}$ discrepancy

d_t^- denotes $\binom{0}{1}$ discrepancy

$$\begin{array}{lccccccccc} A_t: & \ldots & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & \ldots \\ B_t: & \ldots & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \ldots \\ & & & & \uparrow & & & & \uparrow & & \\ & & & & d_t^+ & & & & d_t^- & & \end{array}$$

Challenge: canceling the original two discrepancies, not creating new discrepancies and still constructing a successful coupling.

$$\boxed{\text{Recurrence}} \Rightarrow \boxed{\begin{array}{c}\{d_t^+ \text{ and } d_t^- \in Range(\sigma), q(\sigma) > 0\} \\ \text{will happen infinitely often}\end{array}}$$

Coupling. When d_t^+ and $d_t^- \in Range(\sigma_0) = R$, where $q(\sigma_0) > 0$, we pick a **cycle** σ_R

s.t. $\boxed{q(\sigma_R) > 0 \text{ and } \sigma_R(A_t) = B_t}$

Rates: Let $m(R) = \min_{\sigma: Range(\sigma)=R}\{q(\sigma)\}$.
Then $\begin{pmatrix} A_t \\ B_t \end{pmatrix}$ transforms into

$$\begin{cases} \begin{pmatrix} \sigma_R^2(A_t) \\ \sigma_R(B_t) \end{pmatrix} = \begin{pmatrix} \sigma_R(B_t) \\ \sigma_R(B_t) \end{pmatrix} \text{ with rate } m(R), \\ \\ \begin{pmatrix} \sigma_R^3(A_t) \\ \sigma_R^2(B_t) \end{pmatrix} = \begin{pmatrix} \sigma_R^2(B_t) \\ \sigma_R^2(B_t) \end{pmatrix} \text{ with rate } m(R), \\ \dots \\ \begin{pmatrix} \sigma_R^{|R|-1}(A_t) \\ \sigma_R^{|R|-2}(B_t) \end{pmatrix} = \begin{pmatrix} \sigma_R^{|R|-2}(B_t) \\ \sigma_R^{|R|-2}(B_t) \end{pmatrix} \text{ with rate } m(R), \\ \\ \begin{pmatrix} \sigma_R(A_t) \\ \sigma_R^{|R|-1}(B_t) \end{pmatrix} = \begin{pmatrix} B_t \\ A_t \end{pmatrix} \text{ with rate } m(R), \end{cases}$$

Rates(Continued):

$\begin{pmatrix} A_t \\ B_t \end{pmatrix}$ transforms into

$$\begin{cases} \begin{pmatrix} \sigma_R(A_t) \\ \sigma_R(B_t) \end{pmatrix} \text{ with rate } q(\sigma_R) - m(R), \\ \\ \begin{pmatrix} \sigma_R^2(A_t) \\ \sigma_R^2(B_t) \end{pmatrix} \text{ with rate } q(\sigma_R^2) - m(R), \\ \ldots \\ \begin{pmatrix} \sigma_R^{|R|-1}(A_t) \\ \sigma_R^{|R|-1}(B_t) \end{pmatrix} \text{ with rate } q(\sigma_R^{|R|-1}) - m(R), \\ \\ \begin{pmatrix} \sigma(A_t) \\ \sigma(B_t) \end{pmatrix} \text{ with rate } q(\sigma) \begin{array}{c} \text{if } Range(\sigma) = R \\ \text{and } \sigma \neq \sigma_R^i, \text{ all } i. \end{array} \end{cases}$$

The coupling is successful.

At the holding time, the discrepancies will cancel with probability

$$\boxed{\begin{array}{c} \geq \sum_{R: d_t^-, d_t^+ \in R} \frac{m(R)}{z_d(t)} \\ \\ \geq \sum_{R: d_t^-, d_t^+ \in R} \frac{Z(R)}{\mathcal{P}(M_{IV}) M_V z_d(t)} = \frac{1}{\mathcal{P}(M_{IV}) M_V} \end{array}},$$

where

$$Z(R) = \sum_{\sigma: Range(\sigma) = R} q(\sigma),$$

$$z_d(t) := \sum_{\substack{\text{range sets } R: \\ d_t^-, d_t^+ \in R}} Z(R) \leq M_I$$

and $\mathcal{P}(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)!$ denotes the number of permutations of n distinct objects such that each object is displaced.

Example. Let $S = \mathbb{Z}$. Each permutation in $\bigcup_{x\in\mathbb{Z}}\left\{\sigma_x := \overline{(x, x+1, x+2)}, \sigma_x^2 = \sigma_x^{-1}\right\}$ has positive rate q, i.e.

$$\boxed{q(\sigma_x) = q(\sigma_x^{-1}) = q > 0}.$$

Range sets: $R_x = \{x, x+1, x+2\}$ for $x \in \mathbb{Z}$.

Then $M_I = 6q$, $M_{IV} = 3$ and $M_V = 1$.

There $m(R_x) = q$ and $Z(R_x) = 2q$ since $R_x = Range(\sigma_x) = Range(\sigma_x^{-1})$.

Suppose

$$\begin{array}{lcccccccc} A_t: & \dots & 1 & 1 & 1 & 0 & 0 & 1 & 0 & \dots \\ B_t: & \dots & 1 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ & & & \uparrow & \uparrow & \uparrow & & & & \\ & & & y-1 & y & y+1 & & & & \end{array}$$

$$\boxed{R = R_{y-1} \Rightarrow \sigma_R = \sigma_{y-1}^{-1}},$$

and $\boxed{R = R_y \Rightarrow \sigma_R = \sigma_y}$.

$$
\begin{pmatrix} A_t \\ B_t \end{pmatrix} \to
\begin{cases}
\begin{pmatrix} \sigma_{y-1}(A_t) \\ \sigma_{y-1}^{-1}(B_t) \end{pmatrix} = \begin{matrix} \dots & 0 & 1 & 1 & 0 & \dots \\ \dots & 0 & 1 & 1 & 0 & \dots \\ & \uparrow & \uparrow & \uparrow & & \\ & y-1 & y & y+1 & & \end{matrix} \\[2em]
\begin{pmatrix} \sigma_{y-1}^{-1}(A_t) \\ \sigma_{y-1}(B_t) \end{pmatrix} = \begin{matrix} \dots & 1 & 0 & 1 & 0 & \dots \\ \dots & 1 & 1 & 0 & 0 & \dots \\ & \uparrow & \uparrow & \uparrow & & \\ & y-1 & y & y+1 & & \end{matrix} \\[2em]
\begin{pmatrix} \sigma_{y}^{-1}(A_t) \\ \sigma_{y}(B_t) \end{pmatrix} = \begin{matrix} \dots & 1 & 0 & 0 & 1 & \dots \\ \dots & 1 & 0 & 0 & 1 & \dots \\ & \uparrow & \uparrow & \uparrow & & \\ & y-1 & y & y+1 & & \end{matrix} \\[2em]
\begin{pmatrix} \sigma_{y}(A_t) \\ \sigma_{y}^{-1}(B_t) \end{pmatrix} = \begin{matrix} \dots & 1 & 0 & 1 & 0 & \dots \\ \dots & 1 & 1 & 0 & 0 & \dots \\ & \uparrow & \uparrow & \uparrow & & \\ & y-1 & y & y+1 & & \end{matrix} \\[2em]
\textbf{with rate } q \quad \textbf{each time.}
\end{cases}
$$

Example(Continued). Here

$$z_d(t) = Z(R_y) + Z(R_{y-1}) = 4q.$$

Discrepancies cancel with probability

$$= \frac{2q}{z_d(t)} = \frac{1}{2} = \frac{1}{\mathcal{P}(M_{IV})M_V}.$$

Proof of Proposition 1: transient case.

$f(A)$ bounded harmonic (i.e. $\Omega_t f = f$) and

$$g_n(A) := P^A \left\{ \begin{array}{c} \exists t \in (0,\infty) : \ A_i(t) \neq A_i(t-) \text{ and} \\ A_j(t) \neq A_j(t-) \text{ some } i \neq j \end{array} \right\},$$

where $A_t = (A_1(t), \ldots, A_n(t))$ is the n-point permutation process.

We prove

$$\boxed{|f(A) - C| = |\Omega_t f(A) - C| \leq c\Omega_t g_n(A), \qquad A \in T_n}$$

for some constants C and c, where

$T_n = \{x = (x_1, \ldots, x_n) \in S^n : x_i \neq x_j \text{ for all } i \neq j\}$.

We show (by transience)

$$\lim_{x \to \infty} \Omega_t g_n(A) = 0$$

for all $A \in T_n$.

Question: How far can we extend these methods? Can we prove some of the same results under weaker conditions?