Random Self-Similar Trees: Dynamical Pruning, Invariance, and Criticality

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Trees.

$\mathcal{L}_{\text{plane}}$ - space of finite unlabeled rooted reduced binary trees with edge lengths and planar embedding.

The space $\mathcal{L}_{\text{plane}}$ includes the empty tree $\phi = \{\rho\}$ comprised of a root vertex ρ and no edges.

$d(x,y)$: the length of the minimal path within T between x and y.

The length of a tree T is the sum of the lengths of its edges:

$$\text{length}(T) = \sum_{i=1}^{\#T} l_i.$$

The height of a tree T is the maximal distance between the root and a vertex:

$$\text{height}(T) = \max_{1 \le i \le \#T} d(v_i, \rho).$$

Partial ordering.

Consider a tree $T \in \mathcal{L}_{\text{plane}}$ and a point $x \in T$. Let $\Delta_{x,T}$ denote all points of T descendant to x, including x. Then $\Delta_{x,T}$ is itself a tree in $\mathcal{L}_{\text{plane}}$ with root at x.

Let (T_1, d) and (T_2, d) be two metric rooted trees, and let ρ_1 denote the root of T_1. A function

$$f : (T_1, d) \to (T_2, d)$$

is an isometry if

$$\text{Image}[f] \subseteq \Delta_{f(\rho_1),T_2}$$

and $\forall x, y \in T_1$,

$$d\big(f(x), f(y)\big) = d(x, y).$$

Partial order: $T_1 \preceq T_2$ if and only if $\exists$ an isometry $f : (T_1, d) \to (T_2, d)$.

Generalized dynamical pruning.

Consider a monotone non-decreasing

$$\varphi : \mathcal{L}_{\text{plane}} \to \mathbb{R},$$

i.e. $\varphi(T_1) \le \varphi(T_2)$ whenever $T_1 \preceq T_2$.

Generalized dynamical pruning operator

$$\mathcal{S}_t(\varphi, T) : \mathcal{L}_{\text{plane}} \to \mathcal{L}_{\text{plane}}$$

induced by φ at any $t \ge 0$:

$$\mathcal{S}_t(\varphi, T) := \rho \cup \Big\{ x \in T \setminus \rho \ : \ \varphi\big(\Delta_{x,T}\big) \ge t \Big\}.$$

$\mathcal{S}_t$ cuts all subtrees for which the value of φ is below threshold t. Here,

$$S_s(T) \preceq S_t(T)$$

whenever $s \ge t$.

Example: Tree height.

Recall: $$\mathcal{S}_t(\varphi, T) := \rho \cup \Big\{x \in T \setminus \rho \ : \ \varphi\big(\Delta_{x,T}\big) \geq t\Big\}.$$

Let the function $\varphi(T)$ equal the height of T:

$$\varphi(T) = \text{height}(T).$$

Semigroup property: $\mathcal{S}_t \circ \mathcal{S}_s = \mathcal{S}_{t+s}$ for any $t, s \geq 0$.

It coincides with the tree erasure introduced by Neveu (1986).

Neveu (1986): established invariance of a critical and sub-critical binary Galton-Watson processes with i.i.d. exponential edge lengths with respect to the tree erasure.

Example: Total tree length.

Recall: $$\mathcal{S}_t(\varphi, T) := \rho \cup \Big\{ x \in T \setminus \rho \ : \ \varphi\big(\Delta_{x,T}\big) \geq t \Big\}.$$

Let the function $\varphi(T)$ equal the total lengths of T:

$$\varphi(T) = \text{length}(T).$$

In this case the pruning operator $\mathcal{S}_t$ coincides with the potential dynamics of Burgers equation, as shown in

M. Arnold, YK, I. Zaliapin (2017) - arXiv:1707.01984

Example: Horton pruning.

Let

$$\varphi(T) = \mathsf{k}(T) - 1,$$

where the Horton-Strahler order $\mathsf{k}(T)$ is the minimal number of Horton prunings $\mathcal{R}$ (cutting the tree leaves and applying series reduction) necessary to eliminate the tree T.

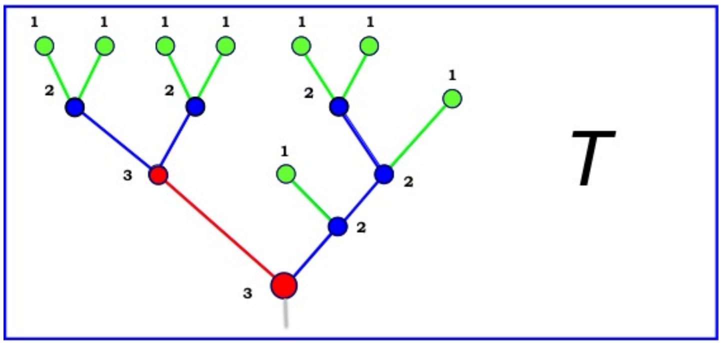
Here,

$$\mathcal{S}_t = \mathcal{R}^{\lfloor t \rfloor}.$$

The Horton-Strahler order is known as the register number as it equals the minimum number of memory registers necessary to evaluate an arithmetic expression described by a tree T.

Horton pruning and Horton-Strahler ordering.

- Pruning $\mathcal{R}(T)$ of a finite tree T cuts the leaves, followed by *series reduction*.
- A chain of the same order vertices with edges connecting to parent vertices is called branch.
- Branches cut at k-th pruning, $\mathcal{R}^{k-1}(T) \setminus \mathcal{R}^{k}(T)$, have order k, $k \geq 1$.
- N_k denotes the number of branches of order k in a finite tree T



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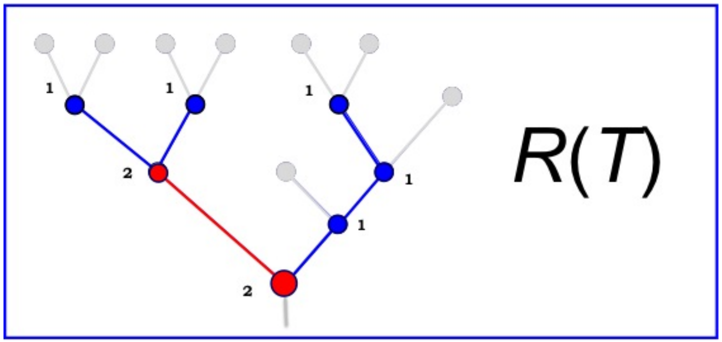

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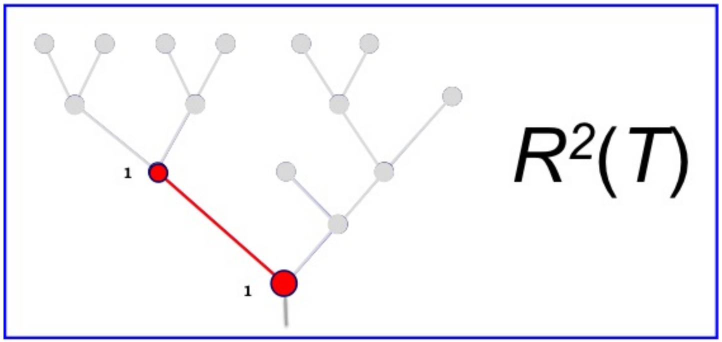

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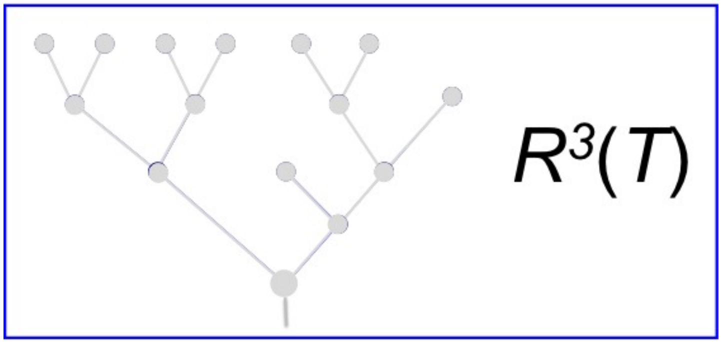

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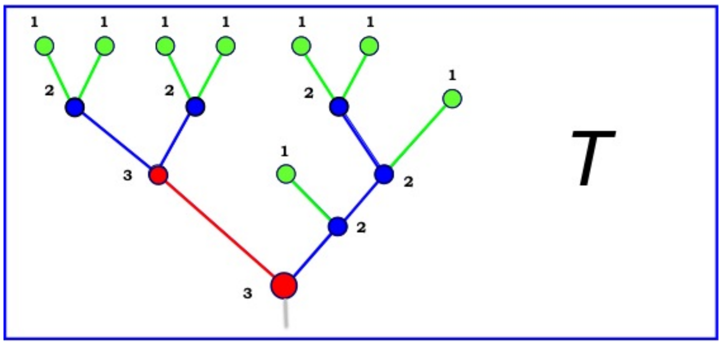
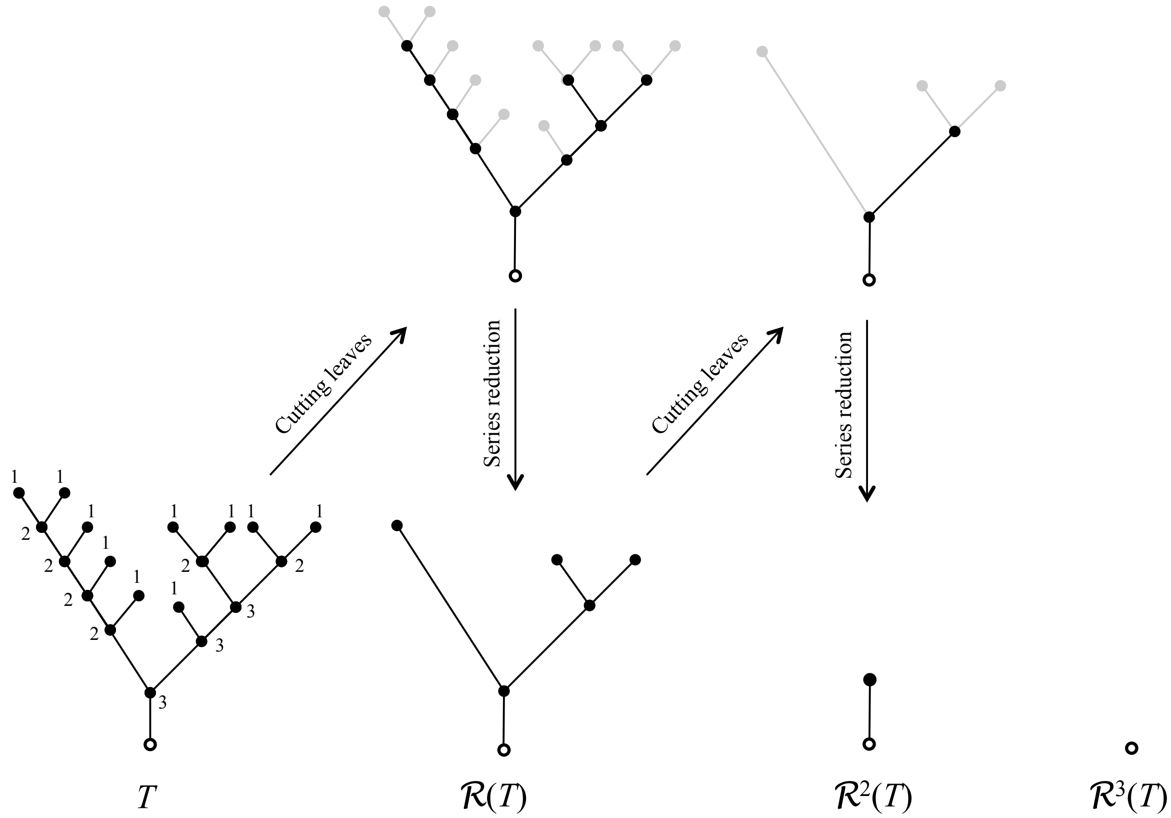
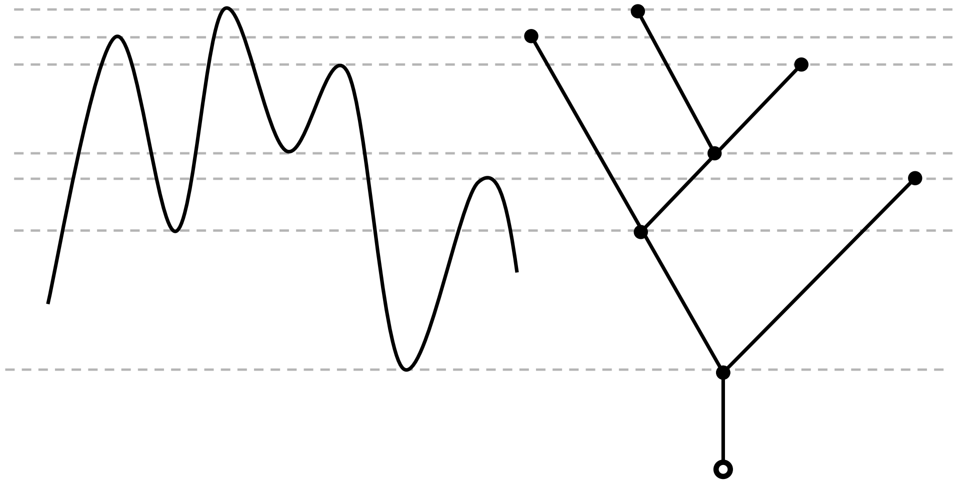


Pruning of a tree mod series reduction



Level-set tree of a function.



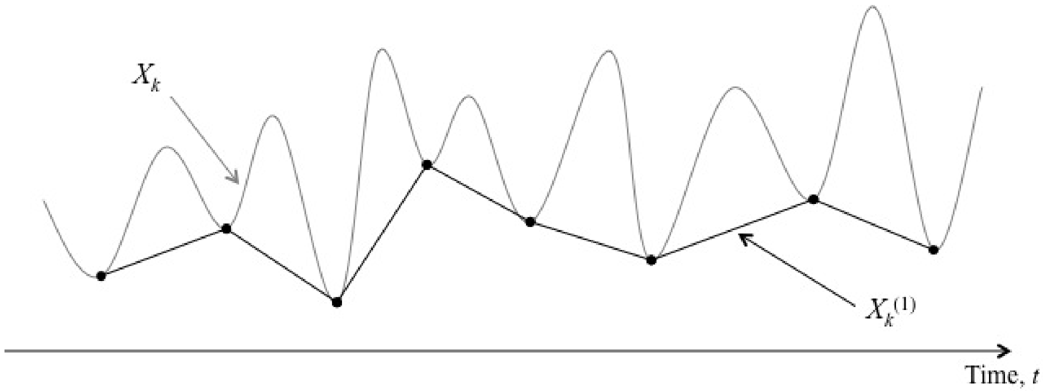
(a) Function X_t (b) Tree LEVEL(X)

Function X_t (panel a) with a finite number of local extrema and its level-set tree level(X) (panel b).

Horton pruning of time series

The transition from a time series X_k to the time series $X_k^{(1)}$ of its local minima corresponds to Horton pruning of the level-set tree level(X).

Zaliapin and YK, CSF (2012).



Exponential critical binary Galton-Watson tree

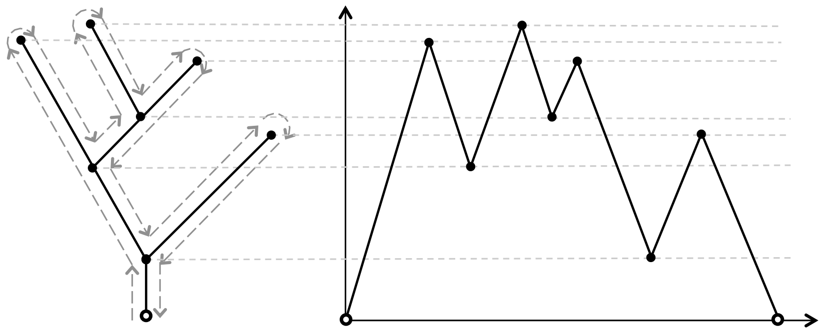
We say that a random tree $T \in \mathcal{L}_{\text{plane}}$ is an exponential critical binary Galton-Watson tree with parameter $\lambda > 0$, and write $T \stackrel{d}{=} \mathsf{GW}(\lambda)$, if

(i) shape(T) is a critical binary Galton-Watson tree;

(ii) the orientation for every pair of siblings in T is uniformly random and symmetric;

(iii) given shape(T), the edges of T are sampled as independent exponential random variables with parameter λ.

Exponential critical binary Galton-Watson tree



(a) Tree T (b) Harris path H_T

The level set tree level(X_t) is an exponential critical binary Galton-Watson tree GW(λ) if and only if the rises and falls of X_t, excluding the last fall, are distributed as independent exponential random variables with parameter $\lambda/2$.

J. Neveu and J. Pitman (1989), J. F. Le Gall (1993)

Invariance under pruning

Theorem. [M. Arnold, YK, I. Zaliapin, 2017]

Let $T \stackrel{d}{=} \mathsf{GW}(\lambda)$ be an exponential critical binary Galton-Watson tree with parameter $\lambda > 0$.

Then, for any monotone non-decreasing function $\varphi : \mathcal{L}_{\text{plane}} \to \mathbb{R}^+$ we have

$$T^t := \big\{\mathcal{S}_t(\varphi, T) | \mathcal{S}_t(\varphi, T) \neq \phi\big\} \stackrel{d}{=} \mathsf{GW}\big(\lambda p_t(\lambda, \varphi)\big),$$

where $p_t(\lambda, \varphi) = \mathsf{P}(\mathcal{S}_t(\varphi, T) \neq \phi)$.

That is, the pruned tree T^t conditioned on surviving is an exponential critical binary Galton-Watson tree with parameter

$$\mathcal{E}_t(\lambda, \varphi) = \lambda p_t(\lambda, \varphi).$$

Invariance under pruning

Theorem. [M. Arnold, YK, I. Zaliapin, 2017]

(a) If $\varphi(T)$ equals the total length of T ($\varphi = \text{length}(T)$), then

$$\mathcal{E}_t(\lambda, \varphi) = \lambda e^{-\lambda t}\Big[I_0(\lambda t) + I_1(\lambda t)\Big].$$

(b) If $\varphi(T)$ equals the height of T ($\varphi = \text{height}(T)$), then

$$\mathcal{E}_t(\lambda, \varphi) = \frac{2\lambda}{\lambda t + 2}.$$

(c) If $\varphi(T) + 1$ equals the Horton-Strahler order of the tree T, then

$$\mathcal{E}_t(\lambda, \varphi) = \lambda 2^{-\lfloor t \rfloor}.$$

Distributional prune-invariance

Definition. Consider a probability measure μ on $\mathcal{L}_{\text{plane}}$ such that $\mu(\phi) = 0$. Let

$$\nu(T) = \mu \circ \mathcal{S}_t^{-1}(T) = \mu\big(\mathcal{S}_t^{-1}(T)\big).$$

Measure μ is called invariant with respect to the pruning operator $\mathcal{S}_t(\varphi, T)$ if for any tree $T \in \mathcal{L}_{\text{plane}}$ we have

$$\mu(T) = \nu(T | T \neq \phi).$$

Also need the invariance of the distribution of edge lengths in the pruned tree $T_t := \mathcal{S}_t(\varphi, T)$.

YK and I. Zaliapin (2017) - arXiv:1608.05032

Open question: finding and classifying all the invariant probability measures μ on $\mathcal{L}_{\text{plane}}$.

Other prune-invariances

YK and Zaliapin (Fractals 2016)

• Coordination and mean self-similarity imply strong Horton law.

YK & Zaliapin (AIHP 2017):

• Established the root-Horton law for the Kingman's coalescent.

• Showed that the tree for Kingman's coalescent is combinatorially equivalent to the level-set tree of iid time series (the two are **one Horton pruning apart**).

Perspectives.

- Time series: extreme values.

- Generalized notions of self-similarity under Horton pruning: **YK and I. Zaliapin (2017) - arXiv:1608.05032**

A class of multi-type branching processes is considered: the Hierarchical Branching Processes

- Generalized dynamical pruning. Burgers equations. **M. Arnold, YK, I. Zaliapin (2017) - arXiv:1707.01984**

- Models of statistical mechanics at criticality.