Random Self-Similar Trees

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collaboration with

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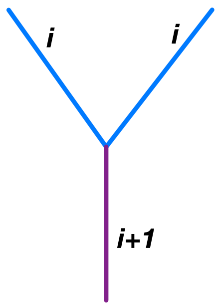
Tribute to Ilya Zaliapin.

I was blessed to have Ilya Zaliapin as a close friend and collaborator with whom we jointly developed the theory of random self-similar trees. I am grateful to Ilya for all the things I learned from him and for his own beautiful scientific world he generously shared with me.

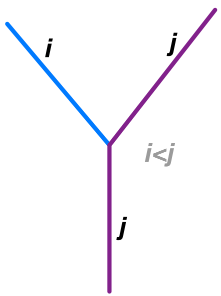
March 6, 1973 - May 2, 2023

The Horton-Strahler orders.



The Horton-Strahler hierarchical ordering scheme was developed by R. E. Horton (1945) and A. N. Strahler (1957) for the analysis of river streams. Each leaf is assigned order 1. At a junction of order i link with an order j link, the new order is determined according to

$$\mathrm{ord}(\ell) = \max(i, j) + \delta_{ij} = \lfloor \log_2(2^i + 2^j) \rfloor.$$



The Horton-Strahler orders are known in computer science as the register function or register number. They are the minimal number of memory registers required for evaluating a binary arithmetic expression [A. P. Ershov, Comm. ACM (1958)].

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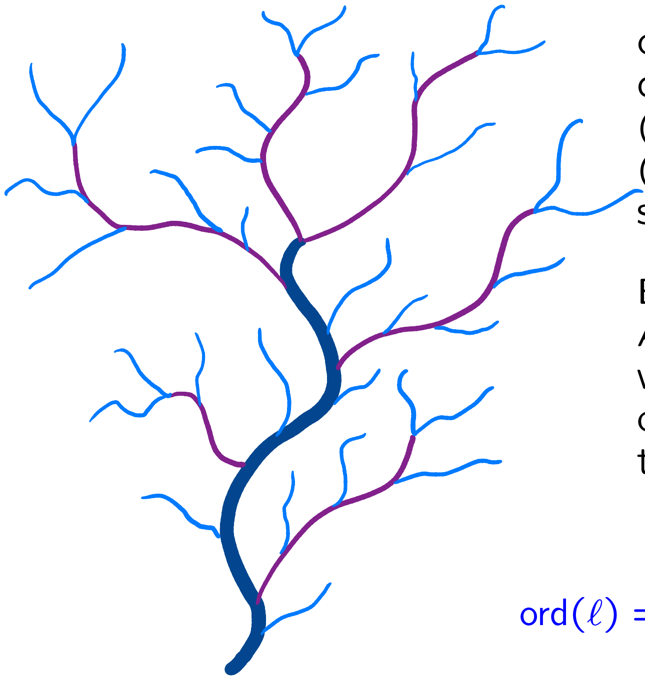

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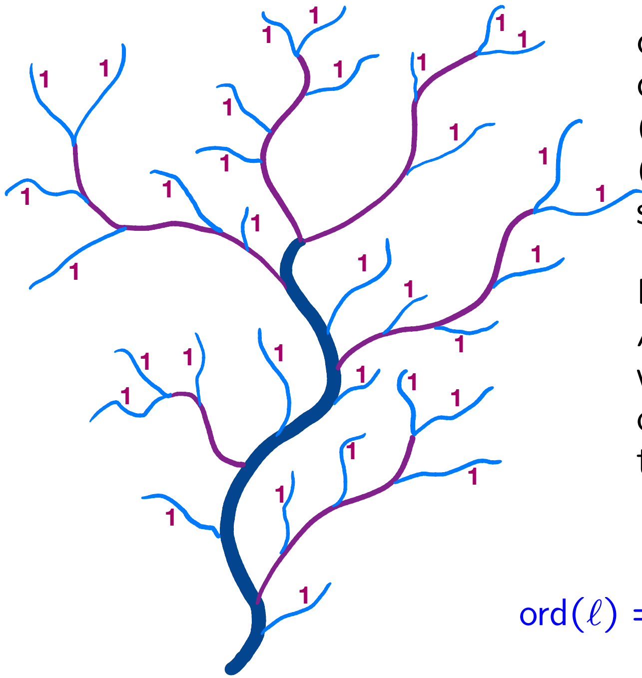

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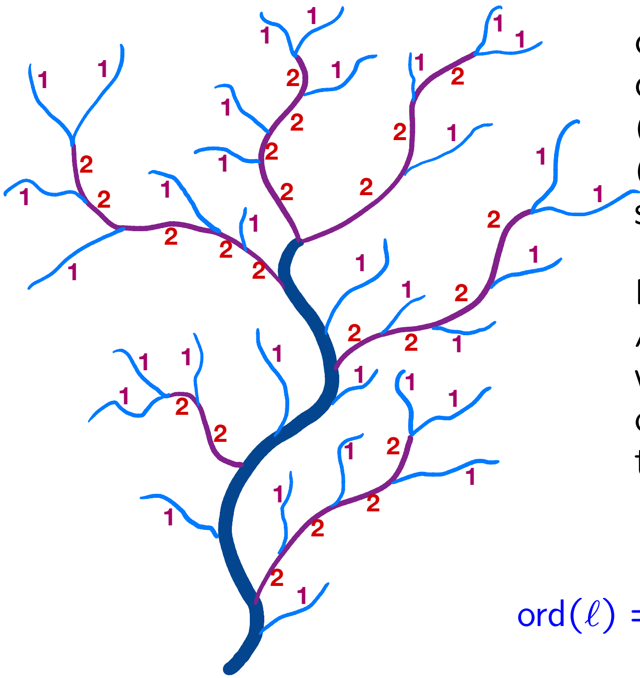

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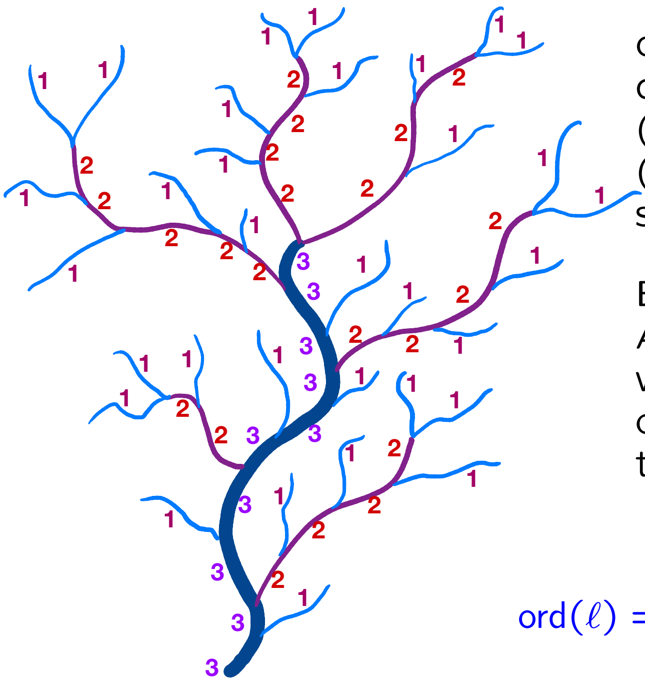

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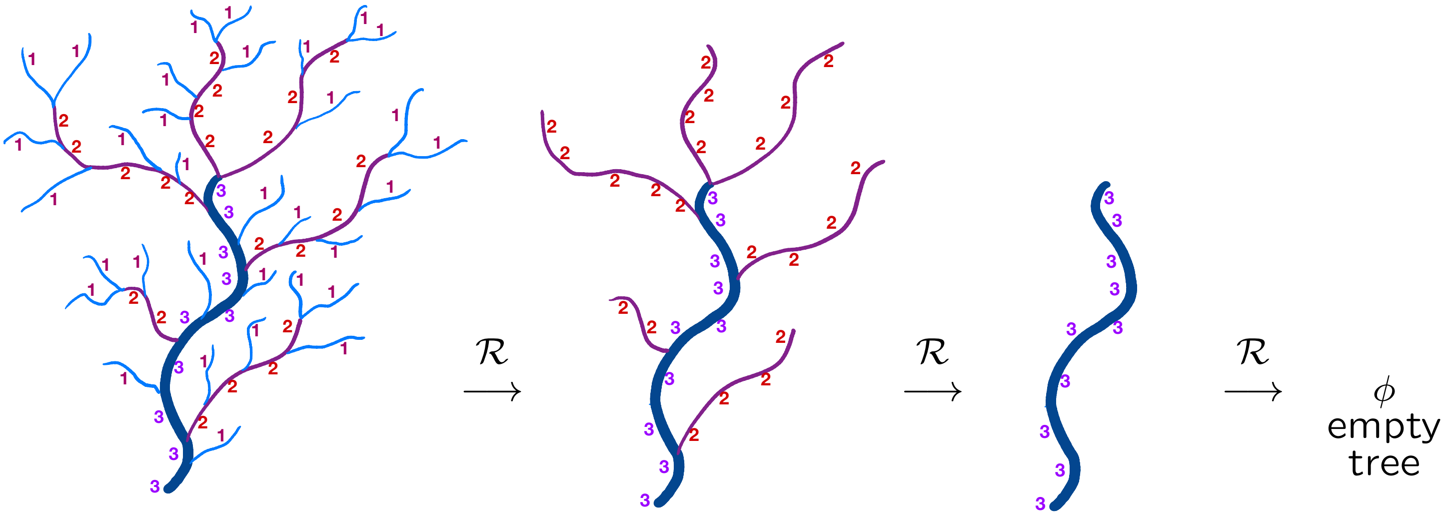
The highest Horton-Strahler order is the order of the entire tree.

The Amazon river has Horton-Strahler order 12.

Horton pruning.

Let T be a reduced (no degree two vertices) rooted tree.

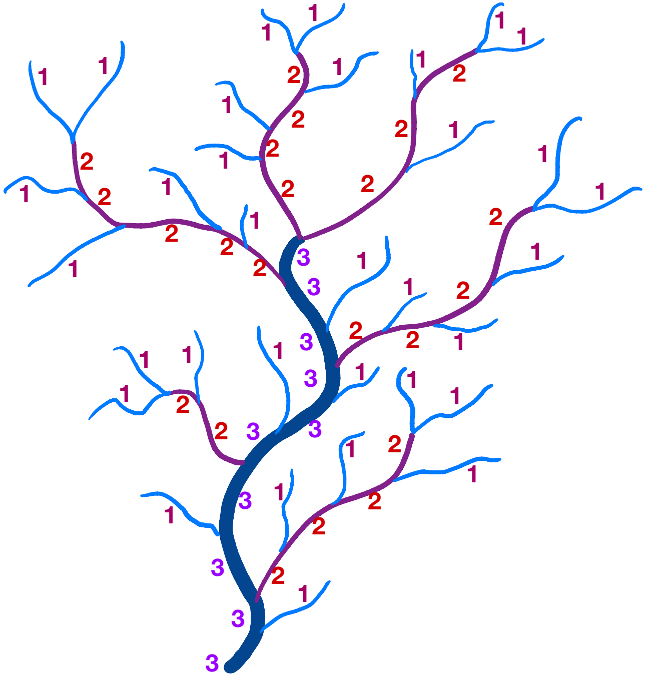


Horton pruning $\mathcal{R}$ is an operation of removing all leaf edges followed by series reduction. Here, $\mathcal{R}(\phi) = \phi$.

Iterating $\mathcal{R}$ induces the Horton-Strahler orders for binary trees and in general: the connected segments in T that were removed after k-th pruning will have the Horton-Strahler order k.

The Horton-Strahler order of T is $\mathrm{ord}(T) = \min\left\{k \geq 0 : \mathcal{R}^k(T) = \phi\right\}$.

The Horton Law.



Let $N_k = N_k[T]$ denote the number of branches of order k in a random tree T.

The Horton Law is satisfied if there exists a parameter $R \geq 2$, called the Horton exponent, such that

$$\frac{N_k}{N_1} \propto R^{1-k}$$

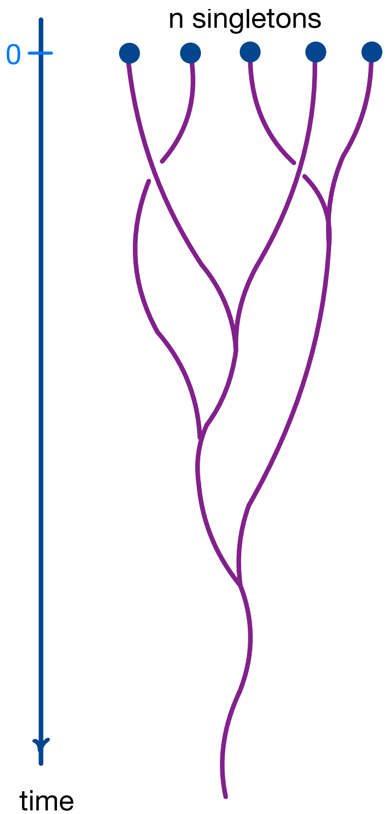
interpreted broadly. For example, it can be convergence in probability or a.s. convergence. The geometric decay can be expressed by taking the limit of the k-th root or the ratio of consecutive terms.

In this example, $N_1 = 33$, $N_2 = 6$, and $N_3 = 1$.

The Horton Law is preserved under Horton pruning: if it holds for a random tree T, it should also hold for $\mathcal{R}(T)$. Indeed,

$$N_k[\mathcal{R}(T)] = N_{k+1}[T].$$

Kingman's Coalescent Tree.



Consider the Kingman's coalescent process that begins with n singletons, where pairs of particles coalesce with rate $\frac{1}{n}$. Let T_n be the tree representing its merger history. It has $N_1[T_n] = n$ leaves.

In [YK and I. Zaliapin, *Ann. I.H.P. Prob.&Stat.* (2017)], the following limit law

$$\frac{N_k[T_n]}{N_1[T_n]} \xrightarrow{p} \mathcal{N}_k$$

is proved in probability for all k.

Determining the hydrodynamic limit yields

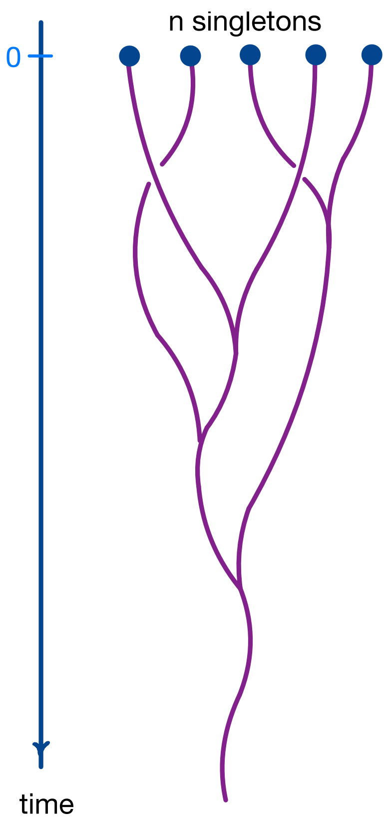
$$\mathcal{N}_k = \frac{1}{2} \int_0^\infty g_k^2(x)\, dx,$$

where $g_k(x)$ solve

$$g'_{k+1}(x) - \frac{g_k^2(x)}{2} + g_k(x) g_{k+1}(x) = 0 \quad (x \geq 0)$$

with $g_1(x) = \frac{2}{x+2}$ and $g_k(0) = 0$ for $k \geq 2$.

Kingman's Coalescent Tree.



Theorem. [YK and I. Zaliapin, *Ann. I.H.P. Prob.&Stat.* (2017)]

$$\lim_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R \quad \text{with} \quad 2 \le R \le 4.$$

Thus, we proved a variant of Horton law:

$$\frac{N_k[T_n]}{N_1[T_n]} \xrightarrow{p} \mathcal{N}_k \propto R^{1-k}$$

The numerical solutions of the ODEs suggest

$$\lim_{k\to\infty} \frac{\mathcal{N}_k}{\mathcal{N}_{k+1}} = \lim_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R$$

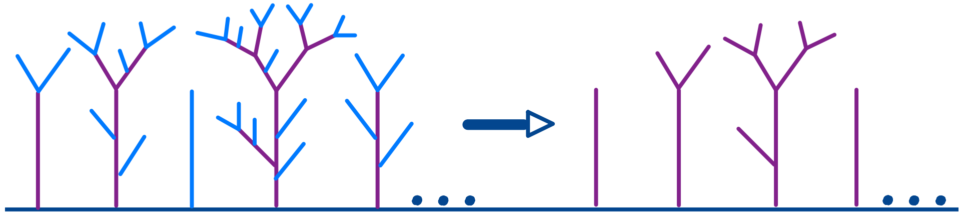
and $\lim_{k\to\infty} (\mathcal{N}_k R^k) = \text{Const.}$ as seen on log-scale with

$$R = 3.043827\ldots.$$

Horton prune-invariance

Recall that the Horton Law $\frac{N_k}{N_1} \propto R^{1-k}$ with the Horton exponent $R \geq 2$ is preserved under Horton pruning: if it holds for a random tree T then it hold for $\mathcal{R}(T)$.

Thus, the Horton Law is a weak form of invariance under Horton-pruning. There is a stronger form of Horton prune-invariance.



For a measure μ on a space of reduced rooted trees such that $\mu(\phi) = 0$ consider the pushforward measure $\nu = \mathcal{R}_*(\mu)$, i.e.,

$$\nu(T) = \mu \circ \mathcal{R}^{-1}(T) = \mu\Big(\mathcal{R}^{-1}(T)\Big).$$

Measure μ is said to be Horton prune-invariant if

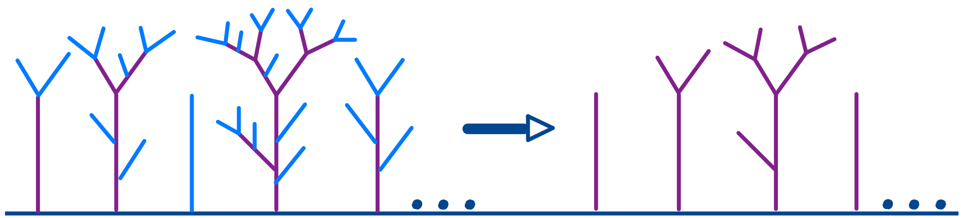
$$\nu\,(T\,|T \neq \phi) = \mu(T) \qquad \forall T \neq \phi.$$

Objective: finding and classifying Horton prune-invariant tree measures.

Attractors

For a tree measure ρ_0 let $\nu_k = \mathcal{R}_*^k(\rho_0)$ denote the pushforward probability measure induced by operator $\mathcal{R}^k$, i.e.,

$$\nu_k(T) = \rho_0 \circ \mathcal{R}^{-k}(T) = \rho_0\big(\mathcal{R}^{-k}(T)\big), \quad \text{and set} \quad \rho_k(T) = \nu_k\left(T \; | T \neq \phi\right).$$



A Horton prune-invariant measure ρ^* is an attractor under Horton pruning if

$$\lim_{k\to\infty} \rho_k = \rho^*.$$

For simplicity, we will use the following notation for this convergence

$$\rho_0 \xrightarrow{\mathcal{R}_*^\infty} \rho^*.$$

Objective: identifying domains of attraction.

Pruning Galton-Watson trees

Consider a Galton-Watson tree measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$.

Theorem. [G. A. Burd, E. C. Waymire, R. D. Winn, *Bernoulli* (2000)]

- Assuming finite second moment $\sum_{k=0}^{\infty} k^2 q_k < \infty$, measure $\mathcal{GW}(\{q_k\})$ is Horton prune-invariant if and only if it is $\mathcal{GW}(q_0 = q_2 = 1/2)$, i.e., critical binary.

- Assume criticality $\sum_{k=0}^{\infty} k q_k = 1$ and finite branching $\Big|\{k : q_k > 0\}\Big| < \infty$.

$$\text{Then,} \qquad \mathcal{GW}(\{q_k\}) \xrightarrow{\mathcal{R}_*^\infty} \mathcal{GW}(q_0 = q_2 = 1/2).$$

- Assume subcriticality $\sum_{k=0}^{\infty} k q_k < 1$, then $\mathcal{GW}(\{q_k\}) \xrightarrow{\mathcal{R}_*^\infty} \mathcal{GW}(q_0 = 1)$.

Moreover, for the Horton prune-invariant measure $\mathcal{GW}(q_0 = q_2 = 1/2)$, they established the Horton law with Horton exponent $R = 4$.

Invariant Galton-Watson measures

For a given $q \in [1/2, 1)$, a critical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with the generating function $Q(z) = \sum_{k=0}^{\infty} q_k z^k$ expressed as

$$Q(z) = z + q(1-z)^{1/q}$$

is called the invariant Galton-Watson (IGW) tree measure with parameter q, and denoted by $\mathcal{IGW}(q)$.

Branching probabilities: $q_0 = q$, $q_1 = 0$, $q_2 = (1-q)/2q$, and

$$q_k = \frac{1-q}{k!\,q} \prod_{i=2}^{k-1} (i - 1/q) \quad (k \geq 3).$$

Here, if $q = 1/2$, then the distribution is critical binary, i.e., $\mathcal{GW}(q_0 = q_2 = 1/2)$.

If $q \in (1/2, 1)$, the distribution is of Zipf type with

$$q_k = \frac{(1-q)\Gamma(k-1/q)}{q\Gamma(2-1/q)\,k!} \sim C k^{-(1+q)/q}, \quad \text{where} \quad C = \frac{1-q}{q\,\Gamma(2-1/q)}.$$

This family of tree measures is also known as stable Galton-Watson trees.

Invariant Galton-Watson measures

Consider $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$ and generating function $Q(x) = \sum_{k=0}^{\infty} q_k x^k$.

Assumption 1. Limit $\displaystyle\lim_{x\to 1-} \frac{Q(x)-x}{(1-x)\big(1-Q'(x)\big)}$ exists.

Theorem [YK and I. Zaliapin, *Bernoulli* (2021)].

• If Assumption 1 is satisfied, then measure $\mathcal{GW}(\{q_k\})$ is Horton prune-invariant if and only if it is $\mathcal{IGW}(q_0)$.

• Assume criticality $\sum_{k=0}^{\infty} k q_k = 1$ and suppose Assumption 1 is satisfied. Then, $\mathcal{GW}(\{q_k\}) \xrightarrow{\mathcal{R}_*^\infty} \mathcal{IGW}(q)$ with $q = \lim_{x\to 1-} \frac{Q(x)-x}{(1-x)(1-Q'(x))}$.

Corollary [YK and I. Zaliapin, *Bernoulli* (2021)].

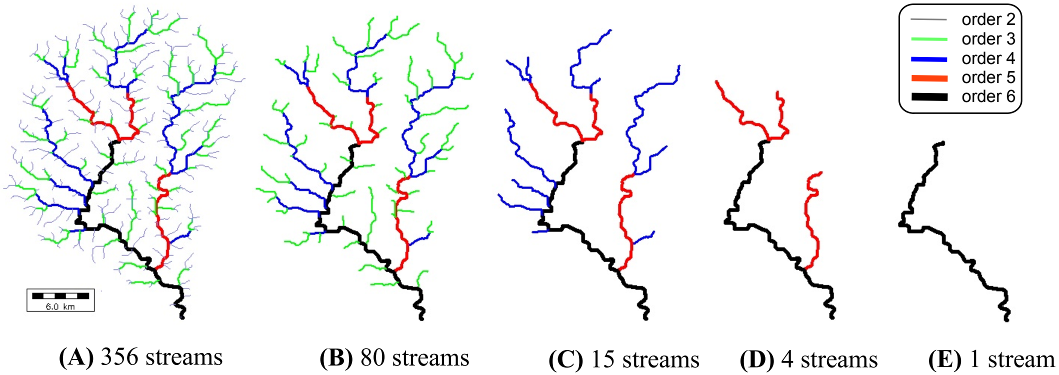
Suppose the offspring distribution q_k is of Zipf type:

$$q_k \sim Ck^{-(\alpha+1)} \quad \text{with} \quad \alpha \in (1,2] \quad \text{and} \quad C > 0.$$

Then, $\mathcal{GW}(\{q_k\}) \xrightarrow{\mathcal{R}_*^\infty} \mathcal{IGW}(q)$ with $q = \frac{1}{\alpha}$.

For $\mathcal{IGW}(q)$ tree, we established the Horton law with $R = (1-q)^{-1/q}$.

The Critical Tokunaga tree.

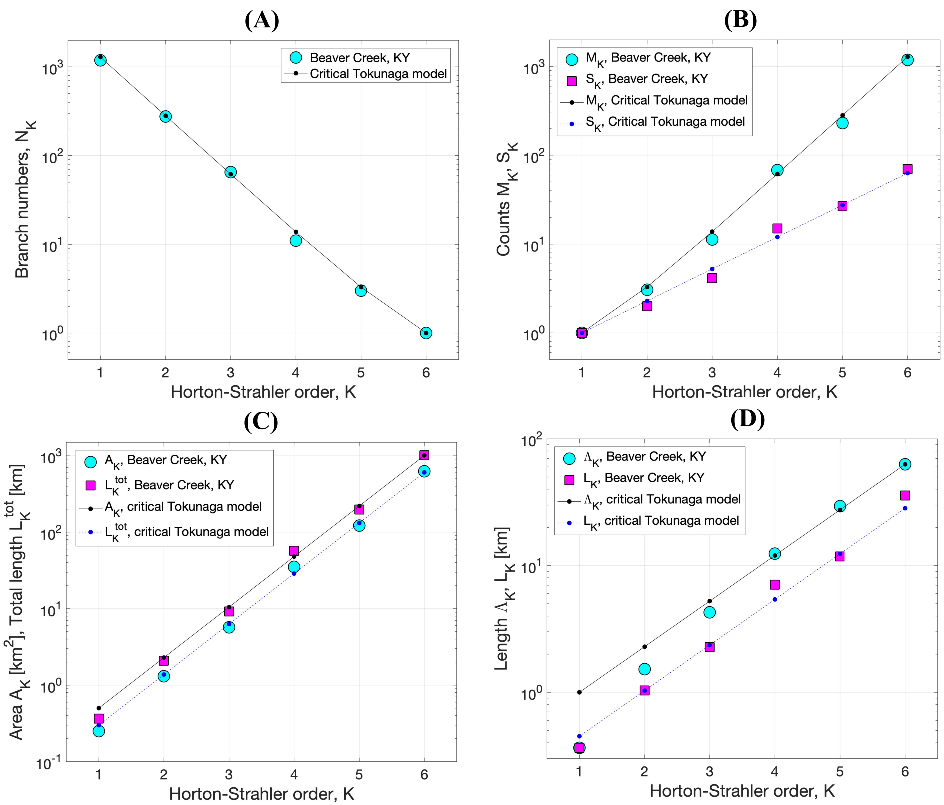


Combining the properties of a river tree model yields a continuous-time multi-type branching process, we named the critical Tokunaga process (model), whose branching structure depends on parameter $c > 1$ and edge-lengths distribution depends on $\gamma > 0$.

For $c = 2$, the critical Tokunaga model yields $\mathcal{GW}(q_0 = q_2 = 1/2)$.

- [YK, I. Zaliapin, E. Foufoula-Georgiou, *Surv. Geophys.* (2022)]
- [YK, I. Zaliapin, E. Foufoula-Georgiou, *Phys. Rev. E* (2022)]
- [YK and I. Zaliapin, *Probability Surveys* (2020)]
- [YK and I. Zaliapin, *Stoc. Proc. Appl.* (2019)]
- [YK and I. Zaliapin, *Chaos* (2018)]

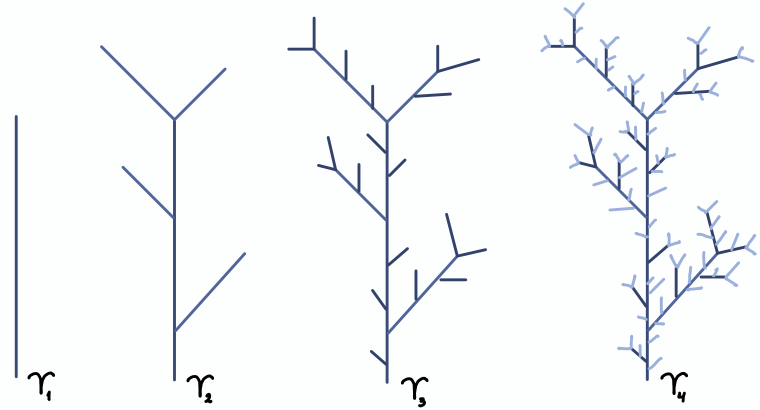
Critical Tokunaga Model closely fits observations.



Critical Tokunaga ($c = 2.3$) fit for hydrological quantities of Beaver Creek, KY.

A random attachment model (RAM).

Consider the following random attachment process $\{\Upsilon_K\}_{K\in\mathbb{N}}$.



- Υ_1 is an I-shaped tree of length $Y_1 \stackrel{d}{\sim} \text{Exp}(\gamma)$.
- Conditioned on Υ_K, tree Υ_{K+1} is obtained as follows:

  (i) Attach new leaf edges to Υ_K at the points sampled with a homogeneous Poisson point process of intensity $\gamma(c-1)$ along the carrier space Υ_K.

  (ii) Attach a pair of new leaf edges to each leaf of Υ_K.

  The lengths of all newly attached leaf edges are i.i.d. exponential random variables with parameter γc^K.

Proving the Horton Law via Martingales.

Observe that each Υ_K is a binary tree of order K. We proved that the critical Tokunaga tree of order K is equivalent to $c^{K-1}\Upsilon_K$.

Let $X_K = N_1[\Upsilon_K]$ (number of leaves) and $Y_K = \text{length}(\Upsilon_K)$.

Lemma [YK and I. Zaliapin, *Probability Surveys* (2020)].

The sequence

$$M_K = R^{1-K}\left(X_K + \gamma(c-1)c^{K-1}Y_K\right) \quad \text{with } K \in \mathbb{N}$$

is a martingale with respect to the process $\left\{\Upsilon_K\right\}_{K\in\mathbb{N}}$.

The above Lemma is used in the proof of the Horton Law.

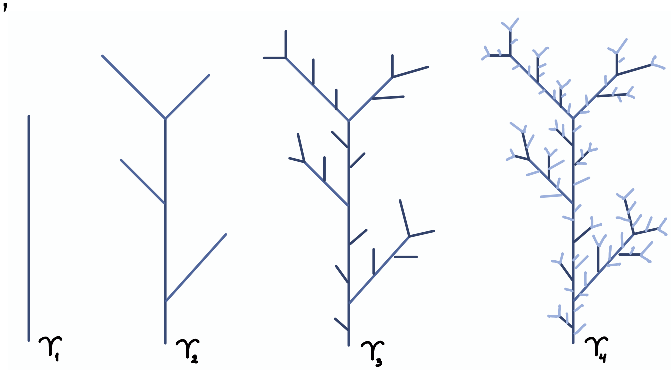
Theorem [YK and I. Zaliapin, *Probability Surveys* (2020)].

$$\frac{N_k[\Upsilon_K]}{N_1[\Upsilon_K]} \xrightarrow{a.s.} R^{1-k} \quad \text{as } K \to \infty, \quad \text{where } R = 2c.$$

Hydrological laws for the RAM.

For the limit $\Upsilon_\infty = \lim_{K\to\infty} \Upsilon_K = \bigcup_{K=1}^{\infty} \Upsilon_K$,

we established



- Fractal dimension:

$$\mathbf{d} = \frac{\log(2c)}{\log c}.$$

- Hack's law: for a link of order i, let A_i denote the local contributing area and Λ_i denote the link's length. Then,

$$\Lambda_i \sim \text{Const.} \times \left(A_i\right)^{\mathbf{h}}, \quad \text{where} \quad \mathbf{h} = \mathbf{d}^{-1} = \frac{\log c}{\log(2c)}.$$

See [YK, I. Zaliapin, E. Foufoula-Georgiou, *Phys. Rev. E* (2022)] and [YK, I. Zaliapin, E. Foufoula-Georgiou, *Surv. Geophys.* (2022)].

Metric Galton-Watson trees.

The following are common notations: for a metric tree T,

$$\text{length}(T) = \text{ the sum of the lengths of edges in } T,$$

$$\text{height}(T) = \text{ the maximal distance to the root } \rho,$$

$$\text{shape}(T) = \text{ combinatorial shape of } T.$$

Continuous Galton-Watson measure: for p.m.f. $\{q_k\}$ and $\lambda > 0$,

$$T \stackrel{d}{=} \mathcal{GW}(\{q_k\}, \lambda) \quad \text{if} \quad \text{shape}(T) \stackrel{d}{=} \mathcal{GW}(\{q_k\})$$

and, conditioned on shape(T), the edges of T are i.i.d. Exp(λ).

Exponential invariant Galton-Watson (IGW) measure: for a given $q \in [1/2, 1)$ and $\lambda > 0$,

$$T \stackrel{d}{=} \mathcal{IGW}(q, \lambda) \quad \text{if} \quad \text{shape}(T) \stackrel{d}{=} \mathcal{IGW}(q)$$

and, conditioned on shape(T), the edges of T are i.i.d. Exp(λ).

Exponential Invariant Galton-Watson trees.

Theorem. [YK, G. Xu, I. Zaliapin, *Adv. Appl. Prob.* (2023)]
Consider $q \in [1/2, 1)$, $\lambda > 0$, and $T \stackrel{d}{=} \mathcal{IGW}(q, \lambda)$. Then,

(a) $\mathsf{P}\Big(\text{length}(T) \le x\Big) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\Gamma(n/q+1)}{n!n!\Gamma(n/q-n+2)}\,(\lambda q)^n x^n.$

Here, for $q = \frac{1}{2}$, $\mathsf{P}\Big(\text{length}(T) \le x\Big) = 1 - e^{-\lambda x}\Big(I_0(\lambda x) + I_1(\lambda x)\Big).$

(b) $\mathsf{P}\Big(\text{length}(T) > x\Big) \sim \frac{1}{(\lambda q)^q\,\Gamma(1-q)}\,x^{-q}.$

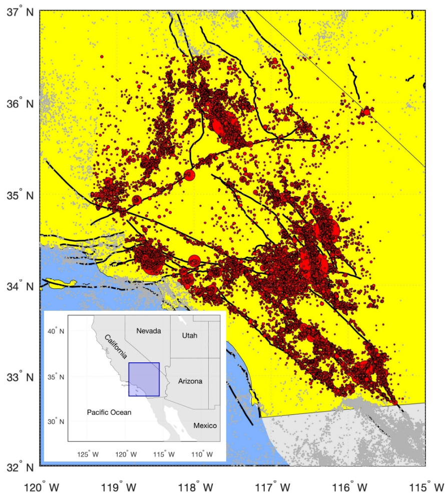
(c) $\mathsf{P}\Big(\text{height}(T) \le x\Big) = 1 - \Big(\lambda(1-q)x + 1\Big)^{-q/(1-q)}.$

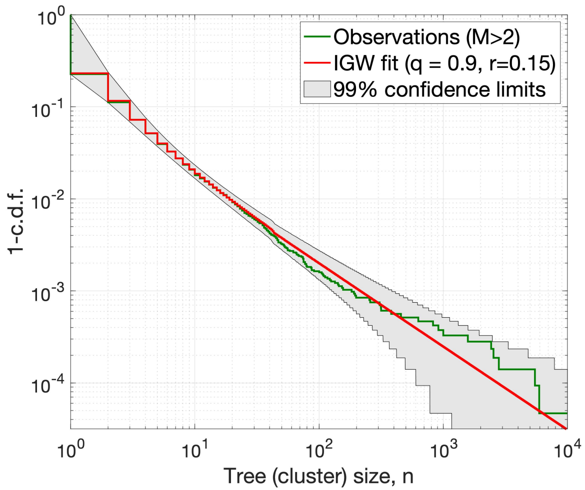
(d) $\mathsf{P}\Big(\#\text{ of edges in } T = n\Big) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \frac{\Gamma(k/q+1)}{k!\,\Gamma(k/q-k+2)}\,q^k$

for $n = 1, 2, \ldots$.

Applications in seismology.

In [YK, I. Zaliapin, Y. Ben-Zion, *Geophys. J. Intl.* (2022)], we analyzed the observed seismicity in southern California and demonstrated that the IGW model provides a close fit to the observed earthquake clusters.

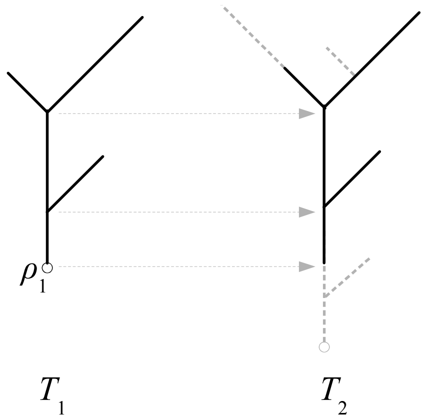




Left: [Hauksson et al. (2012)] Southern California seismicity catalog for 1981-2019 (magnitude ≥ 2);

Right: IGW fit to the empirical cluster sizes.

Generalized dynamical pruning.



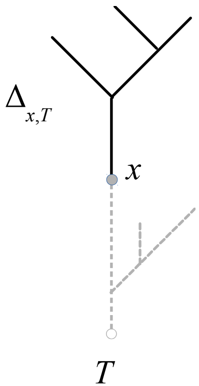
Consider a monotone non-decreasing function $\varphi(T)$ on trees: $\varphi(T_1) \le \varphi(T_2)$ whenever $T_1 \preceq T_2$, i.e., T_1 can be inscribed into T_2 via an isometry.

Generalized dynamical pruning: for any $t \ge 0$, let

$$\mathcal{S}_t(\varphi, T) = \{\text{root } \rho\} \cup \Big\{x \in T : \varphi\big(\Delta_{x,T}\big) \ge t\Big\}$$

It cuts all descendant trees $\Delta_{x,T}$ for which $\varphi\big(\Delta_{x,T}\big)$ is below threshold t. Here,

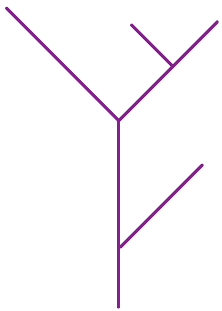
$$\mathcal{S}_s(\varphi, T) \preceq \mathcal{S}_t(\varphi, T) \quad \text{whenever} \quad s \ge t.$$



Example (Tree height). Let $\varphi(T) = \text{height}(T)$, then $\mathcal{S}_t(\varphi, T)$ represents the tree erasure as studied in [J. Neveu, *Adv. Appl. Prob.* (1986)].

Example (Tree length). Let $\varphi(T) = \text{length}(T)$, then $\mathcal{S}_t(\varphi, T)$ represents the potential dynamics of 1D continuum ballistic annihilation studied in [YK and I. Zaliapin, *JSP* (2020)].

Generalized dynamical pruning.



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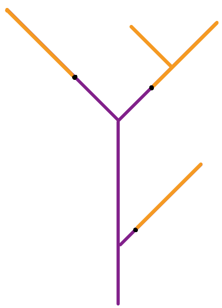


Example (Tree height). Let $\varphi(T) = \text{height}(T)$, then $\mathcal{S}_t(\varphi, T)$ represents the tree erasure as studied in [J. Neveu, *Adv. Appl. Prob.* (1986)].

Example (Tree length). Let $\varphi(T) = \text{length}(T)$, then $\mathcal{S}_t(\varphi, T)$ represents the potential dynamics of 1D continuum ballistic annihilation studied in [YK and I. Zaliapin, *JSP* (2020)].

A related tree reduction.

[T. Duquesne and M. Winkel, *SPA* (2019)] introduced a very general kind of tree reduction, called hereditary reduction, in the context of complete locally compact rooted (CLCR) real trees. The notion of hereditary reduction is a generalization of tree erasure in [J. Neveu, *Adv. Appl. Prob.* (1986)] and trimming in [S. N. Evans, *Saint-Flour Lect.* (2006)] .

[T. Duquesne and M. Winkel, *SPA* (2019)]: tree measures $\mathcal{IGW}(q,\lambda)$ are invariant with respect to hereditary reduction.

Prune-invariance. We established an analogous result.

Theorem. [YK, G. Xu, I. Zaliapin, *Adv. Appl. Prob.* (2023)]

Consider a monotone non-decreasing function $\varphi(T) \geq 0$. Then,

$$T \stackrel{d}{=} \mathcal{IGW}(q,\lambda) \;\;\Rightarrow\;\; \Big\{\mathcal{S}_t(\varphi,T)\Big|\mathcal{S}_t(\varphi,T) \neq \phi\Big\} \stackrel{d}{=} \mathcal{IGW}\Big(q, \lambda p_t^{(1-q)/q}\Big),$$

where $p_t = \mathsf{P}(\varphi(T) > t)$ is assumed to be positive for all $t \geq 0$.

Recall that $p_t = 1 - \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}\Gamma(n/q+1)}{n!n!\Gamma(n/q-n+2)}(\lambda q)^n t^n$ for $\varphi(T) = \text{length}(T)$,

and $p_t = \Big(\lambda(1-q)t+1\Big)^{-q/(1-q)}$ for $\varphi(T) = \text{height}(T)$.

Attraction property of critical Galton-Watson trees

Consider $\mathcal{GW}(\{q_k\}, \lambda)$ with $q_1 = 0$ and generating function $Q(x) = \sum_{k=0}^{\infty} q_k x^k$.

Assumption 1. Limit $\displaystyle\lim_{x\to 1-} \frac{Q(x)-x}{(1-x)\big(1-Q'(x)\big)}$ exists.

Theorem. [YK, G. Xu, I. Zaliapin, *Adv. Appl. Prob.* (2023)]

Consider a monotone non-decreasing function $\varphi(T) \geq 0$.

- Assume criticality $\sum_{k=0}^{\infty} k q_k = 1$ and suppose Assumption 1 is satisfied. Then, for $T \stackrel{d}{\sim} \mathcal{GW}(\{q_k\}, \lambda)$,

$$\lim_{t\to\infty} \mathbb{P}\Big(\text{shape}(\mathcal{S}_t(T)) = \cdot \,\Big|\, \mathcal{S}_t(T) \neq \phi\Big) = \mathcal{IGW}(q) \quad \text{with } q = \lim_{x\to 1-} \frac{Q(x)-x}{(1-x)(1-Q'(x))}.$$

- Assume subcriticality $\sum_{k=0}^{\infty} k q_k < 1$, then for $T \stackrel{d}{\sim} \mathcal{GW}(\{q_k\}, \lambda)$,

$$\lim_{t\to\infty} \mathbb{P}\Big(\text{shape}(\mathcal{S}_t(T)) = \cdot \,\Big|\, \mathcal{S}_t(T) \neq \phi\Big) = \mathcal{GW}(q_0 = 1).$$

Attraction property of critical Galton-Watson trees

Corollary. [YK, G. Xu, I. Zaliapin, *Adv. Appl. Prob.* (2023)]

Consider a critical Galton-Watson measure $\mathcal{GW}(\{q_k\},\lambda)$ with $q_1 = 0$, with offspring distribution q_k of Zipf type:

$$q_k \sim Ck^{-(\alpha+1)} \quad \text{with} \quad \alpha \in (1,2] \quad \text{and} \quad C > 0.$$

Then, for $T \stackrel{d}{\sim} \mathcal{GW}(\{q_k\},\lambda)$,

$$\lim_{t\to\infty} \mathbb{P}\Big(\text{shape}(\mathcal{S}_t(T)) = \cdot \,\Big|\, \mathcal{S}_t(T) \neq \phi\Big) = \mathcal{IGW}(q) \quad \text{with} \quad q = \frac{1}{\alpha}.$$

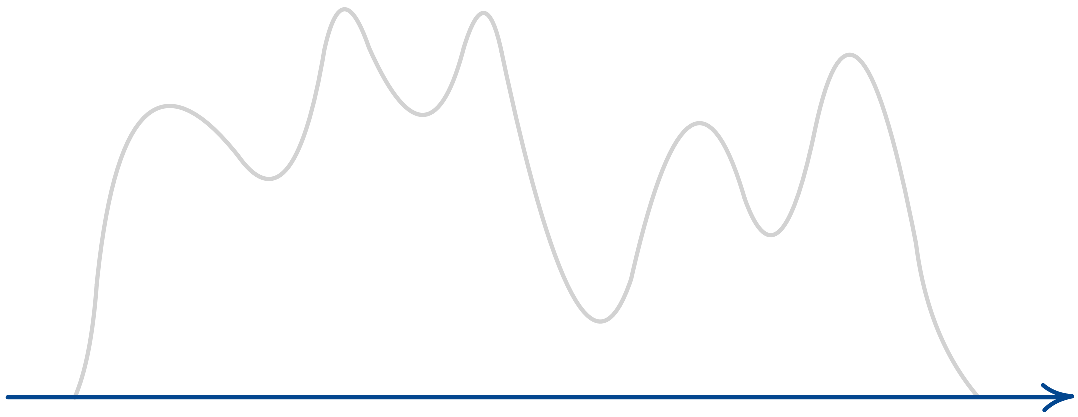
Corollary. [YK, G. Xu, I. Zaliapin, *Adv. Appl. Prob.* (2023)]

Consider a critical Galton-Watson measure $\mathcal{GW}(\{q_k\},\lambda)$ with $q_1 = 0$ such that $\sum\limits_{k=2}^{\infty} k^2 q_k < \infty$.

Then, for $T \stackrel{d}{\sim} \mathcal{GW}(\{q_k\},\lambda)$,

$$\lim_{t\to\infty} \mathbb{P}\Big(\text{shape}(\mathcal{S}_t(T)) = \cdot \,\Big|\, \mathcal{S}_t(T) \neq \phi\Big) = \mathcal{IGW}(1/2) \quad \text{(critical binary)}.$$

Level Set Trees.



Consider a continuous excursion X_t. Its level set tree level(X_t) tracks the branching/termination history of the connected components in the level sets $\mathcal{L}_\alpha = \{t : X_t \geq \alpha\}$ as α increases from 0 up.

Consider excursion $X_t^{(1)}$ obtained by a linear interpolation of the boundary values and the local minima of X_t. We observed:

Proposition. [YK and I. Zaliapin, *Probability Surveys* (2020)].

$$\text{level}\left(X_t^{(1)}\right) = \mathcal{R}\Big(\text{level}(X_t)\Big).$$

This can be iterated: $\text{level}\left(X_t^{(m)}\right) = \mathcal{R}^m\Big(\text{level}(X_t)\Big)$, $m = 1, 2, \ldots$.

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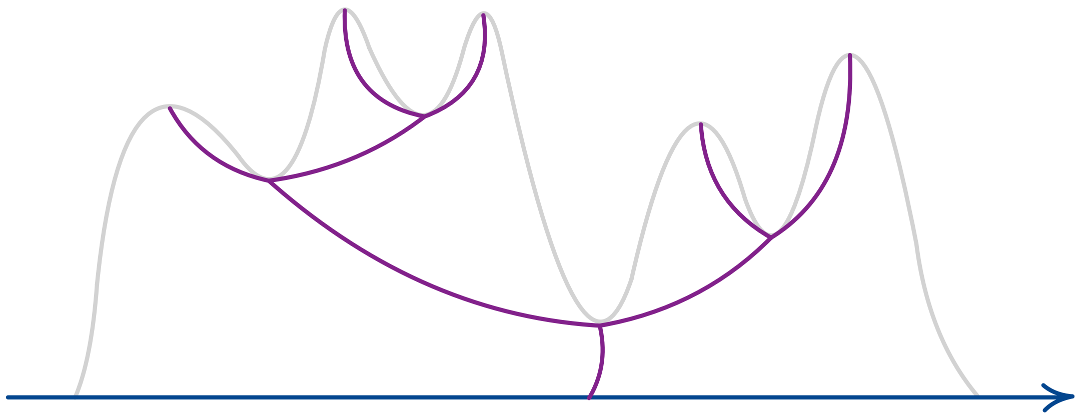

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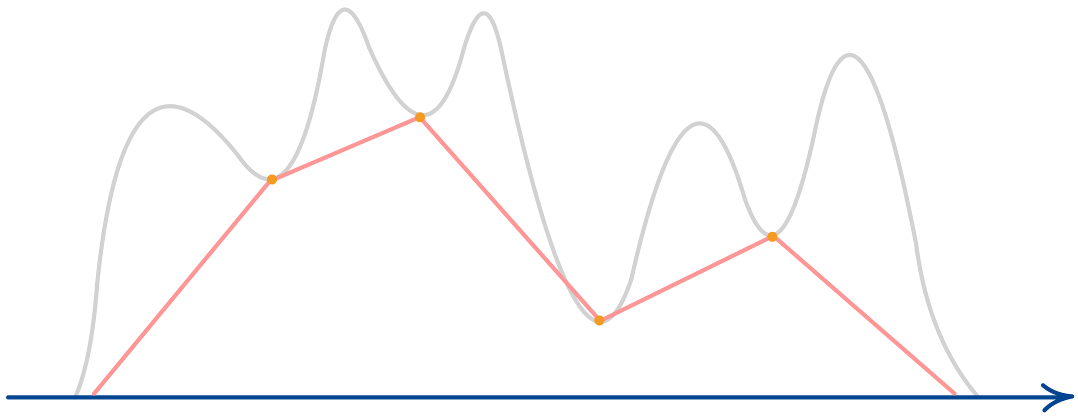

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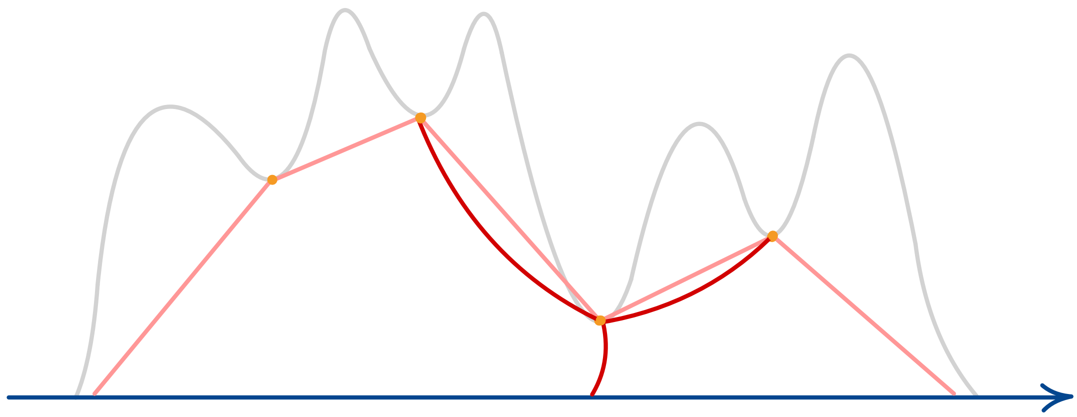

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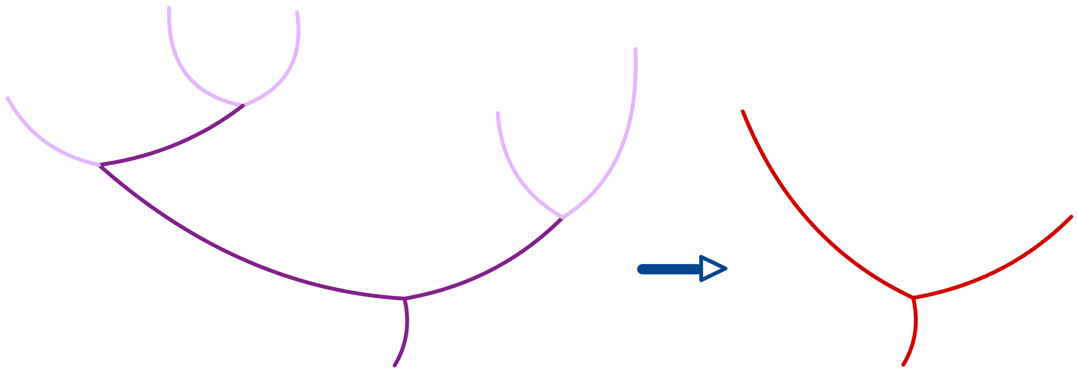

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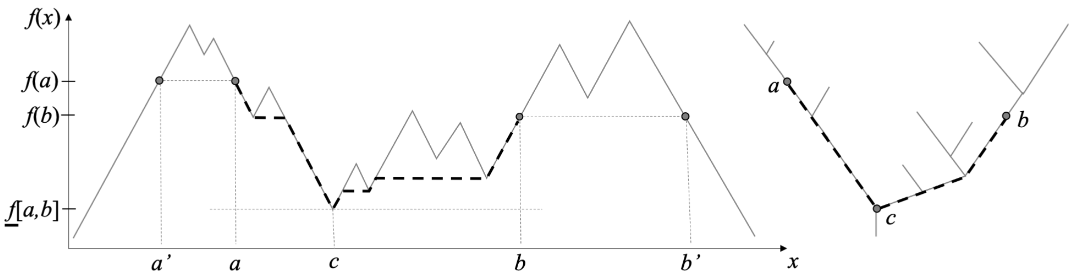

There is a general framework for defining level set trees.

We generalized a well-known result about excursions with Laplace kernels. See [J. Neveu and J. Pitman, *Lect. Notes Math.* (1989)].

Theorem. [YK and I. Zaliapin, *Probability Surveys* (2020)]. Consider a positive excursion X_t induced by a homogeneous random walk on $\mathbb{R}$ with a symmetric atomless transition kernel. Let $T = \text{level}(X_t)$, then shape$(T) \stackrel{d}{\sim} \mathcal{GW}(q_0 = q_2 = 1/2)$.

Moreover, conditioned on shape(T), the edge lengths are i.i.d. if and only if the transition kernel has Laplace distribution.

In particular, this framework yields some of the results in [G. A. Burd, E. C. Waymire, R. D. Winn, *Bernoulli* (2000)].

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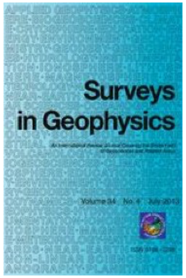

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