A generalization of Abel's binomial theorem

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Abel's binomial theorem.

Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Abel's binomial theorem

$$y^{-1}(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^k (y+n-k)^{n-k-1}$$

Abel's binomial theorem.

$$y^{-1}(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^k (y+n-k)^{n-k-1}$$

Lemma: $\sum_{k=0}^{n} \binom{n}{k} (-1)^k (z-k)^{n-1} = 0 \quad (n = 1, 2, \ldots)$

Induction proof of Abel's binomial theorem:

- Base step: $n = 0$
- Suppose the theorem holds for $n-1$ $(n = 1, 2, \ldots)$:

$$y^{-1}(x+y+n-1)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} (x+k)^k (y+n-1-k)^{n-k-2}$$

Abel's binomial theorem.

- Suppose the theorem holds for $n-1$ $(n = 1, 2, \ldots)$:

$$y^{-1}(x+y+n-1)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} (x+k)^k (y+n-1-k)^{n-k-2}$$

Then, $y^{-1}(x+y+n)^n = y^{-1}(s+y+n-1)^n \Big|_{s=-y-n+1}^{x+1}$

$$= n \int_{-y-n+1}^{x+1} y^{-1}(s+y+n-1)^{n-1} ds$$

$$= n \int_{-y-n+1}^{x+1} \sum_{k=0}^{n-1} \binom{n-1}{k} (s+k)^k (y+n-1-k)^{n-k-2} \, ds$$

$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} \int_{-y-n+1}^{x+1} (s+k)^k ds \, (y+n-1-k)^{n-k-2}$$

Abel's binomial theorem.

- Suppose the theorem holds for $n-1$. Then,

$$y^{-1}(x+y+n)^n = n\sum_{k=0}^{n-1}\binom{n-1}{k}\int_{-y-n+1}^{x+1}(s+k)^k ds\,(y+n-1-k)^{n-k-2}$$

$$=\sum_{k=0}^{n-1} n\binom{n-1}{k}\frac{1}{k+1}(s+k)^{k+1}\bigg|_{s=-y-n+1}^{x+1}(y+n-1-k)^{n-k-2}$$

$$=\sum_{k=0}^{n-1} n\binom{n-1}{k}\frac{1}{k+1}(x+1+k)^{k+1}(y+n-1-k)^{n-k-2}$$

$$-\sum_{k=0}^{n-1} n\binom{n-1}{k}\frac{1}{k+1}(-y-n+1+k)^{k+1}(y+n-1-k)^{n-k-2}$$

Abel's binomial theorem.

- Suppose the theorem holds for $n-1$. Then,

$$y^{-1}(x+y+n)^n = \sum_{k=0}^{n-1} n\binom{n-1}{k}\frac{1}{k+1}(x+1+k)^{k+1}(y+n-1-k)^{n-k-2}$$

$$-\sum_{k=0}^{n-1} n\binom{n-1}{k}\frac{1}{k+1}(-y-n+1+k)^{k+1}(y+n-1-k)^{n-k-2}$$

$$=\sum_{k=0}^{n-1}\binom{n}{k+1}(x+1+k)^{k+1}(y+n-1-k)^{n-k-2}$$

$$-\sum_{k=0}^{n-1}\binom{n}{k+1}(-1)^{k+1}(y+n-1-k)^{n-1}$$

as $n\binom{n-1}{k}\frac{1}{k+1} = n\,\frac{(n-1)!}{k!\,(n-1-k)!}\frac{1}{k+1} = \frac{n!}{(k+1)!\,(n-1-k)!} = \binom{n}{k+1}$

- Suppose the theorem holds for $n-1$. Then,

$$y^{-1}(x+y+n)^n = \sum_{k=0}^{n-1} \binom{n}{k+1} (x+1+k)^{k+1} (y+n-1-k)^{n-k-2}$$

$$- \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^{k+1} (y+n-1-k)^{n-1}$$

$$= \sum_{j=1}^{n} \binom{n}{j} (x+j)^j (y+n-j)^{n-j-1} - \sum_{j=1}^{n} \binom{n}{j} (-1)^j (y+n-j)^{n-1}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (x+j)^j (y+n-j)^{n-j-1} - \sum_{j=0}^{n} \binom{n}{j} (-1)^j (y+n-j)^{n-1}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (x+j)^j (y+n-j)^{n-j-1} \qquad (j=k+1)$$

as $\sum_{k=0}^{n} \binom{n}{k} (-1)^k (z-k)^{n-1} = 0$ for $z = y+n$. $\square$

Abel's binomial theorem.

$$y^{-1}(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^k (y+n-k)^{n-k-1}$$

Lemma: $\sum_{k=0}^{n} \binom{n}{k} (-1)^k (z-k)^{n-1} = 0 \quad (n = 1, 2, \ldots)$

Induction proof of the lemma:

- Base step: $n = 1$
- Suppose the lemma holds for $n-1$ $(n = 2, 3, \ldots)$:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (z-k)^{n-2} = 0$$

Lemma: $\sum_{k=0}^{n} \binom{n}{k} (-1)^k (z-k)^{n-1} = 0 \quad (n = 1, 2, \ldots)$

- Suppose the lemma holds for $n-1$ $(n = 2, 3, \ldots)$:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (z-k)^{n-2} = 0$$

Then, since $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $n > k > 0$,

$$\frac{1}{n-1} \frac{d}{dz} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (z-k)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (z-k)^{n-2}$$

$$= z^{n-2} + (-1)^n (z-n)^{n-2} + \sum_{k=1}^{n-1} \binom{n-1}{k} (-1)^k (z-k)^{n-2} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} (-1)^k (z-k)^{n-2}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (z-k)^{n-2} + \sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^k (z-k)^{n-2}$$

$$= 0 - \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j (z-1-j)^{n-2} = 0 \qquad (j = k-1).$$

Lemma: $\sum_{k=0}^{n} \binom{n}{k}(-1)^k(z-k)^{n-1} = 0 \quad (n = 1, 2, \ldots)$

- Suppose the lemma holds for $n-1$ $(n = 2, 3, \ldots)$:

$$\sum_{k=0}^{n-1} \binom{n-1}{k}(-1)^k(z-k)^{n-2} = 0$$

Then, $\frac{d}{dz} \sum_{k=0}^{n} \binom{n}{k}(-1)^k(z-k)^{n-1} = 0$

and $\sum_{k=0}^{n} \binom{n}{k}(-1)^k(z-k)^{n-1} = c_n$ (constant) for all z.

Plug in $z = 0$ and $z = n$, then, $c_n = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n+k-1}k^{n-1}$

and, as $\binom{n}{j} = \binom{n}{k}$ for $k = n-j$,

$$c_n = \sum_{j=0}^{n} \binom{n}{j}(-1)^j(n-j)^{n-1} = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n+k}k^{n-1} = -c_n$$

Hence, $\sum_{k=0}^{n} \binom{n}{k}(-1)^k(z-k)^{n-1} = c_n = 0.$ □

Abel's binomial theorem: a variation.

$$y^{-1}(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^k (y+n-k)^{n-k-1}$$

Swap x with y, and k with $n-k$. Then,

$$x^{-1}(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-1} (y+n-k)^{n-k}$$

Add the two formulas together:

$$(x^{-1}+y^{-1})(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-1} (y+n-k)^{n-k-1} (x+y+n)$$

Thus,

$$(x^{-1}+y^{-1})(x+y+n)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-1} (y+n-k)^{n-k-1}$$

Abel's binomial theorem: a variation.

$$(x^{-1}+y^{-1})(x+y+n)^{n-1}=\sum_{k=0}^{n}\binom{n}{k}(x+k)^{k-1}(y+n-k)^{n-k-1}$$

Therefore,

$$(x^{-1}+y^{-1})(x+y+n)^{n-1}-x^{-1}(y+n)^{n-1}-y^{-1}(x+n)^{n-1}$$

$$=\sum_{k=1}^{n-1}\binom{n}{k}(x+k)^{k-1}(y+n-k)^{n-k-1}$$

Hence,

$$\sum_{k=1}^{n-1}\binom{n}{k}k^{k-1}(n-k)^{n-k-1}=\lim_{\substack{x\to 0\\ y\to 0}}\sum_{k=1}^{n-1}\binom{n}{k}(x+k)^{k-1}(y+n-k)^{n-k-1}$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x+y+n)^{n-1}-(y+n)^{n-1}}{x}+\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x+y+n)^{n-1}-(x+n)^{n-1}}{y}$$

Abel's binomial theorem: a variation.

$$\sum_{k=1}^{n-1}\binom{n}{k}k^{k-1}(n-k)^{n-k-1} = \lim_{\substack{x\to 0\\ y\to 0}}\frac{(x+y+n)^{n-1}-(y+n)^{n-1}}{x}$$

$$+\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x+y+n)^{n-1}-(x+n)^{n-1}}{y}$$

$$= 2\lim_{h\to 0}\frac{(h+n)^{n-1}-n^{n-1}}{h} = 2\frac{d}{dx}x^{n-1}\Big|_{x=n} = 2(n-1)n^{n-2}.$$

Identity $\sum_{k=1}^{n-1}\binom{n}{k}k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}$ rewrites as

$$n^{n-2} = \frac{1}{2(n-1)}\sum_{k=1}^{n-1}\binom{n}{k}k(n-k)\,k^{k-2}\,(n-k)^{n-k-2}$$ which . . .

Minimal spanning trees.

Identity $\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}$ rewrites as

$$n^{n-2} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)\, k^{k-2}\, (n-k)^{n-k-2}$$

which has everything to do with spanning trees!

In particular,

$T_n =$ the number of spanning trees in a complete graph K_n

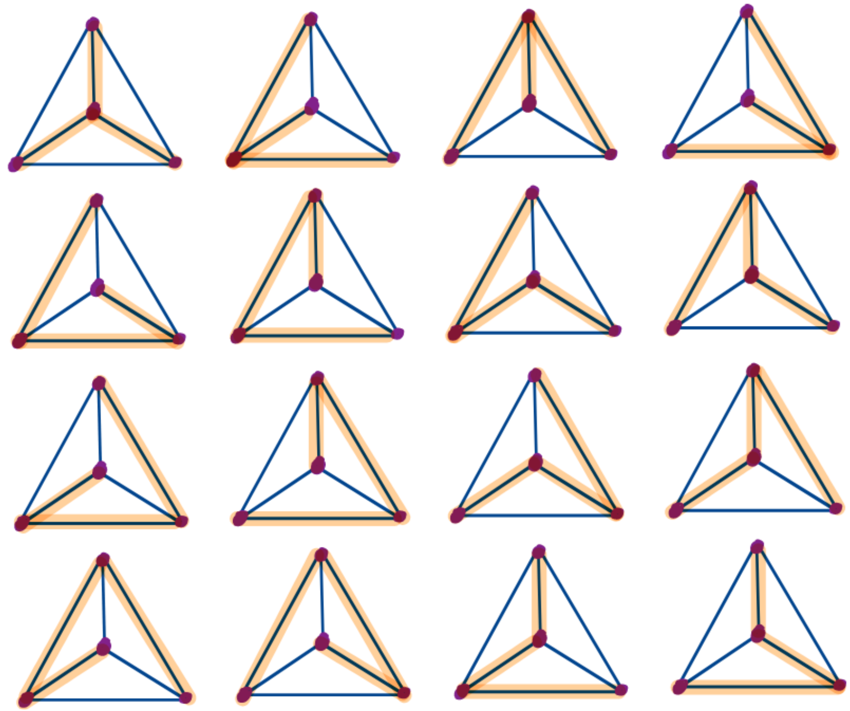
satisfies the following recursion

$$T_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)\, T_k\, T_{n-k}$$

Thus, $T_n = n^{n-2}$

Minimal spanning trees.

$$T_n = n^{n-2} \quad \Rightarrow \quad T_4 = 4^2 = 16$$



Minimal spanning trees.

Abel's binomial theorem yields

$$n^{n-2} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)\, k^{k-2}\, (n-k)^{n-k-2}$$

Equivalently,

$T_n =$ the number of spanning trees in a complete graph K_n satisfies

$$T_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)\, T_k\, T_{n-k}$$

implying $T_n = n^{n-2}$.

Question: Is there a useful generalization?

Minimal spanning trees.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)^\mathsf{T}$ be a column vector, where $x_i \geq 0$ are integers satisfying $\sum_{i=1}^{m} x_i > 0$.

Consider a graph $K_\mathbf{x}(V)$ equipped with edge weights:

- $K_\mathbf{x}(V)$ is a complete graph with $\sum_{i=1}^{m} x_i$ vertices partitioned into m kinds;
- For each $i = 1, \ldots, m$, $K_\mathbf{x}(V)$ has x_i vertices of kind i;
- Let $v_{i,j} = v_{j,i}$ be the weight of each edge connecting a vertex i-th kind to a vertex of j-th kind, and let $V = \left(v_{i,j}\right)$ be the $m \times m$ matrix of weights.

Minimal spanning trees.

Consider a graph $K_{\mathrm{x}}(V)$ equipped with edge weights $V = \left(v_{i,j}\right)$.

If $\mathcal{T}$ is a spanning tree of $K_{\mathrm{x}}(V)$, then the weight of $\mathcal{T}$ is the **product** of the weights of all of its edges.

Let T_{x} denote the weighted spanning tree enumerator of $K_{\mathrm{x}}(V)$, i.e., T_{x} is the sum of weights of all spanning trees of $K_{\mathrm{x}}(V)$.

Theorem (YK and P. T. Otto, 2021).

$$T_{\mathbf{x}} = \frac{1}{2\left(\sum_{i=1}^{m} x_i - 1\right)} \sum_{\mathbf{y},\mathbf{z}:\mathbf{y}+\mathbf{z}=\mathbf{x}} \binom{x_1}{y_1}\binom{x_2}{y_2}\cdots\binom{x_m}{y_m}\left(\mathbf{y}^{\mathsf{T}} V \mathbf{z}\right) T_{\mathbf{y}}\, T_{\mathbf{z}}$$

Minimal spanning trees.

Consider a graph $K_{\mathbf{x}}(V)$ equipped with edge weights $V = \left(v_{i,j}\right)$. Let $T_{\mathbf{x}}$ denote the weighted spanning tree enumerator of $K_{\mathbf{x}}(V)$.

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Example. Let $m = 1$, $\mathbf{x} = x_1 = n$, and $V = v_{1,1} = 1$. Then,

$T_n =$ the number of spanning trees in a complete graph K_n

and

$$T_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)\, T_k\, T_{n-k}$$

Question: Is there a general expression for $T_{\mathrm{x}} = T_{\mathrm{x}}(V)$?

Consider a complete graph K_m consisting of vertices $\{1, \ldots, m\}$ with weights $w_{i,j} = w_{j,i} \geq 0$ assigned to its edges $[i, j]$, let the weight $W(\mathcal{T})$ of a spanning tree $\mathcal{T}$ be the product of the weights of all of its edges.

Denote $\tau(K_m, w_{i,j}) = \sum_{\mathcal{T}} W(\mathcal{T})$ denote the weighted spanning tree enumerator, i.e., the sum of weights of all spanning trees in K_m.

Theorem (YK and P. T. Otto, 2021).

$$T_{\mathrm{x}}(V) = \frac{\tau(K_m, x_i v_{i,j} x_j)}{\prod_{i=1}^{m} x_i} (V\mathbf{x})^{\mathbf{x}-\mathbf{1}}$$

Notation: for vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^m$ let $\mathbf{a}^{\mathbf{b}} = a_1^{b_1} a_2^{b_2} \ldots a_m^{b_m}$
Also, $\mathbf{1} = (1, \ldots, 1)^{\mathsf{T}}$.

Vector-Multiplicative Coalescent Process

Consider m types of particles, $1, \ldots, m$.

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)^{\mathsf{T}}$ be a column vector, where $\alpha_i \geq 0$ are reals satisfying $\sum_{i=1}^{m} \alpha_i > 0$.

Let matrix $V = \left(v_{i,j}\right)$ be nonnegative, irreducible, and symmetric.

Vector-multiplicative coalescent process:

• The process begins with $n \left(\sum_{i=1}^{m} \alpha_i \right) + o(\sqrt{n})$ singletons distributed between the m types so that for each $i = 1, \ldots, m$, there are $\alpha_i n + o(\sqrt{n})$ particles of type i.

• A particle of type i **bonds** with a particle of type j with the intensity (rate) $v_{i,j}/n$. The bonds are formed independently.

Vector-Multiplicative Coalescent Process

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• A particle of type i **bonds** with a particle of type j with the rate $v_{i,j}/n$. The bonds are formed independently.

Hydrodynamic limit as $n\to\infty$:

For $\mathbf{x}=(x_1,x_2,\dots,x_m)^{\mathsf{T}}$ let $\zeta_{\mathbf{x}}(t)$ denote the relative (to n) number of clusters bonding together particles: x_i of type i.

$$\frac{d}{dt}\zeta_{\mathbf{x}}(t)=-\zeta_{\mathbf{x}}(t)\big(\mathbf{x}^{\mathsf{T}}V\boldsymbol{\alpha}\big)+\frac{1}{2}\sum_{\mathbf{y},\mathbf{z}\,:\mathbf{y}+\mathbf{z}=\mathbf{x}}\big(\mathbf{y}^{\mathsf{T}}V\mathbf{z}\big)\zeta_{\mathbf{y}}(t)\,\zeta_{\mathbf{z}}(t),\;\;\zeta_{\mathbf{x}}(0)=\sum_{i=1}^{m}\alpha_i\delta_{\mathbf{e}_i,\mathbf{x}}$$

Vector-Multiplicative Coalescent Process

$$\frac{d}{dt}\zeta_{\mathbf{x}}(t) = -\zeta_{\mathbf{x}}(t)\left(\mathbf{x}^{\mathsf{T}} V \boldsymbol{\alpha}\right) + \frac{1}{2} \sum_{\mathbf{y},\mathbf{z}\,:\mathbf{y}+\mathbf{z}=\mathbf{x}} \left(\mathbf{y}^{\mathsf{T}} V \mathbf{z}\right) \zeta_{\mathbf{y}}(t)\, \zeta_{\mathbf{z}}(t), \quad \zeta_{\mathbf{x}}(0) = \sum_{i=1}^{m} \alpha_i \delta_{\mathbf{e}_i,\mathbf{x}}$$

Solution: Plug in

$$\zeta_{\mathbf{x}}(t) = \boldsymbol{\alpha}^{\mathbf{x}} \frac{T_{\mathbf{x}}}{\mathbf{x}!} e^{-(\mathbf{x}^{\mathsf{T}} V \boldsymbol{\alpha})t}\ t^{x_1+\ldots+x_m-1}, \qquad \text{where } \mathbf{x}! = \prod_{i=1}^{m} x_i!$$

obtaining

$$T_{\mathbf{x}} = \frac{1}{2\left(\sum_{i=1}^{m} x_i\ -1\right)} \sum_{\mathbf{y},\mathbf{z}:\mathbf{y}+\mathbf{z}=\mathbf{x}} \binom{x_1}{y_1}\binom{x_2}{y_2}\cdots\binom{x_m}{y_m}\left(\mathbf{y}^{\mathsf{T}} V \mathbf{z}\right) T_{\mathbf{y}}\, T_{\mathbf{z}}$$

Thus,

$$T_{\mathbf{x}}(V) = \frac{\tau(K_m, x_i v_{i,j} x_j)}{\prod_{i=1}^{m} x_i} (V\mathbf{x})^{\mathbf{x}-\mathbf{1}}$$

Vector-Multiplicative Coalescent Process

$$\frac{d}{dt}\zeta_{\mathbf{x}}(t) = -\zeta_{\mathbf{x}}(t)\big(\mathbf{x}^{\mathsf{T}}V\boldsymbol{\alpha}\big) + \frac{1}{2}\sum_{\mathbf{y},\mathbf{z}\,:\mathbf{y}+\mathbf{z}=\mathbf{x}}\big(\mathbf{y}^{\mathsf{T}}V\mathbf{z}\big)\zeta_{\mathbf{y}}(t)\,\zeta_{\mathbf{z}}(t), \quad \zeta_{\mathbf{x}}(0) = \sum_{i=1}^{m}\alpha_i\delta_{\mathbf{e}_i,\mathbf{x}}$$

Theorem (YK and P. T. Otto, 2021).

$$\zeta_{\mathbf{x}}(t) = \frac{1}{\mathbf{x}!}\boldsymbol{\alpha}^{\mathbf{x}}\,\frac{\tau(K_m, x_i v_{i,j} x_j)}{\prod_{i=1}^{m} x_i}(V\mathbf{x})^{\mathbf{x}-\mathbf{1}}\,e^{-(\mathbf{x}^{\mathsf{T}}V\boldsymbol{\alpha})t}\;t^{x_1+\ldots+x_m-1},$$

where $\mathbf{x}! = \prod_{i=1}^{m} x_i!$ and $\mathbf{a}^{\mathbf{b}} = a_1^{b_1}a_2^{b_2}\ldots a_m^{b_m}$.

- $\tau(K_m, w_{i,j})$ is computed via Kirchhoff's **Weighted Matrix-Tree Theorem**.