

# Occupation times and Bessel densities

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## Abstract

Consider a Markov process with countably many states. In order to find a one-state occupation time distribution, we use a combination of Fourier and Laplace transforms in the way that allows for the inversion of the Fourier transform. We derive a closed-form expression for the occupation time distribution in the case of a simple continuous time random walk on  $\mathbb{Z}$  and represent the one state occupation density of a reversible process as a mixture of Bessel densities.

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## 1 Introduction

The occupation time distributions for continuous-time Markov processes that live on countable state space was a subject of intense research in the 60s, 70s and early 80s. We would like to refer the reader to [5], [6], [13], [17], [19] and [3] for some of the results in the field. With the exception of [3], the main instrument was the multidimensional Laplace transform. More recently, a general expression in infinite sums of the occupation distributions for finite-state Markov processes was produced in [18] and [2] using essentially randomization and order statistics. The occupation times for one-dimensional nearest-neighbor processes were studied by Karlin and McGregor with orthogonal polynomials in [12] following the paper of Darling and Kac [4]. Both papers considered occupation times for Markov processes when  $t$  is taken to  $\infty$ , while the present paper concerns with closed-form expressions for a given fixed time interval  $[0, t]$ .

The questions addressed in this paper originate in the research done by E.Nir et al [16] in the field of single molecule Fluorescence Resonance Energy Transfer (FRET) spectroscopy, where a single molecule fluctuates between multiple states, and the experimental observable depends on a state's occupation time distribution. While working on [16] the authors have noticed that the single state occupation time densities, when computed

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via randomization technique (i.e. multiple infinite sums) can often be represented via modified Bessel functions of the kind

$$I_\rho(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + \rho + 1)} \left(\frac{z}{2}\right)^{2k+\rho} \quad \text{for } \rho > -1$$

It was observed in Section II.7 of Feller [8] that for a continuous-time random walk on  $\mathbb{Z}$ , the first passage time  $T_r$  is distributed according to the Bessel density function  $v_r(t) = e^{-\frac{t}{r}} I_r(t)$ , where  $t > 0$ ,  $r \in \mathbb{Z}$  is a state, and  $I_{-r} = I_r$ . As it was the case with the first passage times, the Bessel density functions for occupation times of a two state Markov process can be obtained as a Poisson randomization of Gamma densities. In this paper, we use spectral theory in an attempt to find an analytic explanation for the relationship between occupation times and Bessel density functions. We will show a connection between a spectral measure of a generator and a Laplace transform of a single state occupation time distribution taken with respect to a time variable  $t$ .

## 2 Approach and results

In this section we will state the results that relate occupation times to the spectral measure of a generator, we will present a closed-form solution for a one-dimensional symmetric random walk in Theorem 2.2, and discuss the connection between occupation time distributions and modified Bessel functions.

### 2.1 Spectral representation and occupation times

Consider an irreducible continuous-time Markov process with generator  $Q$  over the discrete countable state space  $\Omega = \{0, 1, \dots\}$ . Let  $\{\lambda_{i,j}\}_{i,j \in \Omega}$  denote the rates. For a given time interval  $[0, t]$ , the random variable representing the time spent at the site 0 is called an occupation time at 0. Let  $f_k(t, x)$  denote the probability density for the occupation time at state 0 if the continuous-time process commences at state  $k$ . We will use the standard notation for the basis vectors  $e_0 = (1, 0, \dots)^T$  and  $e_j = (0, \dots, 0, 1, 0, \dots)^T$ . Our first result characterizes the single-state occupation time density  $f_0(t, x)$  as follows.

**Theorem 2.1.** *The Laplace transform of  $f_0(t, x)$  w.r.t. time variable  $t$  can be written as*

$$L_{f_0}(s, x) = \frac{1}{sh(s)} \exp\left\{-\frac{x}{h(s)}\right\} \quad \text{for } s > 0,$$

where  $h(s) = -((Q - sI)^{-1}e_0, e_0)$ .

The domain of  $h(s)$  is the resolvent set of  $Q$ . It was observed that if  $Q$  is a bounded reversible Markov process then  $-h(s)$  is a Herglotz function, i.e. it maps  $\mathbb{C}_+$  into  $\mathbb{C}_+$ . As all such functions  $h$  can be represented uniquely as

$$h(s) = - \int_{(-\infty, 0]} \frac{d\mu(x)}{x - s} \quad (\text{Im}(s) \neq 0), \quad (1)$$

where  $\mu$  is a probability measure with compact support in  $(-\infty, 0]$ . See [9], [10], [11], [7], [15], and references therein.

**Example: Two-state Markov processes.** Consider a two-state Markov process with the generator  $Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$ . There  $(Q - sI)^{-1} = \frac{-1}{s^2 + (\lambda + \mu)s} \begin{pmatrix} \mu + s & \lambda \\ \mu & \lambda + s \end{pmatrix}$  and Theorem 2.1 implies

$$\mathbb{L}_{f_0}(s, x) = e^{-x(s+\lambda)} e^{\frac{\lambda\mu x}{s+\mu}} + \frac{\lambda}{s+\mu} e^{-x(s+\lambda)} e^{\frac{\lambda\mu x}{s+\mu}}$$

Now, the equation (29.3.81) in Abramowitz and Stegun [1] contains the following Laplace transforms

$$\int_0^\infty I_0(2\sqrt{at})e^{-pt} dt = \frac{1}{p} e^{\frac{a}{p}} \quad \text{and} \quad \int_0^\infty \frac{1}{\sqrt{t}} I_1(2\sqrt{at})e^{-pt} dt = \frac{1}{\sqrt{a}} (e^{\frac{a}{p}} - 1),$$

which can be rewritten as  $e^{-px} \frac{1}{p} e^{\frac{a}{p}} = \int_x^\infty I_0(2\sqrt{a(t-x)})e^{-pt} dt$  and

$$e^{-px} e^{\frac{a}{p}} = e^{-px} + \sqrt{a} \int_x^\infty \frac{1}{\sqrt{t-x}} I_1(2\sqrt{a(t-x)})e^{-pt} dt.$$

Plugging in  $a = \lambda\mu x$  and  $p = s + \mu$ , we obtain

$$\frac{1}{s+\mu} e^{-(s+\mu)x} e^{\frac{\lambda^2 x}{s+\mu}} = \int_x^\infty I_0(2\sqrt{\lambda\mu x(t-x)})e^{-(s+\mu)t} dt$$

and

$$e^{-x(s+\mu)} e^{\frac{\lambda\mu x}{s+\mu}} = e^{-(s+\mu)x} + \sqrt{\lambda\mu x} \int_x^\infty \frac{I_1(2\sqrt{\lambda\mu x(t-x)})}{\sqrt{t-x}} e^{-(s+\mu)t} dt$$

Thus

$$f_0(t, x) = e^{-\lambda t} \delta_t(x) + \lambda e^{-\lambda x} e^{-\mu(t-x)} I_0(2\sqrt{\lambda\mu x(t-x)}) + \sqrt{\frac{\lambda\mu x}{t-x}} I_1(2\sqrt{\lambda\mu x(t-x)}) e^{-\lambda x} e^{-\mu(t-x)}$$

for  $0 \leq x \leq t$ .

The above equation was originally derived in [17] via two-dimensional Laplace transforms. One can also derive it via randomization, where the infinite sums are easily recognized to be the corresponding modified Bessel functions.

## 2.2 Occupation times for simple random walks and other reversible processes

Surprisingly the one state occupation time density for a one-dimensional simple nearest-neighbor random walk has a closed-form expression in Bessel density functions. For our

next result, we consider

$$Q = \begin{pmatrix} -r & r & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ 0 & 0 & 1 & -2 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \text{where } r > 0.$$

**Theorem 2.2.** *The zero-state occupation time density for a simple nearest-neighbor random walk with the above infinitesimal generator can be expressed via modified Bessel functions as follows:*

$$f_0(t, x) = e^{-rt} \delta_t(x) + r e^{(2-r)x-2t} I_0 \left( 2\sqrt{(t-x)(t+(r-1)x)} \right) \\ + \frac{rt}{\sqrt{(t-x)(t+(r-1)x)}} e^{(2-r)x-2t} I_1 \left( 2\sqrt{(t-x)(t+(r-1)x)} \right)$$

for  $x \in [0, t]$ .

Observe that  $r = 1$  represents the case of a simple continuous-time random walk on  $\mathbb{Z}_+$ , while the zero-state occupation time for  $r = 2$  corresponds to the case of a simple random walk on  $\mathbb{Z}$ .

Now, suppose  $Q$  is a bounded reversible Markov process such as the above random walk. Then  $h$  can be represented uniquely as in (1). The (spectral) probability measure  $\mu$  has bounded support, say  $\text{supp}(\mu) \subset [-K, 0]$ . Let  $m_0, m_1, \dots$  denote the moments of the spectral measure  $\mu$ , i.e.

$$m_j = \int_{(-\infty, 0]} (-x)^j d\mu(x)$$

Then, for any  $z \in \mathbb{C} \setminus (-\infty, 0]$  such that  $|z| > 2K$ ,

$$\frac{1}{h(z)} = \frac{z}{1 - m_1 z^{-1} + m_2 z^{-2} - \dots} = z + m_1 - (m_2 - m_1^2) z^{-1} + \phi(z^{-1}) z^{-2}$$

Recall Theorem 2.1. We will consider the inverse Laplace transform of a function  $F_0(t, x)$ , whose Laplace transform  $L_{F_0}(s, x) = \frac{1}{s} \exp \left\{ -\frac{x}{h(s)} \right\}$ . There  $f_0(t, x) = -\frac{\partial}{\partial x} F_0(t, x)$ . Take  $a > 2K$  and  $0 \leq x \leq t$ , then the inverse Laplace transform

$$F_0(t, x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{z} \exp \left\{ z(t-x) - m_1 x + (m_2 - m_1^2) x z^{-1} - x \phi(z^{-1}) z^{-2} \right\} dz$$

Expanding  $e^{-x\phi(z^{-1})z^{-2}} = 1 + \sum_{k=2}^{\infty} v_k(x) z^{-k}$ , we obtain

$$F_0(t, x) = e^{-m_1 x} I_0 \left( 2\sqrt{(m_2 - m_1^2)(t-x)x} \right) \\ + e^{-m_1 x} \sum_{k=2}^{\infty} v_k(x) \left( \frac{t-x}{(m_2 - m_1^2)x} \right)^{\frac{k}{2}} I_k \left( 2\sqrt{(m_2 - m_1^2)(t-x)x} \right)$$

by (29.3.81) in Abramowitz and Stegun [1]. However, even in the case of a simple random walk, it is not obvious that the above expansion in Bessel functions using the moments of  $\mu$  can be simplified to that of Theorem 2.2. In general, the following question is still to be fully explored: what properties of  $\mu$  would allow for a closed-form expression of the occupation time via Bessel density functions? It seems possible to exploit the connection between spectral probability measure  $\mu$  and the occupation time densities for a wider class of time reversible stochastic processes. Another interesting direction is to learn more about reinforced processes (see [14] and references therein) as they can be interpreted as occupation time driven processes.

Next we would like to discuss the limit behavior of  $f_0(t, x)$  as  $t \rightarrow \infty$ . Given  $t > 0$ , let  $Z(t)$  denote the occupation time variable governed by the distribution  $f_0(t, x)$  as derived in Theorem 2.2. As we will see in the proof of the theorem (equation (4)),

$$h(s) = \frac{2}{(2-r)s + r\sqrt{s^2 + 4s}} = s^{-\alpha} L\left(\frac{1}{s}\right),$$

where  $\alpha = 1/2$  and  $L(1/s)$  is a ‘‘slowly varying’’ function (see [12] and [4]) that converges to  $1/r$  as  $s \rightarrow 0+$ . Hence, as it was the case in Karlin and McGregor [12], this  $h(s)$  satisfies the main assumption for the following limit theorem in Darling and Kac [4]:

$$\lim_{t \rightarrow \infty} P\left(\frac{Z(t)}{h(1/t)} \leq u\right) = G_\alpha(u),$$

where  $G_\alpha(u)$  denotes the Mittag-Liffler distribution,

$$G_\alpha(u) = \frac{1}{\pi\alpha} \int_0^u \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(\alpha n + 1) y^{n-1} \sin(n\pi\alpha) dy$$

Now, since we have obtained the closed form solution in Theorem 2.2, we observe that here, the Darling-Kac limit theorem can be derived as follows. Given  $u > 0$ ,

$$P\left(\frac{Z(t)}{h(1/t)} \leq u\right) = \int_0^u f_0(t, h(1/t)y) h(1/t) dy$$

Since  $h(1/t) = \frac{\sqrt{t}}{r}(1 + o(1))$ ,

$$f_0(t, h(1/t)y) h(1/t) = \sqrt{t} e^{(2-r)h(1/t)y - 2t} [I_0(2t\varphi)(1 + o(1)) + I_1(2t\varphi)(1 + o(1))],$$

where

$$\varphi = \sqrt{(1 - h(1/t)y/t)(1 + (r-1)h(1/t)y/t)} = 1 - \frac{2-r}{2} h(1/t)y/t - \frac{r^2}{8} [h(1/t)]^2 y^2/t^2 + o(t^{-1})$$

We use  $I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}(1 + o(1))$  as in the equation 9.7.1 of [1] to obtain the limit

$$\lim_{t \rightarrow \infty} P\left(\frac{Z(t)}{h(1/t)} \leq u\right) = \frac{1}{\sqrt{\pi}} \int_0^u e^{-\frac{y^2}{4}} dy = G_{1/2}(u)$$

via the dominated convergence theorem.

### 3 Proofs

*Proof. (Theorem 2.1)* Observe that the integral equations relating  $\{f_k(t, x)\}_{k=0,1,\dots}$  can be produced via conditioning as follows:

$$\begin{aligned} f_0(t, x) &= e^{-(\sum_{m: m \neq 1} \lambda_{0,m})t} \delta_t(x) + \sum_{k: k \neq 0} \int_0^t f_k(t-y, x-y) \lambda_{0,k} e^{-(\sum_{m: m \neq 0} \lambda_{0,m})y} dy, \\ f_j(t, x) &= e^{-(\sum_{m: m \neq j} \lambda_{j,m})t} \delta_0(x) + \sum_{k: k \neq j} \int_0^t f_k(t-y, x) \lambda_{j,k} e^{-(\sum_{m: m \neq j} \lambda_{j,m})y} dy \\ &\text{for } j = 1, 2, \dots \end{aligned}$$

We plug in  $\psi = t - y$  into the above equations, take the Fourier transform with respect to  $x$ , and simplify to get

$$\begin{aligned} e^{(\sum_{m: m \neq 0} \lambda_{0,m} - is_2)t} \hat{f}_0(t, s_2) &= 1 + \sum_{k: k \neq 0} \int_0^t \hat{f}_k(\psi, s_2) \lambda_{0,k} e^{(\sum_{m: m \neq 0} \lambda_{0,m} - is_2)\psi} d\psi, \\ e^{(\sum_{m: m \neq j} \lambda_{j,m})t} \hat{f}_j(t, s_2) &= 1 + \sum_{k: k \neq j} \int_0^t \hat{f}_k(\psi, s_2) \lambda_{j,k} e^{(\sum_{m: m \neq j} \lambda_{j,m})\psi} d\psi \\ &\text{for } j = 1, 2, \dots \end{aligned}$$

We differentiate w.r.t. variable  $t$ , and obtain

$$\begin{aligned} \left( \sum_{m: m \neq 0} \lambda_{0,m} - is_2 \right) \hat{f}_0(t, s_2) + \frac{\partial}{\partial t} \hat{f}_0(t, s_2) &= \sum_{k: k \neq 0} \lambda_{0,k} \hat{f}_k(t, s_2), \\ \left( \sum_{m: m \neq j} \lambda_{j,m} \right) \hat{f}_j(t, s_2) + \frac{\partial}{\partial t} \hat{f}_j(t, s_2) &= \sum_{k: k \neq j} \lambda_{j,k} \hat{f}_k(t, s_2) \quad (j = 1, 2, \dots) \end{aligned}$$

We observe that  $\hat{f}_j(0, s_2) = 1$  for all  $j$ . Our next step is to take the Laplace transform w.r.t. variable  $t$ :

$$\begin{aligned} \left( \sum_{m: m \neq 0} \lambda_{0,m} + s_1 - is_2 \right) \mathbf{L}_{\hat{f}_0}(s_1, s_2) &= 1 + \sum_{k: k \neq 0} \lambda_{0,k} \mathbf{L}_{\hat{f}_k}(s_1, s_2), \\ \left( \sum_{m: m \neq j} \lambda_{j,m} + s_1 \right) \mathbf{L}_{\hat{f}_j}(s_1, s_2) &= 1 + \sum_{k: k \neq j} \lambda_{j,k} \mathbf{L}_{\hat{f}_k}(s_1, s_2) \quad (j = 1, 2, \dots) \end{aligned}$$

The above system of equations can be rewritten via the spectral decomposition of the generator operator  $Q$  as follows. Let  $\mathbf{L}_{\hat{f}}(s_1, s_2) = \begin{bmatrix} \mathbf{L}_{\hat{f}_0}(s_1, s_2) \\ \mathbf{L}_{\hat{f}_1}(s_1, s_2) \\ \vdots \end{bmatrix}$  and  $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix}$ . We

proved the following spectral identity

$$(Q - s_1 I) \mathbf{L}_{\hat{f}}(s_1, s_2) = -\mathbf{1} - is_2 \begin{bmatrix} \mathbf{L}_{\hat{f}_0}(s_1, s_2) \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad (2)$$

Thus  $\mathbf{L}_{\hat{f}}(s_1, s_2) = -(Q - s_1 I)^{-1} \mathbf{1} - is_2 \mathbf{L}_{\hat{f}_0}(s_1, s_2) (Q - s_1 I)^{-1} e_0$  and

$$\mathbf{L}_{\hat{f}_0}(s_1, s_2) = -((Q - s_1 I)^{-1} \mathbf{1}, e_0) - is_2 \mathbf{L}_{\hat{f}_0}(s_1, s_2) ((Q - s_1 I)^{-1} e_0, e_0)$$

Therefore the Laplace-Fourier transform of  $f_0$  can be represented as

$$\mathbf{L}_{\hat{f}_0}(s_1, s_2) = \frac{-((Q - s_1 I)^{-1} \mathbf{1}, e_0)}{1 + is_2 ((Q - s_1 I)^{-1} e_0, e_0)} \quad (3)$$

Since  $\frac{1}{s_1} (Q - s_1 I) \mathbf{1} = -\mathbf{1}$ , the expression in (3) can be simplified to

$$\mathbf{L}_{\hat{f}_0}(s_1, s_2) = \frac{1/s_1}{1 - is_2 h(s_1)},$$

where  $h(s) = -((Q - sI)^{-1} e_0, e_0)$ . The Fourier transform can be inverted via complex integration over a lower semi-circle contour with the radius converging to infinity, thus giving us

$$\mathbf{L}_{f_0}(s_1, x) = \frac{1}{s_1 h(s_1)} \exp \left\{ -\frac{x}{h(s_1)} \right\}$$

□

*Proof.* (**Theorem 2.2**) In the case of a simple random walk, the equation (2) translates as

$$\begin{aligned} \mathbf{L}_{\hat{f}_0}(s_1, s_2) &= \frac{1}{r + s_1 - is_2} + \frac{r}{r + s_1 - is_2} \mathbf{L}_{\hat{f}_1}(s_1, s_2) \\ \mathbf{L}_{\hat{f}_1}(s_1, s_2) &= \frac{1}{2 + s_1} + \frac{1}{2 + s_1} \mathbf{L}_{\hat{f}_0}(s_1, s_2) + \frac{1}{2 + s_1} \mathbf{L}_{\hat{f}_2}(s_1, s_2) \\ \mathbf{L}_{\hat{f}_k}(s_1, s_2) &= \frac{1}{2 + s_1} + \frac{1}{2 + s_1} \mathbf{L}_{\hat{f}_{k-1}}(s_1, s_2) + \frac{1}{2 + s_1} \mathbf{L}_{\hat{f}_{k+1}}(s_1, s_2) \quad (k = 1, 2, \dots) \end{aligned}$$

where  $\mathbf{L}_{\hat{f}_k}(s_1, s_2)$  again denotes the Laplace transform in the first variable  $t$  and the Fourier transform in the second variable  $x$  of  $f_k(t, x)$ . The functions  $l_k(s_1, s_2) = \mathbf{L}_{\hat{f}_k}(s_1, s_2) - \frac{1}{s_1}$  satisfy the following recurrence relation

$$l_k(s_1, s_2) = \frac{1}{2 + s_1} l_{k-1}(s_1, s_2) + \frac{1}{2 + s_1} l_{k+1}(s_1, s_2) \quad (k = 1, 2, \dots)$$

Observe that  $\mathbb{L}_{\hat{f}_k}(s_1, 0) = \int_{[0, +\infty)} \int_{\mathbb{R}} e^{-s_1 t} f_k(t, x) dx dt = \frac{1}{s_1}$  and

$$\mathbb{L}_{\hat{f}_k}(s_1, s_2) = \int_{[0, +\infty)} \int_{\mathbb{R}} e^{-s_1 t + i s_2 x} f_k(t, x) dx dt \rightarrow \int_{[0, +\infty)} \int_{\mathbb{R}} e^{-s_1 t + i s_2 x} \delta_0(x) dx dt = \frac{1}{s_1} \quad \text{as } k \rightarrow \infty$$

That is  $l_k(s_1, s_2) \rightarrow 0$  as  $k \rightarrow \infty$  and therefore  $l_k(s_1, s_2) = l_0(s_1, s_2) \left( \frac{2 + s_1 - \sqrt{s_1^2 + 4s_1}}{2} \right)^k$  as  $s_1 > 0$ . Thus the top recurrence equation reads

$$\mathbb{L}_{\hat{f}_0}(s_1, s_2) = \frac{1}{r + s_1 - i s_2} + \frac{r}{r + s_1 - i s_2} \left[ \left( \mathbb{L}_{\hat{f}_0}(s_1, s_2) - \frac{1}{s_1} \right) \frac{2 + s_1 - \sqrt{s_1^2 + 4s_1}}{2} + \frac{1}{s_1} \right]$$

and therefore

$$\mathbb{L}_{\hat{f}_0}(s_1, s_2) = \frac{i}{2s_1} \cdot \frac{(2-r)s_1 + r\sqrt{s_1^2 + 4s_1}}{s_2 + \frac{i}{2}((2-r)s_1 + r\sqrt{s_1^2 + 4s_1})}$$

Once again, using complex integration, we invert the Fourier transform

$$\mathbb{L}_{f_0}(s_1, x) = \frac{(2-r)s_1 + r\sqrt{s_1^2 + 4s_1}}{2s_1} \exp \left\{ -\frac{x}{2} \left( (2-r)s_1 + r\sqrt{s_1^2 + 4s_1} \right) \right\}$$

Here the spectral measure will satisfy

$$\int_{(-\infty, 0]} \frac{d\mu(x)}{x-s} = \frac{-2}{(2-r)s + r\sqrt{s^2 + 4s}} \quad (4)$$

We will now invert the Laplace transform by decomposing  $\mathbb{L}_{f_0}(s_1, x)$  as follows

$$\mathbb{L}_{f_0}(s_1, x) = \frac{2-r}{2} \mathcal{P}_I + \frac{r}{2} \mathcal{P}_{II} + 2r \mathcal{P}_{III} ,$$

where

$$\mathcal{P}_I = \exp \left\{ -\frac{x}{2} (2-r)s_1 \right\} \cdot \exp \left\{ -\frac{x}{2} r \sqrt{s_1^2 + 4s_1} \right\} ,$$

$$\mathcal{P}_{II} = \frac{s_1}{\sqrt{s_1^2 + 4s_1}} \exp \left\{ -\frac{x}{2} (2-r)s_1 \right\} \cdot \exp \left\{ -\frac{x}{2} r \sqrt{s_1^2 + 4s_1} \right\}$$

and

$$\mathcal{P}_{III} = \frac{1}{\sqrt{s_1^2 + 4s_1}} \exp \left\{ -\frac{x}{2} (2-r)s_1 \right\} \cdot \exp \left\{ -\frac{x}{2} r \sqrt{s_1^2 + 4s_1} \right\}$$

We will quote a Laplace transform formula (29.3.91) in [1]:

$$\int_k^\infty e^{-st} e^{-\frac{1}{2}at} I_0 \left( \frac{1}{2}a\sqrt{t^2 - k^2} \right) dt = \frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}} , \quad (k \geq 0)$$

First we will find the inverse-Laplace transform of  $\mathcal{P}_{III}$ . Taking  $s = s_1$ ,  $a = 4$  and  $k = \frac{rx}{2}$  in (29.3.91) of [1], we get

$$\int_{\frac{rx}{2}}^{\infty} e^{-s_1 t} e^{-2t} I_0 \left( 2\sqrt{t^2 - \left(\frac{rx}{2}\right)^2} \right) dt = \frac{e^{-\frac{rx}{2} \sqrt{s_1(s_1+4)}}}{\sqrt{s_1(s_1+4)}}$$

Multiplying both sides of the above equation by  $\exp\left\{-\frac{(2-r)x}{2}s_1\right\}$ , and changing the variable to  $t := t + \frac{(2-r)x}{2}$ , obtain

$$\mathcal{P}_{III} = \int_x^{\infty} e^{-s_1 t} e^{(2-r)x-2t} I_0 \left( 2\sqrt{(t-x)(t+(r-1)x)} \right) dt \quad (5)$$

Therefore, the inverse of  $\mathcal{P}_{III}$  is

$$\mathcal{L}^{-1}(\mathcal{P}_{III}) = e^{(2-r)x-2t} I_0 \left( 2\sqrt{(t-x)(t+(r-1)x)} \right) \cdot \mathbf{1}_{\{x \leq t\}}$$

We differentiate  $\frac{\partial}{\partial t}$  and integrate by parts in (5):

$$\mathcal{L} \left( \frac{\partial}{\partial t} \left[ e^{(2-r)x-2t} I_0 \left( 2\sqrt{(t-x)(t+(r-1)x)} \right) \cdot \mathbf{1}_{\{x \leq t\}} \right] \right) = \mathcal{P}_{II} - e^{-rx} e^{-s_1 x}$$

Hence

$$\begin{aligned} \mathcal{L}^{-1}(\mathcal{P}_{II}) &= e^{-rt} \delta_0(t-x) - 2e^{(2-r)x-2t} I_0 \left( 2\sqrt{(t-x)(t+(r-1)x)} \right) \cdot \mathbf{1}_{\{x \leq t\}} \\ &+ \frac{2t - (2-r)x}{\sqrt{(t-x)(t+(r-1)x)}} e^{(2-r)x-2t} I_1 \left( 2\sqrt{(t-x)(t+(r-1)x)} \right) \cdot \mathbf{1}_{\{x \leq t\}} \end{aligned}$$

In order for us to invert  $\mathcal{P}_I$ , we will need (29.3.96) of [1], that states the following

$$\int_k^{\infty} e^{-st} \frac{ak}{\sqrt{t^2 - k^2}} I_1 \left( a\sqrt{t^2 - k^2} \right) dt = e^{-k\sqrt{s^2 - a^2}} - e^{-ks}, \quad (k > 0)$$

Here we let  $s = s_1 + 2$ ,  $a = 2$  and  $k = \frac{rx}{2}$ , thus obtaining

$$\mathcal{P}_I = e^{-rx} e^{-s_1 x} + \int_{\frac{rx}{2}}^{\infty} e^{-s_1(t + \frac{(2-r)x}{2})} \frac{rx e^{-2t}}{\sqrt{t^2 - \left(\frac{rx}{2}\right)^2}} I_1 \left( 2\sqrt{t^2 - \left(\frac{rx}{2}\right)^2} \right) dt$$

Once again changing the variable to  $t := t + \frac{(2-r)x}{2}$ , get

$$\mathcal{P}_I = e^{-rx} e^{-s_1 x} + \int_x^{\infty} e^{-s_1 t} \frac{rx e^{(2-r)x-2t}}{\sqrt{(t-x)(t+(r-1)x)}} I_1 \left( 2\sqrt{(t-x)(t+(r-1)x)} \right) dt$$

and

$$\mathcal{L}^{-1}(\mathcal{P}_I) = e^{-rt} \delta_0(t-x) + \frac{rx e^{(2-r)x-2t}}{\sqrt{(t-x)(t+(r-1)x)}} I_1 \left( 2\sqrt{(t-x)(t+(r-1)x)} \right) \cdot \mathbf{1}_{\{x \leq t\}}$$

We add up all three terms together, thus proving the theorem.  $\square$

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