CROSS-MULTIPLICATIVE COALESCENT PROCESSES AND APPLICATIONS

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Abstract. We introduce and analyze a novel type of coalescent processes called *cross-multiplicative coalescent* that models a system with two types of particles, A and B. The bonds are formed only between the pairs of particles of opposite types with the same rate for each bond, producing connected components made of particles of both types. We analyze and solve the Smoluchowski coagulation system of equations obtained as a hydrodynamic limit of the corresponding Marcus-Lushnikov process. We establish that the cross-multiplicative kernel is a gelling kernel, and find the gelation time. As an application, we derive the limiting mean length of a minimal spanning tree on a complete bipartite graph $K_{\alpha[n],\beta[n]}$ with partitions of sizes $\alpha[n] = \alpha n + o(\sqrt{n})$ and $\beta[n] = \beta n + o(\sqrt{n})$ and independent edge weights, distributed uniformly over $[0, 1]$.

Contents

Date: October 16, 2020.
2000 *Mathematics Subject Classification.* Primary 60K35; Secondary 82B27.
This work was supported by the FAPESP award 2016/19286-0 and by the NSF award DMS-1412557.

1. Introduction

The coalescence dynamics of clusters with multidimensional weight (mass) vectors was originally considered in Krapivsky and Ben-Naim [24] and Vigil and Ziff [37] in the context of aggregation kinetics with applications to aerosol dynamics and copolymerization kinetics. In this paper, we consider a coalescent process whose clusters have vector-valued weights in $\mathbb{R}_+^2$. The coalescent process begins with $\alpha[n] = \alpha n + o(\sqrt{n})$ singletons of weight $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\beta[n] = \beta n + o(\sqrt{n})$ singletons of weight $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This continuous time Markov process evolves as follows. Each pair of clusters with respective weight vectors $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix}$ has the rate $K(\mathbf{i}, \mathbf{j})/n$ for coalescing into a cluster of weight $\mathbf{i} + \mathbf{j}$, where

$$K(\mathbf{i}, \mathbf{j}) = i_1 j_2 + i_2 j_1$$

is the *cross-multiplicative coalescent kernel* governing the coalescent process. Such process will be called the *cross-multiplicative coalescent process*.

As a physical model, one may consider a system with two types of particles, A and B. The process begins with $\alpha[n]$ particles of type A and $\beta[n]$ particles of type B. Each particle interacts only with the particles of opposite type, with which it may form a bond. The bonds are formed independently, each with rate $1/n$. Thus, the bonds may be formed only between the pairs of particles of opposite types, producing connected components (clusters). In these clusters, each pair of neighbor vertices will be of opposite type. The model can be interpreted as a bond percolation model on a complete bipartite graph $K_{\alpha[n],\beta[n]}$ with the probability of an edge being open $p = 1 - e^{-t/n}$ increasing from zero to one as the time t increases from zero to infinity. See Subsect. 4.4.

Let $\zeta_{i_1,i_2}^{[n]}(t)$ denote the number of the components of weight $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ at time t. The hydrodynamic limit

$$\lim_{n \to \infty} \sup_{s \in [0,T]} \left| n^{-1} \zeta_{i_1,i_2}^{[n]}(s) - \zeta_{i_1,i_2}(s) \right| = 0 \qquad \text{a.s.}$$

for an arbitrary $T > 0$ is established in Sect. 5.3 via the weak convergence results of Kurtz [11, 25] for density dependent population processes. The limiting functions $\zeta_{i_1,i_2}(t)$ indexed by $\mathbb{Z}_+^2 \setminus \{(0,0)\}$ are expressed as the solutions of the following *modified Smoluchowski coagulation system* of differential equations

$$\frac{d}{dt}\zeta_{i_1,i_2}(t) = -(\beta i_1 + \alpha i_2)\zeta_{i_1,i_2}(t) + \frac{1}{2}\sum_{\substack{\ell_1,k_1:\ \ell_1+k_1=i_1,\\ \ell_2,k_2:\ \ell_2+k_2=i_2}} (\ell_1 k_2 + \ell_2 k_1)\zeta_{\ell_1,\ell_2}(t)\zeta_{k_1,k_2}(t) \tag{1}$$

with the initial conditions $\zeta_{i_1,i_2}(0) = \alpha\delta_{1,i_1}\delta_{0,i_2} + \beta\delta_{0,i_1}\delta_{1,i_2}$. The above system has a unique solution as established in the following theorem.

Sect. 3, Theorem 3.2. *The modified Smoluchowski coagulation system of equations* (1) *with the initial conditions* $\zeta_{i_1,i_2}(0) = \alpha\delta_{1,i_1}\delta_{0,i_2} + \beta\delta_{0,i_1}\delta_{1,i_2}$ *has the unique solution*

$$\zeta_{i_1,i_2}(t) = \frac{i_1^{i_2-1} i_2^{i_1-1} \alpha^{i_1}\beta^{i_2}}{i_1! i_2!} e^{-(\beta i_1 + \alpha i_2)t} t^{i_1+i_2-1}.$$

This solution of the coagulation system enables us to establish gelation in the cross-multiplicative coalescent process.

Sect. 3, Corollary 3.7. *The cross-multiplicative kernel is a gelling kernel, with the gelation time given by*

$$T_{gel} = \frac{1}{\sqrt{\alpha\beta}}.$$

We already mentioned the connection of cross-multiplicative coalescece to aggregation kinetics [24, 37]. Besides this, the study of the cross-multiplicative coalescent process is justified by its relation to the Erdős-Rényi process on $K_{\alpha[n],\beta[n]}$. This relation will be used in the applications presented in Sect. 4 of this paper.

1.1. **Applications in minimal spanning trees.** As an application, we attempt to extend the connection between coalescent processes and random graph processes, e.g. Erdős-Rényi random graph evolution as described in Sect. 4. In particular, deriving a formula for the limiting length of the minimal spanning tree in a random graph process in terms of the solutions of the Smoluchowski coagulation equations for the corresponding coalescent process.

It is well known that in many cases the cluster dynamics of a random graph process can be replicated with a coalescent process. For example, the Erdős-Rényi random graph process on K_n can be tied to the n-particle multiplicative coalescent (see Aldous [2]). The connection lies in that the probability of two components merging, at a given time, depends only on the number of edges that connect those two components (rather than other structural properties). There are more elaborate examples.

The cluster dynamics of a coalescent process (without merger history) is traced by an auxiliary process called the Marcus-Lushnikov process. The merger dynamics of such coalescent processes corresponds to a greedy algorithm for finding the minimal spanning tree in the respective random graph process. This observation allows us to express the limiting mean length of a minimal spanning tree in terms of the solutions of the Smoluchowski coagulation equations that represent the hydrodynamic limit of the Marcus-Lushnikov process corresponding to the random graph process.

As a particular application of the proposed general approach we find the asymptotic limit for the mean length of a minimal spanning tree for the complete bipartite graph with partitions of sizes $\alpha[n] = \alpha n + o(\sqrt{n})$ and $\beta[n] = \beta n + o(\sqrt{n})$. See Sect. 4. There, the probability of two components merging at a given time depends only on the number of edges that connect those two components. If connected component C_i and C_j have partition sizes (i_1, i_2) and (j_1, j_2) respectively, then there are $i_1 j_2 + i_2 j_1$ edges which, when opened, would connect C_i and C_j.

Sect. 4, Theorem 4.2. *Let $\alpha, \beta > 0$ and $L_n = L_n(\alpha, \beta)$ be the length of a minimal spanning tree on a complete bipartite graph $K_{\alpha[n],\beta[n]}$ with partitions of sizes*

$$\alpha[n] = \alpha n + o(\sqrt{n}) \quad \textit{and} \quad \beta[n] = \beta n + o(\sqrt{n})$$

and independent uniform edge weights over $[0,1]$. Then

$$\lim_{n\to\infty} E[L_n] = \sum_{i_1,i_2}^{\infty} \int\limits_0^{\infty} \zeta_{i_1,i_2}(t) d(t),$$

where $\zeta_{i_1,i_2}(t)$ indexed by $\mathbb{Z}_+^2 \setminus \{(0,0)\}$ is the solution of the following system of equations

$$\frac{d}{dt}\zeta_{i_1,i_2}(t) = -(\beta i_1 + \alpha i_2)\zeta_{i_1,i_2}(t) + \frac{1}{2} \sum_{\substack{\ell_1,k_1:\ \ell_1+k_1=i_1,\\ \ell_2,k_2:\ \ell_2+k_2=i_2}} (\ell_1 k_2 + \ell_2 k_1)\zeta_{\ell_1,\ell_2}(t)\zeta_{k_1,k_2}(t)$$

with the initial conditions $\zeta_{i_1,i_2}(0) = \alpha\delta_{1,i_1}\delta_{0,i_2} + \beta\delta_{0,i_1}\delta_{1,i_2}$.

Recall that the above system of equations is the modified Smoluchowski coagulation system (18) of the cross-multiplicative coalescent process with the cross-multiplicative kernel introduced in (14).

In Theorem 3.2, the system of equations in Theorem 4.2 is solved. This yields the main result of Sect. 4.

Sect. 4, Theorem 4.3. *Let $\alpha, \beta > 0$, $\gamma = \alpha/\beta$, and $L_n = L_n(\alpha, \beta)$ be the length of a minimal spanning tree on a complete bipartite graph $K_{\alpha[n],\beta[n]}$ with partitions of sizes*

$$\alpha[n] = \alpha n + o(\sqrt{n}) \quad \textit{and} \quad \beta[n] = \beta n + o(\sqrt{n})$$

and independent uniform edge weights over $[0,1]$. Then the limiting mean length of the minimal spanning tree is

$$\lim_{n\to\infty} E[L_n] = \gamma + \frac{1}{\gamma} + \sum_{i_1\geq 1;\ i_2\geq 1} \frac{(i_1+i_2-1)!}{i_1! i_2!} \frac{\gamma^{i_1} i_1^{i_2-1} i_2^{i_1-1}}{(i_1+\gamma i_2)^{i_1+i_2}}.$$

The above result is novel for $\alpha \neq \beta$, when the complete bipartite graph $K_{\alpha[n],\beta[n]}$ is an irregular graph. For $\alpha = \beta$, Theorem 4.3 recovers the result of Frieze and McDiarmid [17], as stated in the following corollary that we also prove in Section 3.

Corollary 1.1 (Sect. 4, Corollary 4.4)**.** *If $\alpha = \beta$, then*

$$\lim_{n\to\infty} E[L_n] = 2\zeta(3).$$

The paper is organized as follows. Sect. 2 provides the background on coalescent processes, multiplicative coalescent, and gelation. In Sect. 3 the cross-multiplicative coalescent process and the corresponding Marcus-Lushnikov process are analyzed. Sect. 4 gives applications of multiplicative and cross-multiplicative coalescent processes in minimal spanning trees. Finally, in Section 5, the weak convergence results of Kurtz [11, 25] are applied to Marcus-Lushnikov processes with multiplicative and cross-multiplicative kernels. The paper concludes with a discussion in Section 6.

2. Background on coalescent processes and gelation

A general finite coalescent process begins with n singletons (clusters of mass one). The cluster formation is governed by a symmetric collision rate kernel $K(i,j) = K(j,i) > 0$. Specifically, a pair of clusters with masses (weights) i and j coalesces at the rate $K(i,j)/n$, independently of the other pairs, to form a new cluster of mass $i+j$. The process continues until there is a single cluster of mass n. See [31, 2, 5, 4, 13] and references therein.

Formally, for a given n consider the space $\mathcal{P}_{[n]}$ of partitions of $[n] = \{1, 2, \dots, n\}$. Let $\Pi_0^{(n)}$ be the initial partition in singletons, and $\Pi_t^{(n)}$ $(t \geq 0)$ be a strong Markov process such that $\Pi_t^{(n)}$ transitions from partition $\pi \in \mathcal{P}_{[n]}$ to $\pi' \in \mathcal{P}_{[n]}$ with rate $K(i,j)/n$ provided that partition π' is obtained from partition π by merging two clusters of π of weights i and j. If $K(i,j) \equiv 1$ for all positive integer masses i and j, the process $\Pi_t^{(n)}$ is known as Kingman's n-coalescent process. If $K(i,j) = i+j$ the process is called n-particle additive coalescent. Finally, if $K(i,j) = ij$ the process is called n-particle multiplicative coalescent. The so called Marcus-Lushnikov process

$$\mathbf{ML}_n(t) = \left(\zeta_1^{[n]}(t), \zeta_2^{[n]}(t), \dots, \zeta_n^{[n]}(t), 0, 0, \dots\right) \tag{2}$$

is an auxiliary process to the corresponding coalescent process that keeps track of the numbers of clusters in each weight category. Here we let $\zeta_k^{[n]}(t)$ denote the number of clusters of mass k in a coalescent process of n particles at time $t \geq 0$. See [27] and [26] for the original papers by Marcus and Lushnikov. The latter work considered the gelation phenomenon emerging in some of the Marcus-Lushnikov processes. The Marcus-Lushnikov process does not differentiate between the clusters of the same weight, and therefore does not keep track of the merger history of the n-particle coalescent process.

The deterministic dynamics of the limiting fractions $\zeta_k(t) = \lim\limits_{n\to\infty} \frac{\zeta_k^{[n]}(t)}{n}$ of clusters of size k is described by the Smoluchowski system of coagulation equations [34] or by its modified version, the Flory equations. See [15, 22, 30, 18, 19]. The general system of Smoluchowski coagulation equations with a positive symmetric kernel $K(i,j)$ is the following mean-field approximation of coalescent dynamics

$$\frac{d}{dt}\zeta_j = -\zeta_j \sum_{i=1}^{\infty} K(i,j)\zeta_i + \frac{1}{2}\sum_{i=1}^{j-1} K(i,j-i)\zeta_i\zeta_{j-i} \qquad (k=1,2,\ldots). \tag{3}$$

One of the important questions in the theory of Smoluchowski equations is whether the *conservation of mass* property

$$\sum_{j=1}^{\infty} j\zeta_j(t) = \sum_{j=1}^{\infty} j\zeta_j(0) \tag{4}$$

holds for all $t \geq 0$, or if there exists a time $T_{gel} < \infty$ after which the total mass $\sum_{j=1}^{\infty} j\zeta_j$ begins to dissipate.

2.1. **Gelation.** The phenomenon of loosing total mass after a certain finite time T_{gel} is called *gelation*. Time $T_{gel} > 0$, if finite as in the multiplicative case, is called the *gelation time*. The kernel function $K(\cdot,\cdot)$ for which such $T_{gel} < \infty$ is called the *gelling kernel*. Informally, the gelation time corresponds to formation of a giant cluster called the *gel*. The gelation phenomenon was studied extensively in the coagulations equations literature. See [1, 2, 36, 23, 26] and references therein. Here, we would like to summarize some (but not all) of the concepts and results concerning the gelation phenomenon.

Consider a general system (3) of Smoluchowski coagulation equations with a positive symmetric kernel $K(i,j)$, and given initial conditions $\zeta_j(0)$. Then, following [28], we use the Smoluchowski equations (3) to obtain

$$\begin{aligned}\frac{d}{dt}\sum_{j=1}^{\infty} j\zeta_j = \sum_{j=1}^{\infty} j\frac{d}{dt}\zeta_j &= -\sum_{i,j=1}^{\infty} jK(i,j)\zeta_j\zeta_i + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{i=1}^{j-1}\big(i+(j-i)\big)K(i,j-i)\zeta_i\zeta_{j-i} \\ &= -\sum_{i,j=1}^{\infty} jK(i,j)\zeta_j\zeta_i + \frac{1}{2}\sum_{i,j=1}^{\infty}\big(i+j\big)K(i,j)\zeta_i\zeta_j \;\; = 0\end{aligned}$$

provided convergence of $\sum\limits_{i,j=1}^{\infty} jK(i,j)\zeta_j\zeta_i$. Thus, letting the gelation time be defined via the following critical transition,

$$T_{gel} := \inf\Big\{t > 0 \;:\; \sum_{i,j=1}^{\infty} jK(i,j)\zeta_j(t)\zeta_i(t) = \infty\Big\}, \tag{5}$$

we have $\frac{d}{dt}\sum_{j=1}^{\infty} j\zeta_j = 0$ for $t \in [0, T_{gel})$, which in turn implies (4) for $t \in [0, T_{gel})$.

Suppose the hydrodynamic limit $\lim_{n\to\infty} \frac{\zeta_k^{[n]}(t)}{n} = \zeta_k(t)$ is established for the Marcus-Lushnikov process with the given kernel $K(i,j)$, where $\zeta_k(t)$ is the solution of a coagulation system of equations. See [30, 18, 19]. Then the definition of gelation time in formula (5) is replaced with

$$T_{gel} := \inf\Big\{t > 0 \ : \ \sum_{j=1}^{\infty} j\zeta_j(t) < \sum_{j=1}^{\infty} j\zeta_j(0)\Big\}. \tag{6}$$

While (5) relies on the explosion of higher moments (often, the second moment $\sum_j j^2\zeta_j$) and (6) concerns the behavior of the first moment, the two definitions of gelation are usually equivalent.

Weak convergence of the Marcus-Lushnikov processes to either a Smoluchowski system or a modified Smoluchowski (Flory) system was explored in Jeon [22], Norris [30], Fournier and Giet [18], and in Fournier and Laurençot [19]. Specifically, it was shown in Fournier and Giet [18] that if $\lim_{i\to\infty} \frac{K(i,j)}{i} = \ell(j) > 0$, then the hydrodynamic limit of the Marcus-Lushnikov process with kernel $K(i,j)$ is the solution to the corresponding modified Smoluchowski (Flory) system. While, in Jeon [22] and Norris [30] it was established that $\lim_{i\to\infty} \frac{K(i,j)}{i} = 0$ implies the hydrodynamic limit of the Marcus-Lushnikov process is the solution to the Smoluchowski system.

The question whether $T_{gel} < \infty$ is the question of whether the gelation phenomenon occurs in a given system of Smoluchowski equations. The first mathematical proof of gelation was produced in McLeod [28] for the multiplicative kernel. Historically, this happened around the time when the formation of a giant cluster in the Erdős-Rényi random graph model (see Sect. 4.1) was proved by P. Erdős and A. Rényi [9]. The overlap in mathematical formulas obtained in the two papers, [28] and [9], representing the two different branches of mathematics is quite remarkable. The work of finding a mathematically solid proof of gelation phenomenon for other conjectured gelling kernels began fifteen years later with the work of Lushnikov [26]. It continued with publications of Ziff [38], Ernst et al. [10], van Dongen and Ernst [36], Jeon [22, 23], Escobedo et al. [12], and many other mathematicians and mathematical physicists. In Spouge [35], gelation is demonstrated numerically for the general bilinear kernel $K(i,j) = A + B(i+j) + Cij$. Aldous [1] proved gelation for $K(i,j) = \frac{2(ij)^\gamma}{(i+j)^\gamma - i^\gamma - j^\gamma}$, where $\gamma \in (1,2)$. While $\gamma = 2$ corresponds to the multiplicative kernel for which, as we know, gelation also occurs. Jeon [23] proved that complete and instantaneous gelation occurs if $K(i,j) \geq ij\psi(i,j)$, where $\psi(i,j)$ is a function increasing in both variables, i and j, such that $\sum_{j=1}^{\infty} \frac{1}{j\psi(i,j)} < \infty$ for all i. This includes $K(i,j) = (ij)^\alpha$, $\alpha > 1$, as a primary example. Finally, Rezakhanlou [32] lists sufficient conditions for each of the three modes of gelation, i.e., simple, instantaneous, and complete.

2.2. **Multiplicative coalescent and its coagulation equations.** Consider the Marcus-Lushnikov process $\mathbf{ML}_n(t)$ corresponding to the multiplicative coalescent process that begins with n singletons, i.e., $K(i,j) = ij$ and $\mathbf{ML}_n(0) = (n, 0, 0, \ldots)$. In this case, the Smoluchowski coagulation equations (3) are stated as follows

$$\frac{d}{dt}\zeta_k = -k\zeta_k \sum_{j=1}^{\infty} j\zeta_j + \frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\zeta_j\zeta_{k-j} \qquad (k = 1, 2, \ldots) \text{ with } \zeta_k(0) = \delta_{1,k}. \tag{7}$$

The dynamics of the total mass $\sum_{j=1}^{\infty} j\zeta_j$ begins with $\sum_{j=1}^{\infty} j\zeta_j(0) = 1$, and following McLeod [28], we have

$$\begin{aligned} \frac{d}{dt}\sum_{j=1}^{\infty} j\zeta_j = \sum_{j=1}^{\infty} j\frac{d}{dt}\zeta_j &= -\sum_{i,j=1}^{\infty} ij^2\zeta_j\zeta_i + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{i=1}^{j-1}\big(i + (j-i)\big)i(j-i)\zeta_i\zeta_{j-i} \\ &= -\sum_{i,j=1}^{\infty} ij^2\zeta_j\zeta_i + \frac{1}{2}\sum_{i,j=1}^{\infty}\big(i+j\big)ij\zeta_i\zeta_j = 0 \end{aligned} \tag{8}$$

provided convergence of $\sum\limits_{j=1}^{\infty} j^2\zeta_j(t)$. Thus, there exists a time $T_{gel} \in (0, \infty]$, defined as the time such that the following conservation of mass formula (4) is satisfied up to T_{gel}, i.e., $\sum\limits_{j=1}^{\infty} j\zeta_j(t) = 1$.

Next, we want to modify the system (7) since the decay rate of $k\zeta_k \sum_{j=1}^{\infty} j\zeta_j$ in (7) does not include the gravitation of clusters of size k towards all the rest of the clusters. The problem is that a cluster of an exceptionally large size, say ϵn, in a single quantity will not be accounted for in (7). Yet, such a large cluster has to contribute $\epsilon k\zeta_k$ to the decay rate of ζ_k. Replacing the decay rate $k\zeta_k \sum_{j=1}^{\infty} j\zeta_j$ with $k\zeta_k$ would resolve this issue as the new rate accounts for the gravitation of a cluster of a given size k towards all clusters in the Marcus-Lushnikov process, whose weights add up to $n - k = n\big(1 + O(n^{-1})\big)$.

Thus, as it was done in [28], the Smoluchowski coagulation system (7) reduces to

$$\frac{d}{dt}\zeta_k = -k\zeta_k + \frac{1}{2}\sum_{j=1}^{k-1} j(k-j)\zeta_j\zeta_{k-j} \qquad (k = 1, 2, \ldots) \text{ with } \zeta_k(0) = \delta_{1,k} \tag{9}$$

which is solved explicitly:

$$\zeta_k(t) = \frac{k^{k-2}t^{k-1}}{k!}e^{-kt} \quad \text{ for } t \geq 0. \tag{10}$$

Now, since (9) is obtained from (7) by substituting $\sum_{j=1}^{\infty} j\zeta_j(t) = 1$, the solutions of systems (7) and (9) coincide as long as the conservation of mass holds, i.e., for all $t \in [0, T_{gel})$.

The above system of equations (9) is called *modified Smoluchowski system* (see Fournier and Giet [18]), and is also known as the *Flory coagulation system* of equations (named after Flory [15]). See also [19] for the analysis of a broad class of Smoluchowski and Flory systems, where the kernel $K(i,j) = i^\alpha j + ij^\alpha$, $\alpha \in (0,1]$.

Importantly, it is well known that the hydrodynamic limit of the Marcus-Lushnikov process $\lim\limits_{n\to\infty} \frac{\zeta_k^{[n]}(t)}{n} = \zeta_k(t)$ is the solution (10) of the modified Smoluchowski system (9) for the multiplicative kernel.

For the multiplicative kernel $K(i,j) = ij$ the *gelation time* T_{gel} is finite [28], and therefore, $K(i,j) = ij$ is a *gelling kernel*. Indeed, applying Stirling's approximation, we have the series

$$\sum_{j=1}^{\infty} j^2 \zeta_j(s) = \sum_{j=1}^{\infty} \frac{j^j s^{j-1}}{j!} e^{-js} = \frac{1}{\sqrt{2\pi} s} \sum_{j=1}^{\infty} \frac{e^{-j(s-\ln s-1)}}{\sqrt{j}} \big(1 + o(j^{-1})\big) \tag{11}$$

converging for all positive $s \neq 1$ and diverging for $s = 1$. Hence, the second moment $\sum\limits_{j=1}^{\infty} j^2 \zeta_j(t)$ in (8) converges for $t \in [0,1)$ and diverges for $t = 1$, i.e.,

$$T_{gel} := \inf\Big\{ t > 0 \ : \ \sum_{j=1}^{\infty} j^2 \zeta_j(t) = \infty \Big\} = 1.$$

Thus, the conservation of mass (4) is satisfied until the explosion of the second moment at $t = 1$.

Moreover, for $t > 0$, consider

$$x(t) := \min\{x > 0 \ : \ xe^{-x} = te^{-t}\}, \tag{12}$$

i.e., $x(t)$ is the unique $x \in (0,1]$ such that $xe^{-x} = te^{-t}$. Obviously, $x(t) = t$ for $0 < t \leq 1$.

We know that for $t < 1$, $\sum\limits_{k=1}^{\infty} k\zeta_k(t) = 1$ implies $\sum\limits_{k=1}^{\infty} \frac{k^{k-1}(te^{-t})^k}{k!} = t$. Thus, for all $t > 0$, $\sum\limits_{k=1}^{\infty} \frac{k^{k-1}(te^{-t})^k}{k!} = x(t)$. Hence, the first moment of the solutions ζ_k of (9) equals

$$\sum_{k=1}^{\infty} k\zeta_k(t) = \frac{1}{t} \sum_{k=1}^{\infty} \frac{k^{k-1}(te^{-t})^k}{k!} = \frac{x(t)}{t}. \tag{13}$$

and therefore, the solutions of (9) satisfy

$$\begin{cases} \sum\limits_{k=1}^{\infty} k\zeta_k(t) = 1 & \text{if } t \leq 1, \\ \sum\limits_{k=1}^{\infty} k\zeta_k(t) < 1 & \text{if } t > 1. \end{cases}$$

Compare this to the mass conservation property in the Marcus-Lushnikov processes:

$$\sum_{k=1}^{\infty} k \frac{\zeta_k^{[n]}(t)}{n} = 1 \quad \forall t \geq 0.$$

The above can be restated using the following alternative definition of the gelation time

$$T_{gel} := \inf \left\{ t > 0 \ : \ \sum_{j=1}^{\infty} j\zeta_j(t) < 1 \right\} = 1,$$

where ζ_k are the solutions of (9).

3. The cross-multiplicative coalescent process

In this section we analyze the cross-multiplicative coalescent process. We are motivated by the need to extend the theory and applications of coalescent processes to the particle system, described in the introduction, where not all pairs of particles interact with each other. Specifically, each particle may bond only with the particles of the opposite type.

For given $\alpha, \beta > 0$, we consider two integer valued functions, $\alpha[n] = \alpha n + o(\sqrt{n})$ and $\beta[n] = \beta n + o(\sqrt{n})$. We will examine a coalescent process where the weight of each cluster is a two-dimensional (weight) vector $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$. Here, $i_1, i_2 \geq 0$ and $i_1 + i_2 > 0$. Each cluster of weight $\mathbf{i}$ consists of i_1 particles of type A and i_2 particles of type B. The coalescent process begins with $\alpha[n] + \beta[n]$ singletons, of which $\alpha[n]$ singletons are of weight $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the other $\beta[n]$ singletons are of weight $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The coalescence kernel is defined by

$$K(\mathbf{i}, \mathbf{j}) := i_1 j_2 + i_2 j_1 \tag{14}$$

for any pair of clusters with weight vectors $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix}$. Each pair of clusters of respective weights $\mathbf{i}$ and $\mathbf{j}$ would coalesce into a cluster of weight $\mathbf{i} + \mathbf{j}$ with rate $K(\mathbf{i}, \mathbf{j})/n$. The last merger will create a cluster of weight $\begin{bmatrix} \alpha[n] \\ \beta[n] \end{bmatrix}$. We will call this *cross-multiplicative coalescent process*, and the kernel $K(\mathbf{i}, \mathbf{j})$ defined in (14) will be referred to as the *cross-multiplicative kernel*.

3.1. Coagulation equations.

Consider the Marcus-Lushnikov process $\mathbf{ML}_n(t)$ that keeps track of cluster counts in the above defined cross-multiplicative coalescent process that begins with $\alpha[n] + \beta[n]$ singletons of the two types, $\alpha[n]$ of weight $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\beta[n]$ of weight $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Specifically, let $\zeta_{i_1,i_2}^{[n]}(t)$ denote the number of components of weight $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ at time

t. Then $\mathbf{ML}_n(t)$ is the process with coordinates $\zeta^{[n]}_{i_1,i_2}(t)$, i.e.

$$\mathbf{ML}_n(t) = \left(\zeta^{[n]}_{i_1,i_2}(t)\right)_{i_1,i_2}$$

with the starting values $\zeta^{[n]}_{1,0}(0) = \alpha[n]$, $\zeta^{[n]}_{0,1}(0) = \beta[n]$, and $\zeta^{[n]}_{i_1,i_2}(0) = 0$ for all other pairs (i_1, i_2).

The Smoluchowski coagulation equations for the Marcus-Lushnikov process $\mathbf{ML}_n(t)$ with cross-multiplicative kernel are written as follows:

$$\frac{d}{dt}\zeta_{i_1,i_2}(t) = -\zeta_{i_1,i_2}(t)\sum_{j_1,j_2}(i_1j_2 + i_2j_1)\zeta_{j_1,j_2}(t) + \frac{1}{2}\sum_{\substack{\ell_1,k_1:\ \ell_1+k_1=i_1,\\ \ell_2,k_2:\ \ell_2+k_2=i_2}}(\ell_1k_2 + \ell_2k_1)\zeta_{\ell_1,\ell_2}(t)\zeta_{k_1,k_2}(t) \tag{15}$$

with the initial conditions $\zeta_{i_1,i_2}(0) = \alpha\delta_{1,i_1}\delta_{0,i_2} + \beta\delta_{0,i_1}\delta_{1,i_2}$.

A reduced system of differential equations corresponding to the above Smoluchowski coagulation equations (15) will be given in (18). It will take into account the mass conservation property of the above Marcus-Lushnikov process $\mathbf{ML}_n(t)$, and therefore will represent the smaller cluster dynamics over the whole time interval $[0, \infty)$.

First, we notice that here the initial total mass is $\sum\limits_{i_1,i_2}(i_1 + i_2)\zeta_{i_1,i_2}(0) = \alpha + \beta$. Moreover, the initial total 'left mass' (type A) is $\sum\limits_{i_1,i_2} i_1\zeta_{i_1,i_2}(0) = \alpha$ and the initial total 'right mass' (type B) is $\sum\limits_{i_1,i_2} i_2\zeta_{i_1,i_2}(0) = \beta$.

Next, we consider the rate of change for the total left mass and the total right mass, and use (15) to obtain

$$\begin{aligned}\frac{d}{dt}\sum_{i_1,i_2} i_1\zeta_{i_1,i_2}(t) \;&=\; -\sum_{i_1,i_2,j_1,j_2} i_1(i_1j_2 + i_2j_1)\zeta_{i_1,i_2}(t)\zeta_{j_1,j_2}(t)\\ &\quad +\frac{1}{2}\sum_{\ell_1,k_1,\ell_2,k_2}(\ell_1 + k_1)(\ell_1k_2 + \ell_2k_1)\zeta_{\ell_1,\ell_2}(t)\zeta_{k_1,k_2}(t) \;=0\end{aligned} \tag{16}$$

and

$$\begin{aligned}\frac{d}{dt}\sum_{i_1,i_2} i_2\zeta_{i_1,i_2}(t) \;&=\; -\sum_{i_1,i_2,j_1,j_2} i_2(i_1j_2 + i_2j_1)\zeta_{i_1,i_2}(t)\zeta_{j_1,j_2}(t)\\ &\quad +\frac{1}{2}\sum_{\ell_1,k_1,\ell_2,k_2}(\ell_2 + k_2)(\ell_1k_2 + \ell_2k_1)\zeta_{\ell_1,\ell_2}(t)\zeta_{k_1,k_2}(t) \;=0\end{aligned} \tag{17}$$

whenever $\sum\limits_{i_1,i_2}(i_1 + i_2)^2\zeta_{i_1,i_2}(t)$ converges.

Here, for $t < T_{gel}$, $\sum\limits_{j_1,j_2} j_1\zeta_{j_1,j_2}(t) = \alpha$ and $\sum\limits_{j_1,j_2} j_2\zeta_{j_1,j_2}(t) = \beta$. Therefore, for any i_1 and i_2, $\sum\limits_{j_1,j_2}(i_1j_2 + i_2j_1)\zeta_{j_1,j_2}(t) = \beta i_1 + \alpha i_2$. Thus we can consider the following *modified Smoluchowski coagulation system of equations*:

$$\frac{d}{dt}\zeta_{i_1,i_2}(t) = -(\beta i_1 + \alpha i_2)\zeta_{i_1,i_2}(t) + \frac{1}{2}\sum_{\substack{\ell_1,k_1:\ \ell_1+k_1=i_1,\\ \ell_2,k_2:\ \ell_2+k_2=i_2}}(\ell_1k_2 + \ell_2k_1)\zeta_{\ell_1,\ell_2}(t)\zeta_{k_1,k_2}(t) \tag{18}$$

with the initial conditions $\zeta_{i_1,i_2}(0) = \alpha\delta_{1,i_1}\delta_{0,i_2} + \beta\delta_{0,i_1}\delta_{1,i_2}$. Once again, the solutions of Smoluchowski coagulation system (15) and the above modified Smoluchowski coagulation system (18) will match up until T_{gel}. Consequently, the solution (23) of the modified Smoluchowski system of equations (18) is used in Sect. 3.3 for establishing the finiteness of the gelation time and for finding its value, T_{gel}.

In Sect. 5.3 we establish that the solution to the above modified Smoluchowski coagulation system (18) is the hydrodynamic limit of the Marcus-Lushnikov process $\mathbf{ML}_n(t)$ with cross-multiplicative kernel. Specifically, in equation (64), it is shown that

$$\lim_{n\to\infty}\sup_{s\in[0,T]}\left|n^{-1}\zeta^{[n]}_{i_1,i_2}(s) - \zeta_{i_1,i_2}(s)\right| = 0 \qquad \text{a.s.}$$

for any given $T > 0$ and all $i_1, i_2 \geq 1$, where $\zeta_{i_1,i_2}(t)$ solves the modified Smoluchowski coagulation system (18).

3.2. **The unique solution of the modified Smoluchowski coagulation system.** Next, we want to find the solution $\zeta_{i_1,i_2}(t)$ of the reduced system (18) for all $t \geq 0$. Here we observe that $\zeta_{1,0}(t) = \alpha e^{-\beta t}$ and $\zeta_{0,1}(t) = \beta e^{-\alpha t}$, and extend the approach of McLeod [28] by considering the solutions of the following form

$$\zeta_{i_1,i_2}(t) = \alpha^{i_1}\beta^{i_2}S_{i_1,i_2}e^{-(\beta i_1+\alpha i_2)t}t^{i_1+i_2-1} \tag{19}$$

and plugging them into equation (18). After cancelations, we arrive with the following recursion

$$(i_1 + i_2 - 1)S_{i_1,i_2} = \frac{1}{2}\sum_{\substack{\ell_1,k_1:\ \ell_1+k_1=i_1,\\ \ell_2,k_2:\ \ell_2+k_2=i_2}}(\ell_1k_2 + \ell_2k_1)S_{\ell_1,\ell_2}S_{k_1,k_2} \tag{20}$$

with initial conditions $S_{i,0} = S_{0,i} = \delta_{1,i}$, and $S_{i_1,i_2} = S_{i_2,i_1}$.

In the next lemma we state the explicit solution to the recursion relation (20) which we prove using a generalization of Abel's binomial theorem.

Lemma 3.1. *The system of equations (20) with the initial conditions $S_{i,0} = S_{0,i} = \delta_{1,i}$ has the following unique solution*

$$S_{i_1,i_2} = \frac{i_1^{i_2-1}i_2^{i_1-1}}{i_1!i_2!}. \tag{21}$$

Note that the numerator $i_1^{i_2-1}i_2^{i_1-1}$ in (21) is the total number of spanning trees in K_{i_1,i_2}. See [3].

Proof. In Theorem 1.1(3) of [20], F. Huang and B. Liu generalize Abel's binomial theorem as follows:

$$\begin{aligned}
&\sum_{k_1=0}^{i_1}\sum_{k_2=0}^{i_2}\binom{i_1}{k_1}\binom{i_2}{k_2}\big(v+zi_1-zk_1\big)^{k_2-1}\big(-z(i_1-k_1)\big)^{i_2-k_2}(-zk_2)^{k_1}(u+zk_2)^{i_1-k_1-1}\\
&\qquad=\frac{[uv-i_1i_2z^2]u^{i_1-1}v^{i_2-1}}{(v+i_1z)(u+i_2z)} \qquad (22)
\end{aligned}$$

Then, we use (22) with $z=-1$ to confirm our candidate solution satisfies (20) by plugging it into the right hand side of (20) as follows.

$$\begin{aligned}
\frac{1}{2}\sum_{\substack{\ell_1,k_1:\ \ell_1+k_1=i_1,\\ \ell_2,k_2:\ \ell_2+k_2=i_2}}(\ell_1k_2+\ell_2k_1)S_{\ell_1,\ell_2}S_{k_1,k_2} &= \sum_{\substack{\ell_1,k_1:\ \ell_1+k_1=i_1,\\ \ell_2,k_2:\ \ell_2+k_2=i_2}}\ell_1k_2S_{\ell_1,\ell_2}S_{k_1,k_2}\\
&= \sum_{\substack{\ell_1,k_1:\ \ell_1+k_1=i_1,\\ \ell_2,k_2:\ \ell_2+k_2=i_2,\\ (k_1,k_2),(\ell_1,\ell_2)\neq(0,0)}}\frac{\ell_1^{\ell_2}\ell_2^{\ell_1-1}k_1^{k_2-1}k_2^{k_1}}{\ell_1!\ell_2!k_1!k_2!}
\end{aligned}$$

$$\begin{aligned}
&=\frac{1}{i_1!i_2!}\sum_{\substack{k_1:\ 0\le k_1\le i_1,\\ k_2:\ 0\le k_2\le i_2,\\ (k_1,k_2)\neq(0,0),(i_1,i_2)}}\binom{i_1}{k_1}\binom{i_2}{k_2}k_1^{k_2-1}(i_1-k_1)^{i_2-k_2}k_2^{k_1}(i_2-k_2)^{i_1-k_1-1}\\
&=\frac{1}{i_1!i_2!}\lim_{\substack{v\to i_1,\\ u\to i_2}}\Bigg\{\sum_{k_1=0}^{i_1}\sum_{k_2=0}^{i_2}\binom{i_1}{k_1}\binom{i_2}{k_2}\big(v-i_1+k_1\big)^{k_2-1}(i_1-k_1)^{i_2-k_2}k_2^{k_1}(u-k_2)^{i_1-k_1-1}\\
&\qquad\qquad -\frac{i_1^{i_2}u^{i_1-1}}{v-i_1}-\frac{i_2^{i_1}v^{i_2-1}}{u-i_2}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&=\frac{1}{i_1!i_2!}\lim_{\substack{v\to i_1,\\ u\to i_2}}\left\{\frac{[uv-i_1i_2]u^{i_1-1}v^{i_2-1}}{(v-i_1)(u-i_2)}-\frac{i_1^{i_2}u^{i_1-1}}{v-i_1}-\frac{i_2^{i_1}v^{i_2-1}}{u-i_2}\right\}\\
&=\frac{1}{i_1!i_2!}\lim_{\substack{v\to i_1,\\ u\to i_2}}\left\{\frac{i_1v^{i_2-1}u^{i_1-1}}{v-i_1}+\frac{u^{i_1}v^{i_2-1}}{u-i_2}-\frac{i_1^{i_2}u^{i_1-1}}{v-i_1}-\frac{i_2^{i_1}v^{i_2-1}}{u-i_2}\right\}
\end{aligned}$$

Hence,

$$\frac{1}{2}\sum_{\substack{\ell_1,k_1:\ \ell_1+k_1=i_1,\\ \ell_2,k_2:\ \ell_2+k_2=i_2}}(\ell_1k_2+\ell_2k_1)S_{\ell_1,\ell_2}S_{k_1,k_2} = \frac{1}{i_1!i_2!}\lim_{\substack{v\to i_1,\\ u\to i_2}}\left\{i_1u^{i_1-1}\frac{v^{i_2-1}-i_1^{i_2-1}}{v-i_1}+v^{i_2-1}\frac{u^{i_1}-i_2^{i_1}}{u-i_2}\right\}$$
$$= \frac{1}{i_1!i_2!}\big((i_2-1)\cdot i_1^{i_2-1}i_2^{i_1-1}+i_1\cdot i_1^{i_2-1}i_2^{i_1-1}\big)$$
$$= (i_1+i_2-1)\frac{i_1^{i_2-1}i_2^{i_1-1}}{i_1!i_2!}$$
$$= (i_1+i_2-1)S_{i_1,i_2}$$

thus completing the proof. □

The solution of equations (18) follows from (19) and Lemma 3.1.

Theorem 3.2. *The modified Smoluchowski coagulation system of equations* (18) *with the initial conditions* $\zeta_{i_1,i_2}(0)=\alpha\delta_{1,i_1}\delta_{0,i_2}+\beta\delta_{0,i_1}\delta_{1,i_2}$ *has the unique solution*

$$\zeta_{i_1,i_2}(t) = \frac{i_1^{i_2-1}i_2^{i_1-1}\alpha^{i_1}\beta^{i_2}}{i_1!i_2!}e^{-(\beta i_1+\alpha i_2)t}t^{i_1+i_2-1}. \tag{23}$$

3.3. **Gelation in the cross-multiplicative coalescent process.** Next, we prove the finiteness of the gelation time that, following the approach in (6), we define as

$$T_{gel} := \inf\Big\{t>0\ :\ \sum_{i_1,i_2}(i_1+i_2)\zeta_{i_1,i_2}(t)<\alpha+\beta\Big\}.$$

Let

$$s(u,v) := \sum_{(i_1,i_2)\in\mathbb{Z}_+^2\setminus\{(0,0)\}} S_{i_1,i_2}u^{i_1}v^{i_2} = \sum_{(i_1,i_2)\in\mathbb{Z}_+^2\setminus\{(0,0)\}}\frac{i_1^{i_2-1}i_2^{i_1-1}}{i_1!i_2!}u^{i_1}v^{i_2} \tag{24}$$

be the generating function of S_{i_1,i_2}. The recurrence relation (20) implies

$$u\frac{\partial s}{\partial u}+v\frac{\partial s}{\partial v}-s = uv\frac{\partial s}{\partial u}\frac{\partial s}{\partial v} \tag{25}$$

with the initial conditions $\frac{\partial}{\partial u}s(0,1)=\frac{\partial}{\partial v}s(1,0)=1$.

Lemma 3.3. *Consider the Smoluchowski coagulation system of equations* (15) *with the initial conditions* $\zeta_{i_1,i_2}(0)=\alpha\delta_{1,i_1}\delta_{0,i_2}+\beta\delta_{0,i_1}\delta_{1,i_2}$. *Then, a phase transition occurs at*

$$\inf\Big\{t>0\ :\ \sum_{i_1,i_2}(i_1+i_2)^2\zeta_{i_1,i_2}(t)=\infty\Big\} = \frac{1}{\sqrt{\alpha\beta}}.$$

Note that the above phase transition corresponds to the gelation times as defined in (5).

Proof. We will follow the approach in [1, 2] and [38]. Let

$$E(t) := \sum_{i_1,i_2} i_1^2 \zeta_{i_1,i_2}(t), \quad F(t) := \sum_{i_1,i_2} i_1 i_2 \zeta_{i_1,i_2}(t), \quad \text{and} \quad G(t) := \sum_{i_1,i_2} i_2^2 \zeta_{i_1,i_2}(t)$$

denote all the second order moments of $\zeta_{i_1,i_2}(t)$. By differentiating as in (16) and (17), we obtain

$$\frac{d}{dt}E(t) = 2E(t)F(t), \quad \frac{d}{dt}F(t) = E(t)G(t) + F^2(t), \quad \text{and} \quad \frac{d}{dt}G(t) = 2G(t)F(t)$$

with the initial conditions $E(0) = \alpha$, $F(0) = 0$, and $G(0) = \beta$. We require the finiteness of all third order moments when deriving the above differential equations for the second order moments. Here the first and the third equations yield $E(t) = \frac{\alpha}{\beta}G(t)$. Hence the system reduces to

$$\frac{d}{dt}E(t) = 2E(t)F(t) \quad \text{and} \quad \frac{d}{dt}F(t) = \frac{\beta}{\alpha}E^2(t) + F^2(t),$$

and therefore,

$$\frac{d}{dt}\left(\sqrt{\frac{\beta}{\alpha}}E(t) + F(t)\right) = \left(\sqrt{\frac{\beta}{\alpha}}E(t) + F(t)\right)^2.$$

Thus,

$$\sqrt{\frac{\beta}{\alpha}}E(t) + F(t) = \frac{1}{\frac{1}{\sqrt{\alpha\beta}} - t}$$

for $t < \frac{1}{\sqrt{\alpha\beta}}$. The statement of the lemma follows from the fact that all functions obtained as all-order partial derivatives of the series (24) have the same domain of convergence. □

For given $\alpha, \beta > 0$ and $t > 0$, define

$$\big(x(t), y(t)\big) := \min\left\{(x,y) \ : \ xe^{-y} = \alpha t e^{-\beta t}, \ ye^{-x} = \beta t e^{-\alpha t}\right\}, \tag{26}$$

where the minimum in one coordinate implies the minimum in another as x and y solving

$$xe^{-y} = u \quad \text{and} \quad ye^{-x} = v \tag{27}$$

for $u, v > 0$ are mutually monotonous, e.g. $x = ue^y$.

Proposition 3.4. *For given $u, v > 0$, consider the system* (27)*. Then, the following holds.*

(i) *Depending on the values of u and v, the system* (27) *may have one, two, or no solutions.*
(ii) *If the system* (27) *has a unique solution, then the solution should satisfy $xy = 1$.*
(iii) *If the system* (27) *has two solutions, then the smallest solution should satisfy $xy < 1$, and the largest solution should satisfy $xy > 1$.*

Proof. First, observe that $x = ue^{ve^x}$, and statement (i) follows from the convexity of ue^{ve^x}.

Next, suppose (x_1, y_1) and (x_2, y_2) are two solutions of (27). Then

$$x_1 e^{-y_1} = x_2 e^{-y_2} \quad \text{and} \quad y_1 e^{-x_1} = y_2 e^{-x_2}. \tag{28}$$

We express x_2 in terms of x_1 and y_1, obtaining $x_2 = x_1 e^{y_1(e^{x_2 - x_1} - 1)}$. We notice that there is a unique solution $x = x_1$ of

$$x = x_1 e^{y_1(e^{x - x_1} - 1)} \tag{29}$$

if and only if $x = x_1$ is the root of $1 = xy_1 e^{x - x_1}$. This yields statement (ii).

Finally, suppose there are two distinct solutions of (29), and $x_2 > x_1$ (implying $y_2 > y_1$). Then, there is a local extremum $x \in (x_1, x_2)$, satisfying $1 = xy_1 e^{x - x_1} > x_1 y_1$.

Similarly, suppose there are two distinct solutions of (29), and $x_2 < x_1$ (implying $y_2 < y_1$). Then, there is a local extremum $x \in (x_2, x_1)$, satisfying $1 = xy_1 e^{x - x_1} < x_1 y_1$. Hence, statement (iii). □

Prop. 3.4 immediately yields the following corollary concerning the functions defined in (26).

Corollary 3.5. *For given $\alpha, \beta > 0$ and $t > 0$, consider $\big(x(t), y(t)\big)$ as defined in* (26). *Then,*

- *$x(t) = \alpha t$ and $y(t) = \beta t$ for all $t \leq \frac{1}{\sqrt{\alpha\beta}}$;*
- *$x(t) < \alpha t$ and $y(t) < \beta t$ for all $t > \frac{1}{\sqrt{\alpha\beta}}$.*

Next, we derive an analogue to the equation (13) that was proved in McLeod [28] for the regular multiplicative kernel.

Lemma 3.6. *Consider the solution $\zeta_{i_1,i_2}(t)$ of the modified Smoluchowski coagulation system of equations* (18) *with the initial conditions $\zeta_{i_1,i_2}(0) = \alpha\delta_{1,i_1}\delta_{0,i_2} + \beta\delta_{0,i_1}\delta_{1,i_2}$, as found in Theorem 3.2. Then,*

$$\sum_{i_1,i_2} i_1 \zeta_{i_1,i_2}(t) = \sum_{i_1,i_2} \frac{i_1^{i_2} i_2^{i_1 - 1} \alpha^{i_1} \beta^{i_2}}{i_1! i_2!} e^{-(\beta i_1 + \alpha i_2)t} t^{i_1 + i_2 - 1} = \frac{x(t)}{t}$$

and

$$\sum_{i_1,i_2} i_2 \zeta_{i_1,i_2}(t) = \sum_{i_1,i_2} \frac{i_1^{i_2 - 1} i_2^{i_1} \alpha^{i_1} \beta^{i_2}}{i_1! i_2!} e^{-(\beta i_1 + \alpha i_2)t} t^{i_1 + i_2 - 1} = \frac{y(t)}{t},$$

where $x(t)$ and $y(t)$ are the functions defined in (26).

Proof. Observe that

$$\sum_{i_1,i_2} i_1 \zeta_{i_1,i_2}(t) = \alpha t e^{-\beta t} \frac{\partial s}{\partial u}\big(\alpha t e^{-\beta t}, \beta t e^{-\alpha t}\big)$$

and

$$\sum_{i_1,i_2} i_2 \zeta_{i_1,i_2}(t) = \beta t e^{-\alpha t} \frac{\partial s}{\partial v}\big(\alpha t e^{-\beta t}, \beta t e^{-\alpha t}\big).$$

By (16), (17), and Lemma 3.3, we have

$$\alpha t = \alpha t e^{-\beta t} \frac{\partial s}{\partial u}\big(\alpha t e^{-\beta t}, \beta t e^{-\alpha t}\big) \quad \text{and} \quad \beta t = \beta t e^{-\alpha t} \frac{\partial s}{\partial v}\big(\alpha t e^{-\beta t}, \beta t e^{-\alpha t}\big) \tag{30}$$

$\forall \alpha, \beta > 0$ and $\forall t < \frac{1}{\sqrt{\alpha\beta}}$. Now, since the function $s(u, v)$ does not depend on the values of α and β, (30) implies

$$x = x e^{-y} \frac{\partial s}{\partial u}\big(x e^{-y}, y e^{-x}\big) \qquad \text{and} \qquad y = y e^{-x} \frac{\partial s}{\partial v}\big(x e^{-y}, y e^{-x}\big)$$

for all $xy < 1$. Hence, by Prop. 3.4 , we have

$$x = u \frac{\partial s}{\partial u}(u, v) \qquad \text{and} \qquad y = v \frac{\partial s}{\partial v}(u, v)$$

whenever (x, y) is the smallest solution of (27). The equations

$$x(t) = \alpha t e^{-\beta t} \frac{\partial s}{\partial u}\big(\alpha t e^{-\beta t}, \beta t e^{-\alpha t}\big) \;\text{ and }\; y(t) = \beta t e^{-\alpha t} \frac{\partial s}{\partial v}\big(\alpha t e^{-\beta t}, \beta t e^{-\alpha t}\big) \qquad \forall t \geq 0, \tag{31}$$

with $x(t)$ and $y(t)$ defined in (26), follow from Proposition 3.4. □

Corollary 3.7. *The cross-multiplicative kernel defined in* (14) *is a gelling kernel, and the gelation time corresponding to the Smoluchowski coagulation system of equations* (15) *with the initial conditions* $\zeta_{i_1,i_2}(0) = \alpha\delta_{1,i_1}\delta_{0,i_2} + \beta\delta_{0,i_1}\delta_{1,i_2}$ *equals*

$$T_{gel} = \frac{1}{\sqrt{\alpha\beta}}.$$

Proof. Lemma 3.6 and Proposition 3.4 imply that the mass of the system in (18) is conserved until $\frac{1}{\sqrt{\alpha\beta}}$, after which time it begins to dissipate, i.e.,

$$\begin{cases} \sum\limits_{i_1,i_2} (i_1 + i_2)\zeta_{i_1,i_2}(t) = \alpha + \beta & \text{if } t \leq \frac{1}{\sqrt{\alpha\beta}}; \\ \sum\limits_{i_1,i_2} (i_1 + i_2)\zeta_{i_1,i_2}(t) < \alpha + \beta & \text{if } t > \frac{1}{\sqrt{\alpha\beta}}. \end{cases}$$

□

Recall that we considered two alternative definitions of gelation time in Sect. 2.1. Definition (5) would often describe the time of the explosion of a higher moment while definition (6) is based on the loss of total mass after gelation. Comparing Lemma 3.3 with Corollary 3.7, we confirm the equivalence of the two alternative definitions of the gelation time for the cross-multiplicative kernel, i.e.,

$$\inf\Big\{t > 0 \;:\; \sum_{i_1,i_2} (i_1+i_2)^2 \zeta_{i_1,i_2}(t) = \infty\Big\} = T_{gel} = \inf\Big\{t > 0 \;:\; \sum_{i_1,i_2} (i_1+i_2)\zeta_{i_1,i_2}(t) < \alpha+\beta\Big\}.$$

4. Applications in minimal spanning trees

In this section we demonstrate how the coagulation equations for the multiplicative and cross-multiplicative coalescent processes and the weak convergence results of Section 5 can be used for finding the lengths of the minimal spanning trees on the complete graph K_n and on the complete bipartite graph $K_{\alpha[n],\beta[n]}$ respectively. The main result of this section Theorem 4.3 is proved using Marcus-Lushnikov processes and coagulation equations in Sect. 4.6. The proof of Theorem 4.3 is preceded by the proof of Theorem 4.1 in Sect. 4.3, a well-known result of Frieze [16] that we use to illustrate the approach.

We recall the following quote from Aldous [1]: *It turns out that there is a large scientific literature relevant to the Marcus-Lushnikov process, mostly focusing on its deterministic approximation. Curiously, this literature has been largely ignored by random graph theorists.* The broader goal of this section is in bridging the gap between the theory of the Smoluchowski coagulation equations for the Marcus-Lushnikov processes and the random graph theory. Here we concentrate on analyzing the length of the minimal spanning tree as the prime example that demonstrates the usefulness of the Marcus-Lushnikov processes and the coalescence theory in general for answering questions about random graphs. We recall that the asymptotic limit for the mean length of a minimal spanning tree on K_n with independent uniform edge weights over $[0,1]$ was derived in Frieze [16]. There, it is shown to be $\lim\limits_{n\to\infty} E[L_n] = \zeta(3) = \sum\limits_{k=1}^{\infty} \frac{1}{k^3}$. In [17], the mean length of the minimal spanning tree for the complete bipartite graph $K_{n,n}$ with independent edge weights distributed uniformly over $[0,1]$ was shown to have asymptotic limit $\lim\limits_{n\to\infty} E[L_n] = 2\zeta(3)$. In Beveridge et al [6], the minimal spanning tree problem was addressed for d-regular graphs. In Sect. 4.5 and 4.6, we will find the mean length of the minimal spanning tree in the case of a complete bipartite graph $K_{\alpha[n],\beta[n]}$ via a connection between the coalescence theory and the random graph theory. Note that $K_{\alpha[n],\beta[n]}$ is an irregular graph when $\alpha \neq \beta$.

4.1. **Relation of Erdős-Rényi process on K_n to multiplicative coalescent.** Recall that Erdős-Rényi random graph is a model on a complete graph of n vertices, K_n, where each edge e of $\binom{n}{2}$ edges there is an associated uniform random variable U_e over $[0,1]$. The random variables $\{U_e\}_e$ are assumed to be independent. For the "time" parameter $p \in [0,1]$, an edge e is considered "open" if $U_e \leq p$. Erdős-Rényi random graph $G(n,p)$ will consist of all n vertices and all open edges at time p. The number of open edges is a binomial random variable with parameters $\binom{n}{2}$ and p, and mean value $\binom{n}{2}p \sim \frac{pn^2}{2}$. As we increase p, more and more edges open up, new clusters are created, and cluster merges occur. Thus Erdős-Rényi random graph model can be viewed as a dynamical model that describes an evolution of a random graph [9].

If we condition on the number of edges in $G(n,p)$, the graph structure will no longer depend on p. Let $\xi_{n,N}$ be the number of components in an Erdős-Rényi random graph with n vertices and N edges. For $t > 0$, letting $N \sim \frac{tn}{2}$, Theorem 6 in [9] by P. Erdős and

A. Rényi states that

$$\frac{E[\xi_{n,N}]}{n} = \frac{1}{t}\sum_{k=1}^{\infty}\frac{k^{k-2}(te^{-t})^k}{k!} + \mathcal{R}_t, \tag{32}$$

where the error term is

$$\mathcal{R}_t = \begin{cases} O\left(\frac{1}{n}\right) & \text{if } 0<t<1 \\ O\left(\frac{\log n}{n}\right) & \text{if } t=1 \\ o(1) & \text{if } t>1 \end{cases}.$$

There $\varphi(t) = \frac{1}{t}\sum\limits_{k=1}^{\infty}\frac{k^{k-2}(te^{-t})^k}{k!}$ reaches its maximum at $t=1$, and $\varphi(1) = \sum\limits_{k=1}^{\infty}\frac{k^{k-2}e^{-k}}{k!} = \frac{1}{2}$. Recall the function $x(t)$ defined in (12). It was pointed out by P. Erdős and A. Rényi that $\varphi(t)$ in the equation (32) can be represented via $x(t)$ as follows,

$$\varphi(t) = \frac{x(t) - \frac{x^2(t)}{2}}{t}.$$

Observe that here, since we are letting $N \sim \frac{tn}{2}$, parameter t is essentially equivalent to np. So t is a scaled time parameter.

Let the number of vertices in a connected component of a random graph be referred to as a weight of the cluster (or cluster size). In Sect. 2.2 we considered the Marcus-Lushnikov process

$$\mathbf{ML}_n(t) = \left(\zeta_1^{[n]}(t), \zeta_2^{[n]}(t), \ldots, \zeta_n^{[n]}(t), 0, 0, \ldots\right)$$

corresponding to the multiplicative coalescent process of n particles that begins with n singletons, i.e., $\mathbf{ML}_n(0) = (n, 0, 0, \ldots)$. As observed in [2], the process $\mathbf{ML}_n(t)$ describes cluster size dynamics of the Erdős-Rényi random graph process $G(n,p)$ with $p = 1-e^{-t/n}$. Here the scaled time parameter in the Erdős-Rényi process is $np = n\left(1-e^{-t/n}\right) \sim t$. Thus the time scale is consistent with the one used in [9] by P. Erdős and A. Rényi.

Recall the function ζ_k in (10) that solves (9). As we know, in the Erdős-Rényi process, the gelation time $T_{gel} = 1$ of the Marcus-Lushnikov process with the multiplicative kernel corresponds to a time after which a single giant component emerges, and continues to absorb components of smaller size. Indeed, in [9], P. Erdős and A. Rényi showed that the cycles are rare for a given fixed $t > 0$, and the clusters of size k at time t consist mainly of *isolated trees of order* k. Specifically, if τ_k denotes the number of isolated trees of order k, Theorem 4b in [9] asserts that

$$\lim_{n\to\infty}\frac{kE[\tau_k]}{n} = \frac{k^{k-2}t^{k-1}}{k!}e^{-kt} = \zeta_k(t) \tag{33}$$

and

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{\infty} kE[\tau_k]}{n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} kE[\tau_k]}{n} = \frac{x(t)}{t}, \tag{34}$$

where $x(t)$ is defined in (12). Moreover, Theorem 9b in [9] proves the emergence of one giant component after time $t = 1$. There, if we let $\gamma_n(t)$ denote the size of the greatest component at time t, then

$$\lim_{n\to\infty} \frac{\gamma_n(t)}{n} = 1 - \frac{x(t)}{t} \qquad \text{in probability.}$$

So the dynamics of $g(t) := 1 - \sum_{k=1}^{\infty} k\zeta_k(t) = 1 - \frac{x(t)}{t}$ represents the asymptotic size of the giant component.

4.2. **The length of the minimal spanning tree in K_n.** Recall that in the construction of the Erdős-Rényi random graph model, each edge e of the complete graph K_n had a random variable U_e associated with it. Here we consider U_e to be uniform over $[0,1]$. However, in general, various types of probability distributions are considered in the extensive literature on the topic. Now, thinking of U_e as the length of the edge e, one can construct a minimal spanning tree on K_n. Let random variable L_n denote the length of such minimal spanning tree. The asymptotic limit of the mean value of L_n was considered in Frieze [16]. There, the results (33) and (34) from P. Erdős and A. Rényi [9] are used in proving the following limit

$$\lim_{n\to\infty} E[L_n] = \int_0^{\infty} \frac{x(t)}{t} dt = \sum_{k=1}^{\infty} \int_0^{\infty} \frac{k^{k-2}t^{k-1}}{k!} e^{-kt} dt = \zeta(3), \tag{35}$$

where $\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} = 1.202\ldots$ is the value of the Riemann zeta function at 3.

Consider a coalescent process with a kernel $K(i,j)$ for which $T_{gel} < \infty$ has been proved. See [1, 23]. Then for a corresponding random graph model, we use the following S. Janson's formula [21]

$$\lim_{n\to\infty} E[L_n] = \lim_{n\to\infty} \int_0^{1} E[\kappa(G(n,p))] dp - 1, \tag{36}$$

where $\kappa(G(n,p))$ is the number of components in the Erdős-Rényi random graph $G(n,p)$, and prove the following statement.

Theorem 4.1. *Let L_n denote the length of the the minimal spanning tree in K_n, where edge weights are independent and uniform random variables on $[0,1]$. Then*

$$\lim_{n\to\infty} E[L_n] = \sum_{k=1}^{\infty}\int_0^{\infty} \zeta_k(t)dt, \tag{37}$$

where $\zeta_k(t)$ are the solutions (10) *of the corresponding system of the modified Smoluchowski coagulation equations* (9).

Observe that the above equation (37) reproduces the result (35) of Frieze [16]. Indeed, plugging (10) into (37), we obtain

$$\begin{aligned}
\lim_{n\to\infty} E[L_n] &= \sum_{k=1}^{\infty}\int_0^{\infty} \zeta_k(t)dt = \sum_{k=1}^{\infty}\int_0^{\infty} \frac{k^{k-2}t^{k-1}}{k!}e^{-kt}dt \\
&= \sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^{\infty} \frac{k^k t^{k-1}}{\Gamma(k)}e^{-kt}dt = \sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(3).
\end{aligned}$$

Theorem 4.1 will be proved in Sect. 4.3. There, we give a novel proof to this well known result [16]. The proof utilizes only the modified Smoluchowski coagulation equations (9) and the weak convergence results that appear in Section 5 of this paper..

Here is the heuristics behind the proof of Theorem 4.1 presented in Sect. 4.3. We already observed that the Marcus-Lushnikov process $\mathbf{ML}_n(t)$ corresponding to the multiplicative coalescent process that begins with n singletons is equivalent to the cluster size dynamics in the process $G(n, 1-e^{-t/n})$. Here

$$\begin{aligned}
\lim_{n\to\infty} E[L_n] &= \lim_{n\to\infty}\int_0^{1} E[\kappa(G(n,p))]\,dp - 1 = \lim_{n\to\infty}\int_0^{\infty}\frac{1}{n}E[\kappa(G(n,1-e^{-t/n}))]e^{-t/n}dt - 1 \\
&= \lim_{n\to\infty}\int_0^{\infty}\sum_{k=1}^{\infty}\frac{1}{n}E[\kappa^{er}(k,n,1-e^{-t/n})]e^{-t/n}dt - 1 \\
&= \lim_{n\to\infty}\int_0^{\infty}\sum_{k=1}^{\infty}\frac{E[\zeta_k^{[n]}(t)]}{n}e^{-t/n}dt - 1,
\end{aligned}$$

where $\kappa^{er}(k,n,p)$ is the number of components of size k in $G(n,p)$ and $p = 1 - e^{-t/n}$. Therefore, one could informally calculate the limit as follows:

$$\begin{aligned}\lim_{n\to\infty} E[L_n] &= \sum_{k=1}^{\infty} \int_0^{\infty} \zeta_k(t)dt + \lim_{n\to\infty} \int_{T_{gel}}^{\infty} \frac{1}{n} e^{-t/n} dt - 1 \\ &= \sum_{k=1}^{\infty} \int_0^{\infty} \zeta_k(t)dt + \lim_{n\to\infty} e^{-T_{gel}/n} - 1 \ = \sum_{k=1}^{\infty} \int_0^{\infty} \zeta_k(t)dt. \end{aligned} \tag{38}$$

Here $\int\limits_{T_{gel}}^{\infty} \frac{1}{n} e^{-t/n} dt$ represents the emergence of one giant component at time $T_{gel} = 1$.

4.3. **Proof of Theorem 4.1.** Here we give a rigorous proof using the idea behind the approach in formula (38). Note that unlike the original proof in Frieze [16], our proof will not rely on knowing the distribution of sizes and the geometry of clusters in the Erdős-Rényi process as provided in [9]. Nor will it require knowing anything about large clusters or the emergence of a unique giant component at time $T_{gel} = 1$. All that we use is the weak convergence results of Kurtz [11, 25] that we applied to the Marcus-Lushnikov processes in Section 5.

Proof. Observe that

$$\lim_{t\to\infty} \sum_{k=1}^{\infty} k\zeta_k(t) = \lim_{t\to\infty} \sum_{k=1}^{\infty} \frac{k^{k-1}t^{k-1}}{k!} e^{-kt} = \lim_{t\to\infty} \frac{x(t)}{t} = 0. \tag{39}$$

Thus, for any given $\epsilon \in (0, 1/4)$, we can fix $T \gg T_{gel}$ so large that

$$\sum_{k=1}^{\infty} k\zeta_k(T) \le \frac{\epsilon}{2}. \tag{40}$$

Notice that the above inequality (40) ties T to ϵ.

Fix integer $K > 0$. By the equation (62) proved in Sect. 5.2 we know that

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left| \sum_{k=1}^{K} n^{-1}\zeta_k^{[n]}(s) - \sum_{k=1}^{K} \zeta_k(s) \right| = 0 \qquad \text{a.s.}$$

Thus, the probability of the complement of the event

$$Q^{\epsilon}_{K,T,n} := \left\{ \sum_{k=1}^{K} \frac{k\zeta_k^{[n]}(T)}{n} \le \epsilon \right\} \tag{41}$$

is decreasing to zero as $n \to \infty$. Moreover,

$$q^{\epsilon}_{K,T}(n) := P(\overline{Q^{\epsilon}_{K,T,n}}) = O(n^{-2})$$

by Proposition 5.3 in Sect. 5.4 as $\sum\limits_{k=1}^{K} \frac{k\zeta_k^{[n]}(0)}{n} - \sum\limits_{k=1}^{K} k\zeta_k(0) = 0$.

We will split $\int\limits_0^\infty \sum\limits_{k=1}^{\infty} \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n}dt$ as follows.

$$
\begin{aligned}
\int\limits_0^\infty \sum_{k=1}^{\infty} \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n}dt &= \int\limits_0^T \sum_{k=1}^{K} \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n}dt && \textbf{(Term I)}\\
&+ \int\limits_0^T \sum_{k=K+1}^{\infty} \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n}dt && \textbf{(Term II)}\\
&+ \big(1-q_{K,T}^{\epsilon}(n)\big) \int\limits_T^\infty \sum_{k=1}^{K} \frac{E[\zeta_k^{[n]}(t) \mid Q_{K,T,n}^{\epsilon}]}{n} e^{-t/n}dt && \textbf{(Term III)}\\
&+ \big(1-q_{K,T}^{\epsilon}(n)\big) \int\limits_T^\infty \sum_{k=K+1}^{\infty} \frac{E[\zeta_k^{[n]}(t) \mid Q_{K,T,n}^{\epsilon}]}{n} e^{-t/n}dt && \textbf{(Term IV)}\\
&+ q_{K,T}^{\epsilon}(n) \int\limits_T^\infty \sum_{k=1}^{\infty} \frac{E[\zeta_k^{[n]}(t) \mid \overline{Q_{K,T,n}^{\epsilon}}]}{n} e^{-t/n}dt && \textbf{(Term V)}
\end{aligned}
\tag{42}
$$

Next, we estimate the terms **I-V** in (42).

Term I. As it is proven in (61) of Section 5, $\lim\limits_{n\to\infty} \sup\limits_{s\in[0,T]} \Big| n^{-1}\zeta_k^{[n]}(s) - \zeta_k(s) \Big| = 0$ *a.s.* on $[0,T]$ for all $k = 1, 2, \ldots, K$. Therefore,

$$
\lim_{n\to\infty} \int\limits_0^T \sum_{k=1}^{K} \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n}dt = \sum_{k=1}^{K} \int\limits_0^T \zeta_k(t)dt.
$$

Term II. Observe that,

$$
\sum_{k=K+1}^{\infty} \frac{\zeta_k^{[n]}(t)}{n} \le \frac{1}{Kn} \sum_{k=K+1}^{\infty} k\zeta_k^{[n]}(t) = \frac{1}{K}\left(1 - \sum_{k=1}^{K} \frac{k\zeta_k^{[n]}(t)}{n}\right) \le \frac{1}{K}.
$$

Thus,

$$
\int\limits_0^T \sum_{k=K+1}^{\infty} \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n}dt = O\left(\frac{T}{K}\right)
$$

regardless of the value of $n > 0$.

Term III. Recall that in the theory of Marcus-Lushnikov processes the *gel* is the set of all "large" clusters. By analogy, we define the *K-gel* to be the collection of all clusters of mass bigger than K. Let $M_{K\,gel}(t)$ denote the total mass of all clusters in the *K-gel* at time $t \geq 0$.

Now, conditioning on the event $Q^{\epsilon}_{K,T,n}$, the mass of the *K-gel* is $M_{K\,gel}(t) \geq (1-\epsilon)n$ for all $t \geq T$. Thus each cluster not in *K-gel* will be gravitating toward the *K-gel* with the rate of at least $\frac{M_{K\,gel}(t)}{n} \geq 1-\epsilon$. Consider a cluster that was not in *K-gel* at time T. Let $T+L$ be the time it becomes a part of the *K-gel.* Then, its contribution to the integral $\int\limits_T^\infty \sum\limits_{k=1}^K \frac{E[\zeta_k^{[n]}(t) \mid Q^{\epsilon}_{K,T,n}]}{n} e^{-t/n} dt$ is at most

$$\int\limits_T^\infty \frac{E[\mathbf{1}_{[T,T+L]}(t) \mid Q^{\epsilon}_{K,T,n}]}{n} e^{-t/n} dt \leq \frac{\int\limits_T^\infty E[\mathbf{1}_{[T,T+L]}(t) \mid Q^{\epsilon}_{K,T,n}] dt}{n} e^{-T/n}$$

$$= \frac{E[L \mid Q^{\epsilon}_{K,T,n}]}{n} e^{-T/n} \leq \frac{1}{(1-\epsilon)n},$$

where

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}.$$

The number of clusters not in *K-gel* at time $t \geq T$ is

$$\sum_{k=1}^K \zeta_k^{[n]}(t) \leq \sum_{k=1}^K k\zeta_k^{[n]}(t) \leq \epsilon n.$$

Therefore,

$$\int\limits_T^\infty \sum_{k=1}^K \frac{E[\zeta_k^{[n]}(t) \mid Q^{\epsilon}_{K,T,n}]}{n} e^{-t/n} dt \leq \frac{\epsilon n}{(1-\epsilon)n} = \frac{\epsilon}{1-\epsilon} < 2\epsilon.$$

Term IV. We let $\mathcal{C} = \{C_1, C_2, C_3, \dots, C_M\}$ denote the set of all clusters that ever exceeded mass K in the whole history of the process $\left\{\mathbf{ML}_n(t)\right\}_{t \in [0,\infty)}$. There are less than n/K such clusters, i.e., $M < n/K$. For each C_i, the emergence time a_i is the time when a pair of clusters of mass not exceeding K mergers into a new cluster C_i of mass greater than K. We enumerate these clusters in the order they emerge.

Let $M_i(t)$ denote the mass of cluster C_i at time t. Consider a pair of clusters, C_i and C_j, coexisting in the *K-gel* at time t, each of mass smaller than $n/2$. We split their merger rate into two by saying that C_i absorbs C_j with rate $\frac{1}{2n}M_i(t)M_j(t)$, and C_j absorbs C_i with rate $\frac{1}{2n}M_i(t)M_j(t)$. In other words, C_i and C_j merge with rate $\frac{1}{n}M_i(t)M_j(t)$, and which one of the two clusters absorbs the other is decided with a toss of an independent fair coin.

There is a finite stopping time

$$t^* = \min\{t \geq 0 \ : \ \exists C_i \in \mathcal{C} \ \text{ with } M_i(t) \geq n/2\}$$

when a cluster C_{i^*} has its mass $M_{i^*}(t^*) \geq n/2$. After t^*, the rules of interactions of cluster C_{i^*} with the other clusters in $\mathcal{C}$ change as follows. For $t > t^*$, C_{i^*} absorbs C_j with rate $\frac{1}{n}M_{i^*}(t)M_j(t)$, while C_{i^*} itself cannot be absorbed by any other cluster in $\mathcal{C}$.

Let b_i denote the time when cluster C_i is absorbed by another cluster in collection $\mathcal{C}$. Naturally, there will be only one survivor C_{i^*} with $b_{i^*} = \infty$. Let $J_i = [a_i, b_i) \cap [T, \infty)$ denote the lifespan of cluster C_i. Note that a cluster C_i from the set $\mathcal{C}$ existing at time $t \in [a_i, b_i)$ is absorbed into one of the clusters in the *K-gel* with the total instantaneous rate of

$$\lambda_i(t) \geq \frac{1}{2n} M_i(t)\big(M_{K\,gel}(t) - M_i(t)\big).$$

Conditioning on the event $Q^\epsilon_{K,T,n}$ defined in (41), we have that if $M_i(t) < n/2$ for $t \in J_i$, then the rate of absorption of C_i into the *K-gel* is

$$\lambda_i(t) \geq \frac{1}{2n} M_i(t)\left((1-\epsilon)n - \frac{1}{2}n\right) \geq \frac{1}{2n} M_i(t)\left(\frac{3}{4}n - \frac{1}{2}n\right) \geq \frac{1}{8}M_i(t) > \frac{K}{8}.$$

Next,

$$\int\limits_T^\infty \sum_{k=K+1}^{\infty} \frac{E[\zeta_k^{[n]}(t) \mid Q^\epsilon_{K,T,n}]}{n} e^{-t/n} dt = \int\limits_T^\infty \frac{1}{n} e^{-t/n} dt + \mathcal{E} \tag{43}$$

where $\int\limits_T^\infty \frac{1}{n}e^{-t/n}dt$ is due to the event $Q^\epsilon_{K,T,n}$ which guarantees the existence of at least one component from $\mathcal{C}$ in the *K-gel* for all $t \in [T, \infty)$ and the second term $\mathcal{E}$ is responsible for all the times $t \geq T$ when the number of clusters in the *K-gel* is greater than one. The term $\mathcal{E}$ is bounded as follows

$$\mathcal{E} \leq \int\limits_T^\infty \frac{E\left[\sum\limits_{i:\ i\neq i^*} \mathbf{1}_{J_i}(t) \;\Big|\; Q^\epsilon_{K,T,n}\right]}{n} e^{-t/n} dt.$$

Now, each cluster C_i is gravitating towards the rest of the *K-gel* with the rate of at least $K/8$. Thus, for each $i \neq i^*$,

$$\int\limits_T^\infty \frac{E\left[\mathbf{1}_{J_i}(t) \mid Q^\epsilon_{K,T,n}\right]}{n} e^{-t/n} dt \leq \frac{E[|J_i| \mid Q^\epsilon_{K,T,n}]}{n} e^{-\frac{T}{n}} \leq \frac{8}{nK}.$$

Hence, since the cardinality of set $\mathcal{C}$ is $M < n/K$,

$$\mathcal{E} < \frac{n}{K} \cdot \frac{8}{nK} = \frac{8}{K^2},$$

and from (43), we obtain

$$\int_T^\infty \sum_{k=K+1}^\infty \frac{E[\zeta_k^{[n]}(t) \mid Q_{K,T,n}^\epsilon]}{n} e^{-t/n} dt = 1 + O(K^{-2}) + O\left(\frac{T}{n}\right) \quad \text{as } n \to \infty,$$

where the term $O(K^{-2})$ does not depend on the value of $n > 0$.

Term V. Here

$$q_{K,T}^\epsilon(n) \int_T^\infty \sum_{k=1}^\infty \frac{E[\zeta_k^{[n]}(t) \mid \overline{Q_{K,T,n}^\epsilon}]}{n} e^{-t/n} dt \;\le n q_{K,T}^\epsilon(n) \int_T^\infty \frac{1}{n} e^{-t/n} dt \;\le n q_{K,T}^\epsilon(n) = O(n^{-1}).$$

Finally, by putting together the analysis in **Terms I-V** in the equation (42), we obtain for a given fixed $\epsilon \in (0, 1/4)$, sufficiently large fixed $T \gg T_{gel}$ satisfying (40), and arbitrarily large K,

(44)

$$\int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n} dt = \sum_{k=1}^K \int_0^T \zeta_k(t)dt + 1 + O\left(\frac{T}{K}\right) + O(K^{-2}) + O(\epsilon) + O\left(\frac{T}{n}\right) + O(n^{-1}),$$

which, when we increase n to infinity will yield

$$\begin{aligned}\limsup_{n\to\infty} \left| \int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n} dt - \sum_{k=1}^\infty \int_0^\infty \zeta_k(t)dt - 1 \right| &= \sum_{k=K+1}^\infty \int_0^T \zeta_k(t)dt + \sum_{k=1}^\infty \int_T^\infty \zeta_k(t)dt \\ &\quad + O\left(\frac{T}{K}\right) + O(K^{-2}) + O(\epsilon).\end{aligned}$$

Consequently, taking $\limsup\limits_{K\to\infty}$, we obtain

$$\limsup_{n\to\infty} \left| \int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n} dt - \sum_{k=1}^\infty \int_0^\infty \zeta_k(t)dt - 1 \right| = \sum_{k=1}^\infty \int_T^\infty \zeta_k(t)dt + O(\epsilon).$$

Finally, formula (40) guarantees that decreasing ϵ down to zero will propel T to $+\infty$, and

$$\lim_{n\to\infty} \int_0^\infty \sum_{k=1}^\infty \frac{E[\zeta_k^{[n]}(t)]}{n} e^{-t/n} dt = \sum_{k=1}^\infty \int_0^\infty \zeta_k(t)dt + 1.$$

Thus we confirmed formula (37) for the case of the multiplicative coalescent process. $\square$

4.4. **Relation of Erdős-Rényi process on $K_{\alpha[n],\beta[n]}$ to cross-multiplicative coalescent.** Let $\alpha, \beta > 0$ be given, and consider two integer valued functions, $\alpha[n] = \alpha n + o(\sqrt{n})$ and $\beta[n] = \beta n + o(\sqrt{n})$. Next, we introduce the Erdős-Rényi random graph process on the bipartite graph $K_{\alpha[n],\beta[n]}$ with $\alpha[n]$ vertices on the left side and $\beta[n]$ vertices on the right side. In this random graph process on $K_{\alpha[n],\beta[n]}$, for each edge e of $\alpha[n]\beta[n] = \alpha\beta n^2 + o(n\sqrt{n})$ edges we have an associated random variable U_e, distributed uniformly on $[0,1]$. The random variables $\{U_e\}_e$ are assumed to be independent. For the "time" parameter $p \in [0,1]$, an edge e is considered "open" if $U_e \leq p$. Erdős-Rényi random graph $G(n,p)$ will consist of all n vertices and all open edges at time p.

In this Erdős-Rényi random graph process, the probability of two components merging at a given time depends only on the number of edges that connect those two components. If connected component C_i and C_j have partition sizes (i_1, i_2) and (j_1, j_2) respectively, then there are $i_1 j_2 + i_2 j_1$ edges which, when opened, would connect C_i and C_j. Therefore, the cross-multiplicative coalescent process represents the cluster dynamics of the above Erdős-Rényi random graph process on the bipartite graph $K_{\alpha[n],\beta[n]}$ under the time change $p = 1 - e^{-t/n}$. This coalescent process representation is obtained by letting each cluster connecting i_1 vertices on the left side of the bipartite graph with i_2 vertices on the right side of the bipartite graph be assigned a two-dimensional weight vector $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$. Then, the Marcus-Lushnikov process $\zeta^{[n]}_{i_1,i_2}(t)$ corresponding to the cross-multiplicative coalescent process will count the number of clusters with the weight vector $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ at time t.

4.5. **The length of the minimal spanning tree on $K_{\alpha[n],\beta[n]}$ via $\zeta_{i_1,i_2}(t)$.** Consider the Erdős-Rényi random graph model on a complete bipartite graph $K_{\alpha[n],\beta[n]}$. Let us interpret U_e as the length of edge e. Then one can construct a minimal spanning tree on $K_{\alpha[n],\beta[n]}$. Let random variable L_n denote the length of such minimal spanning tree. We want to represent the asymptotic limit of the mean value of L_n via $\zeta_{i_1,i_2}(t)$.

For a random graph process $G(n,p)$ over $K_{\alpha[n],\beta[n]}$, Lemma 1 in Beveridge et al [6] implies

$$E[L_n] = \int_0^1 E[\kappa(G(n,p))]dp - 1, \tag{45}$$

where $\kappa(G(n,p))$ is the number of components in the random graph process $G(n,p)$ at time p. This will be used in Sect. 4.6 for proving the following theorem.

Theorem 4.2. *Let $\alpha, \beta > 0$ and $L_n = L_n(\alpha, \beta)$ be the length of a minimal spanning tree on a complete bipartite graph $K_{\alpha[n],\beta[n]}$ with partitions of sizes*

$$\alpha[n] = \alpha n + o(\sqrt{n}) \quad \textit{and} \quad \beta[n] = \beta n + o(\sqrt{n})$$

and independent uniform edge weights over $[0,1]$*. Then*

$$\lim_{n\to\infty} E[L_n] = \sum_{i_1,i_2}^{\infty} \int_0^{\infty} \zeta_{i_1,i_2}(t)dt. \tag{46}$$

where $\zeta_{i_1,i_2}(t)$ *indexed by* $\mathbb{Z}_+^2 \setminus \{(0,0)\}$ *is the solution of the modified Smoluchowski coagulation system* (18) *with the initial conditions* $\zeta_{i_1,i_2}(0) = \alpha\delta_{1,i_1}\delta_{0,i_2} + \beta\delta_{0,i_1}\delta_{1,i_2}$.

Observe that if we plug-in the solutions (19) of the reduced system of Smoluchowski coagulation equations (18) into the right hand side of (46), we get

$$\begin{aligned}
\sum_{i_1,i_2}^{\infty} \int_0^{\infty} \zeta_{i_1,i_2}(t)dt &= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \sum_{i_1\ge 1:\ i_2\ge 1} \alpha^{i_1}\beta^{i_2} S_{i_1,i_2} \int_0^{\infty} t^{i_1+i_2-1} e^{-(\beta i_1+\alpha i_2)t} dt \\
&= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \sum_{i_1\ge 1:\ i_2\ge 1} \frac{\alpha^{i_1}\beta^{i_2} S_{i_1,i_2}}{(\beta i_1+\alpha i_2)^{i_1+i_2}} (i_1+i_2-1)! \\
&= \gamma + \frac{1}{\gamma} + \sum_{i_1\ge 1:\ i_2\ge 1} \frac{\gamma^{i_1} S_{i_1,i_2}}{(i_1+\gamma i_2)^{i_1+i_2}} (i_1+i_2-1)! \qquad (47)
\end{aligned}$$

with $\gamma = \frac{\alpha}{\beta}$.

Next, by combining Lemma 3.1 with (47) we obtained the following important theorem.

Theorem 4.3. *Let* $\alpha, \beta > 0$, $\gamma = \alpha/\beta$, *and* $L_n = L_n(\alpha,\beta)$ *be the length of a minimal spanning tree on a complete bipartite graph* $K_{\alpha[n],\beta[n]}$ *with partitions of sizes*

$$\alpha[n] = \alpha n + o(\sqrt{n}) \quad \text{and} \quad \beta[n] = \beta n + o(\sqrt{n})$$

and independent uniform edge weights over $[0,1]$*. Then the limiting mean length of the minimal spanning tree is*

$$\lim_{n\to\infty} E[L_n] = \gamma + \frac{1}{\gamma} + \sum_{i_1\ge 1;\ i_2\ge 1} \frac{(i_1+i_2-1)!}{i_1!i_2!} \frac{\gamma^{i_1} i_1^{i_2-1} i_2^{i_1-1}}{(i_1+\gamma i_2)^{i_1+i_2}}.$$

Theorem 4.3 is consistent with [17], where it was shown that for $\alpha = \beta$, $\lim_{n\to\infty} E[L_n] = 2\zeta(3)$. Indeed, we have the following Corollary reproducing the results in [17]. Observe however that for $\alpha \neq \beta$ the bipartite graph is irregular and the results in Frieze and McDiarmid [17] no longer apply.

Corollary 4.4. *If* $\gamma = 1$*, then*

$$\lim_{n\to\infty} E[L_n] = 2\zeta(3).$$

Proof. Abel's binomial theorem [7, 33] states that

$$\sum_{k=0}^{n} \binom{n}{k} (x-kz)^{k-1}(y+kz)^{n-k} = x^{-1}(x+y)^n.$$

Plugging-in $x = nz \neq 0$, $y = 0$, and $i = n-k$, we obtain

$$\sum_{i=0}^{n} \binom{n}{i} i^{n-i-1}(n-i)^i = n^{n-1}$$

and therefore,

$$\sum_{i_1,i_2:\ i_1+i_2=n} i_1 S_{i_1,i_2} = \sum_{i_1,i_2:\ i_1+i_2=n} i_1 \frac{i_1^{i_2-1} i_2^{i_1-1}}{i_1! i_2!} = \frac{n^{n-1}}{n!}.$$

Hence,

$$n \cdot \sum_{i_1,i_2:\ i_1+i_2=n} S_{i_1,i_2} = \sum_{i_1,i_2:\ i_1+i_2=n} (i_1+i_2) S_{i_1,i_2} = 2 \sum_{i_1,i_2:\ i_1+i_2=n} i_1 S_{i_1,i_2} = 2\frac{n^{n-1}}{n!}$$

and

$$\sum_{i_1,i_2:\ i_1+i_2=n} S_{i_1,i_2} = 2\frac{n^{n-2}}{n!}.$$

Plugging the above into (47) with $\gamma = 1$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} E[L_n] &= 2 + \sum_{i_1\geq 1:\ i_2 \geq 1} \frac{S_{i_1,i_2}}{(i_1+i_2)^{i_1+i_2}} (i_1+i_2-1)! \\
&= 2 + \sum_{n=2}^{\infty} \left(\sum_{i_1,i_2:\ i_1+i_2=n} \frac{S_{i_1,i_2}}{n^n} \right) (n-1)! \\
&= 2 + \sum_{n=2}^{\infty} 2\frac{n^{n-2}}{n!} \cdot \frac{1}{n^n} (n-1)! \\
&= 2 + \sum_{n=2}^{\infty} \frac{2}{n^3} = 2\zeta(3). \qquad (48)
\end{aligned}$$

Thus confirming the results in [17]. □

4.6. **Proof of Theorem 4.2.** Let us give a rigorous proof of Theorem 4.2. Here, we will follow the strategy used for proving Theorem 4.1 in Sect. 4.3.

Proof. Observe that

$$\lim_{t\to\infty} \sum_{i_1,i_2} i_1 \zeta_{i_1,i_2}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \sum_{i_1,i_2} i_2 \zeta_{i_1,i_2}(t) = 0. \qquad (49)$$

Indeed, by plugging in $\zeta_{i_1,i_2}(t)$ as in (23), we obtain

$$\frac{d}{dt}\sum_{i_1,i_2} i_1\zeta_{i_1,i_2}(t) = \sum_{i_1,i_2} i_1\zeta_{i_1,i_2}(t)\left(\frac{i_1+i_2-1}{t} - (\beta i_1 + \alpha i_2)\right) \leq -(\alpha\wedge\beta)\sum_{i_1,i_2} i_1\zeta_{i_1,i_2}(t)$$

for $t > \frac{1}{\alpha\wedge\beta}$. Thus, $\sum\limits_{i_1,i_2} i_1\zeta_{i_1,i_2}(t)$, and similarly $\sum\limits_{i_1,i_2} i_2\zeta_{i_1,i_2}(t)$, would decrease to zero exponentially fast when $t > \frac{1}{\alpha\wedge\beta}$.

Now, having established (49), for any given $\epsilon \in (0, 1/4)$, we can fix $T \gg T_{gel}$ so large that

$$\sum_{i_1,i_2} i_1\zeta_{i_1,i_2}(t) \leq \frac{\alpha\epsilon}{2} \quad \text{and} \quad \sum_{i_1,i_2} i_2\zeta_{i_1,i_2}(t) \leq \frac{\beta\epsilon}{2}. \tag{50}$$

Notice that the above inequalities (50) ties T to ϵ.

Fix integers $K_1 > 0$ and $K_2 > 0$, and let $R := R(K_1, K_2) = \{1, 2, \dots, K_1\} \times \{1, 2, \dots, K_2\}$. By the equation (65) in Sect. 5.3 we have

$$\lim_{n\to\infty}\sup_{s\in[0,T]}\left|n^{-1}\sum_R \zeta^{[n]}_{i_1,i_2}(s) - \sum_R \zeta_{i_1,i_2}(s)\right| = 0 \qquad \text{a.s.}$$

Thus, the probability of the complement of the event

$$Q^\epsilon_{R,T,n} := \left\{\sum_{\mathbf{i}\in R} i_1\frac{\zeta^{[n]}_{i_1,i_2}(T)}{n} \leq \frac{3}{4}\alpha\epsilon \quad \text{and} \quad \sum_{\mathbf{i}\in R} i_2\frac{\zeta^{[n]}_{i_1,i_2}(T)}{n} \leq \frac{3}{4}\beta\epsilon\right\} \tag{51}$$

is decreasing to zero as $n\to\infty$. Moreover,

$$q^\epsilon_{R,T}(n) := P(\overline{Q^\epsilon_{R,T,n}}) = O(n^{-2})$$

by Proposition 5.3 in Sect. 5.4 since

$$\lim_{n\to\infty}\sqrt{n}\left(\sum_{\mathbf{i}\in R} i_1\frac{\zeta^{[n]}_{i_1,i_2}(0)}{n} - \sum_{\mathbf{i}\in R} i_1\zeta_{i_1,i_2}(0)\right) = \lim_{n\to\infty}\sqrt{n}\big(\alpha[n]/n - \alpha\big) = 0$$

and

$$\lim_{n\to\infty}\sqrt{n}\left(\sum_{\mathbf{i}\in R} i_2\frac{\zeta^{[n]}_{i_1,i_2}(0)}{n} - \sum_{\mathbf{i}\in R} i_2\zeta_{i_1,i_2}(0)\right) = \lim_{n\to\infty}\sqrt{n}\big(\beta[n]/n - \beta\big) = 0.$$

We know from (45) that

$$\lim_{n\to\infty} E[L_n] = \lim_{n\to\infty}\int_0^1 E[\kappa(G(n,p))]dp - 1 = \lim_{n\to\infty}\int_0^\infty \sum_{i_1,i_2}\frac{E[\zeta^{[n]}_{i_1,i_2}(t)]}{n}e^{-t/n}dt - 1$$

provided the latter limit exists.

We will split $\int\limits_0^\infty \sum\limits_{i_1,i_2} \frac{E[\zeta^{[n]}_{i_1,i_2}(t)]}{n} e^{-t/n}dt$ as follows.

$$\begin{aligned}
\int\limits_0^\infty \sum_{i_1,i_2} \frac{E[\zeta^{[n]}_{i_1,i_2}(t)]}{n} e^{-t/n}dt = & \int\limits_0^T \sum_{\mathbf{i}\in R} \frac{E[\zeta^{[n]}_{i_1,i_2}(t)]}{n} e^{-t/n}dt && \textbf{(Term I)}\\
& + \int\limits_0^T \sum_{\mathbf{i}\notin R} \frac{E[\zeta^{[n]}_{i_1,i_2}(t)]}{n} e^{-t/n}dt && \textbf{(Term II)}\\
& + \big(1-q^\epsilon_{R,T}(n)\big) \int\limits_T^\infty \sum_{\mathbf{i}\in R} \frac{E[\zeta^{[n]}_{i_1,i_2}(t) \mid Q^\epsilon_{R,T,n}]}{n} e^{-t/n}dt && \textbf{(Term III)}\\
& + \big(1-q^\epsilon_{R,T}(n)\big) \int\limits_T^\infty \sum_{\mathbf{i}\notin R} \frac{E[\zeta^{[n]}_{i_1,i_2}(t) \mid Q^\epsilon_{R,T,n}]}{n} e^{-t/n}dt && \textbf{(Term IV)}\\
& + q^\epsilon_{R,T}(n) \int\limits_T^\infty \sum_{i_1,i_2} \frac{E[\zeta^{[n]}_{i_1,i_2}(t) \mid \overline{Q^\epsilon_{R,T,n}}]}{n} e^{-t/n}dt && \textbf{(Term V)}
\end{aligned} \tag{52}$$

Next, we estimate the terms **I-V** in (52).

Term I. As we establish in (64) of Section 5, $\lim\limits_{n\to\infty} \sup\limits_{s\in[0,T]} \Big| n^{-1}\zeta^{[n]}_{i_1,i_2}(s) - \zeta_{i_1,i_2}(s)\Big| = 0$ *a.s.* on $[0,T]$ for all $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in R$. Therefore,

$$\lim_{n\to\infty} \int\limits_0^T \sum_{\mathbf{i}\in R} \frac{E[\zeta^{[n]}_{i_1,i_2}(t)]}{n} e^{-t/n}dt = \sum_{\mathbf{i}\in R} \int\limits_0^T \zeta_{i_1,i_2}(t)dt.$$

Term II. Observe that,

$$\begin{aligned}
\sum_{\mathbf{i}\notin R} \frac{\zeta^{[n]}_{i_1,i_2}(t)}{n} & \le \frac{1}{n}\sum_{i_1>K_1}\sum_{i_2} \zeta^{[n]}_{i_1,i_2}(t) + \frac{1}{n}\sum_{i_1}\sum_{i_2>K_2} \zeta^{[n]}_{i_1,i_2}(t)\\
& \le \frac{1}{K_1 n}\sum_{i_1>K_1}\sum_{i_2} i_1\zeta^{[n]}_{i_1,i_2}(t) + \frac{1}{nK_2}\sum_{i_1}\sum_{i_2>K_2} i_2\zeta^{[n]}_{i_1,i_2}(t)\\
& \le \frac{\alpha[n]}{K_1 n} + \frac{\beta[n]}{nK_2} \le 2\frac{\alpha}{K_1} + 2\frac{\beta}{K_2}
\end{aligned}$$

for all n large enough. Thus,

$$\int_0^T \sum_{\mathbf{i}\notin R} \frac{E[\zeta^{[n]}_{i_1,i_2}(t)]}{n} e^{-t/n} dt = O\left(\frac{T}{K_1}\right) + O\left(\frac{T}{K_2}\right).$$

Term III. We define the *R-gel* to be the collection of all clusters whose mass vector is not in R. Let

$$M_{R\,gel}(t) = \begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix} \tag{53}$$

denote the total mass vector of all clusters in the *R-gel* at time $t \geq 0$.

Now, conditioning on the event $Q^{\epsilon}_{R,T,n}$, we have $m_1(t) \geq \alpha(1-\epsilon)n$ and $m_2(t) \geq \beta(1-\epsilon)n$ for all $t \geq T$, and n large enough. Thus each cluster in R will be gravitating toward the *R-gel* with the rate of at least $(\alpha \wedge \beta)(1-\epsilon)$. Consider a cluster in R at time T. Let $T+L$ be the time it becomes a part of the *R-gel*. Then, its contribution to the integral $\int_T^\infty \sum_{\mathbf{i}\in R} \frac{E[\zeta^{[n]}_{i_1,i_2}(t) \mid Q^{\epsilon}_{R,T,n}]}{n} e^{-t/n} dt$ is at most

$$\int_T^\infty \frac{E[\mathbf{1}_{[T,T+L]}(t) \mid Q^{\epsilon}_{R,T,n}]}{n} e^{-t/n} dt \leq \frac{E[L \mid Q^{\epsilon}_{R,T,n}]}{n} e^{-T/n} \leq \frac{1}{(\alpha \wedge \beta)(1-\epsilon)n}.$$

The number of clusters in R at time $t \geq T$ is

$$\sum_{\mathbf{i}\in R} \zeta^{[n]}_{i_1,i_2}(t) \leq \sum_{\mathbf{i}\in R} (i_1+i_2)\zeta^{[n]}_{i_1,i_2}(t) \leq (\alpha+\beta)\epsilon n.$$

Therefore,

$$\int_T^\infty \sum_{\mathbf{i}\in R} \frac{E[\zeta^{[n]}_{i_1,i_2}(t) \mid Q^{\epsilon}_{R,T,n}]}{n} e^{-t/n} dt \leq \frac{(\alpha+\beta)\epsilon n}{(\alpha \wedge \beta)(1-\epsilon)n} = \frac{2\epsilon}{1-\epsilon} < 3\epsilon.$$

Term IV. We let $\mathcal{C} = \{C_1, C_2, C_3, \dots, C_M\}$ denote the set of all clusters whose mass vectors ever exceeded K_1 in the first coordinate and/or ever exceeded K_2 in the second coordinate in the history of the process $\mathbf{ML}_n(t)$, i.e., all clusters that were ever a part of *R-gel*. The number of clusters in $\mathcal{C}$ is less than $\alpha[n]/K_1 + \beta[n]/K_2$. For each C_i, the emergence time a_i is the time of a merger of a pair of clusters in R, resulting in appearance of a new cluster C_i in *R-gel*. We enumerate these clusters in the order they emerge.

Let $M_i(t) = \begin{bmatrix} m_{1,i}(t) \\ m_{2,i}(t) \end{bmatrix}$ denote the mass vector of cluster C_i at time t. Consider a pair of clusters, C_i and C_j, coexisting in the *R-gel* at time t, such that $m_{1,i}, m_{1,j} < \alpha n/2$ and $m_{2,i}, m_{2,j} < \beta n/2$. We split their merger rate into two by saying that C_i absorbs C_j with rate $\frac{1}{2n}\big(m_{1,i}(t)m_{2,j}(t)+m_{2,i}(t)m_{1,j}(t)\big)$, and C_j absorbs C_i with rate $\frac{1}{2n}\big(m_{1,i}(t)m_{2,j}(t)+m_{2,i}(t)m_{1,j}(t)\big)$.

There is a finite stopping time

$$t^*=\min\{t\ge 0\ :\ \exists C_i\in\mathcal{C}\ \text{ with } m_{1,i}(t)\ge \alpha n/2\ \text{ or }\ m_{2,i}(t)\ge\beta n/2\}$$

when a cluster C_{i^*} has its mass vector satisfying either $m_{1,i^*}(t^*)\ge\alpha n/2$ or $m_{2,i^*}(t^*)\ge\beta n/2$. After time t^* the rules of interactions of cluster C_{i^*} with the other clusters in $\mathcal{C}$ change as follows. For $t>t^*$, C_{i^*} absorbs C_j with rate $\frac{1}{n}\big(m_{1,i^*}(t)m_{2,j}(t)+m_{2,i^*}(t)m_{1,j}(t)\big)$, while C_{i^*} itself cannot be absorbed by any other cluster in $\mathcal{C}$.

Let b_i denote the time when cluster C_i is absorbed by another cluster in collection $\mathcal{C}$. Naturally, there will be only one survivor C_{i^*} with $b_{i^*}=\infty$. Let $J_i=[a_i,b_i)\cap[T,\infty)$ denote the lifespan of cluster C_i. Note that a cluster C_i from the collection $\mathcal{C}$ existing at time $t\in[a_i,b_i)$ is absorbed into one of the clusters in the *R-gel* with the total instantaneous rate of

$$\lambda_i(t)\ge\frac{1}{2n}\Big(m_{1,i}(t)\big(m_2(t)-m_{2,i}(t)\big)+m_{2,i}(t)\big(m_1(t)-m_{1,i}(t)\big)\Big),$$

where $m_1(t)$ and $m_2(t)$ are as defined in (53). Conditioning on the event $Q^\epsilon_{R,T,n}$ defined in (51), we have that if $m_{1,i}(t)<\alpha n/2$ and $m_{2,i}(t)<\beta n/2$ for $t\in J_i$, then the rate of absorption of C_i into the *R-gel* is

$$\begin{aligned}\lambda_i(t)&\ge\frac{1}{2n}m_{1,i}(t)\beta\left((1-\epsilon)n-\frac12 n\right)+\frac{1}{2n}m_{2,i}(t)\alpha\left((1-\epsilon)n-\frac12 n\right)\\&\ge\frac{1}{2n}m_{1,i}(t)\beta\left(\frac34 n-\frac12 n\right)+\frac{1}{2n}m_{2,i}(t)\alpha\left(\frac34 n-\frac12 n\right)\\&\ge\frac{m_{1,i}(t)\beta+m_{2,i}(t)\alpha}{8}>\frac{K_1\beta+K_2\alpha}{8}.\end{aligned}$$

Next,

$$\int\limits_T^\infty\sum_{\mathbf{i}\notin R}\frac{E[\zeta^{[n]}_{i_1,i_2}(t)\mid Q^\epsilon_{R,T,n}]}{n}e^{-t/n}dt=\int\limits_T^\infty\frac1n e^{-t/n}dt+\mathcal{E}\tag{54}$$

where $\int\limits_T^\infty\frac1n e^{-t/n}dt$ is due to the event $Q^\epsilon_{R,T,n}$ which guarantees the existence of at least one component from $\mathcal{C}$ in the *R-gel* for all $t\in[T,\infty)$ and the second term $\mathcal{E}$ is responsible for all the times $t\ge T$ when the number of clusters in the *R-gel* is greater than one. The term $\mathcal{E}$ is bounded as follows

$$\mathcal{E}\le\int\limits_T^\infty\frac{E\left[\sum\limits_{i:\ i\neq i^*}\mathbf{1}_{J_i}(t)\ \Big|\ Q^\epsilon_{R,T,n}\right]}{n}e^{-t/n}dt.$$

Now, each cluster C_i is gravitating towards the rest of the *R-gel* with the rate of at least $\frac{K_1\beta+K_2\alpha}{8}$. Thus, for each $i \neq i^*$,

$$\int\limits_T^\infty \frac{E\left[\mathbf{1}_{J_i}(t) \mid Q^\epsilon_{R,T,n}\right]}{n} e^{-t/n}dt \leq \frac{E[|J_i| \mid Q^\epsilon_{R,T,n}]}{n} e^{-\frac{T}{n}} \leq \frac{8}{n(K_1\beta+K_2\alpha)}.$$

Hence, since the cardinality of set $\mathcal{C}$ is $M < \alpha[n]/K_1 + \beta[n]/K_2$,

$$\mathcal{E} < (\alpha[n]/K_1 + \beta[n]/K_2) \cdot \frac{8}{n(K_1\beta+K_2\alpha)} = \frac{8(\alpha/K_1+\beta/K_2)}{K_1\beta+K_2\alpha} + o(1),$$

and from (54), we obtain

$$\int\limits_T^\infty \sum_{\mathbf{i}\notin R} \frac{E[\zeta^{[n]}_{i_1,i_2}(t) \mid Q^\epsilon_{R,T,n}]}{n} e^{-t/n}dt = 1 + O(K_1^{-2}) + O(K_2^{-2}) + O\left(\frac{T}{n}\right) + o(1) \quad \text{as } n \to \infty.$$

Term V. Here

$$\begin{aligned} q^\epsilon_{R,T}(n) \int\limits_T^\infty \sum_{i_1,i_2} \frac{E[\zeta^{[n]}_{i_1,i_2}(t) \mid \overline{Q^\epsilon_{R,T,n}}]}{n} e^{-t/n}dt &\leq q^\epsilon_{R,T}(n) \int\limits_T^\infty \frac{\alpha[n]+\beta[n]}{n} e^{-t/n}dt \\ &\leq (\alpha[n]+\beta[n]) q^\epsilon_{R,T}(n) \ = O(n^{-1}) \end{aligned}$$

as $q^\epsilon_{R,T}(n) = O(n^{-2})$.

Finally, by putting together the analysis in **Terms I-V** in the equation (52), we obtain for a given fixed $\epsilon \in (0, 1/4)$, sufficiently large fixed $T \gg T_{gel}$ satisfying (50), and arbitrarily large K_1 and K_2,

$$\begin{aligned} \int\limits_0^\infty \sum_{i_1,i_2} \frac{E[\zeta^{[n]}_{i_1,i_2}(t)]}{n} e^{-t/n}dt = &\sum_{\mathbf{i}\in R(K_1,K_2)} \int\limits_0^T \zeta_{i_1,i_2}(t)dt + 1 + O\left(\frac{T}{K_1}\right) + O\left(\frac{T}{K_2}\right) \\ &+ O(K_1^{-2}) + O(K_2^{-2}) + O(\epsilon) + O\left(\frac{T}{n}\right) + O(n^{-1}), \end{aligned} \tag{55}$$

which when we increase n to infinity will yield

$$\lim_{n\to\infty} \int\limits_0^\infty \sum_{i_1,i_2} \frac{E[\zeta^{[n]}_{i_1,i_2}(t)]}{n} e^{-t/n}dt = \sum_{i_1,i_2} \int\limits_0^\infty \zeta_{i_1,i_2}(t)dt + 1.$$

$\square$

5. Hydrodynamic limits for Marcus-Lushnikov processes

In [25] and [11], a certain class of Markov processes, called *density dependent population processes*, was considered. These are jump Markov processes which depend on a certain parameter n which can be interpreted depending on the context of a model. Usually it represents the population size. Many coalescent processes can be restated as a case of density dependent population processes if all cluster weights are integers. There, the total mass n is the parameter representing the population size. Specifically, we may assume that the coalescent process starts with n clusters of unit mass each (aka singletons). In Kurtz [25] and in Chapter 11 of Ethier and Kurtz [11], the law of large numbers and the central limit theorems were established for such density dependent population processes as $n \to \infty$. In this section we will adopt these weak limit laws for the multiplicative and cross-multiplicative coalescent processes.

5.1. **Density dependent population processes.** We first formulate the framework for the convergence result of Kurtz as stated in Theorem 2.1 in Chapter 11 of [11] (Theorem 8.1 in [25]). There, the *density dependent population processes* are defined as continuous time Markov processes with state spaces in $\mathbb{Z}^d$, and transition intensities represented as follows

$$q^{(n)}(k, k+\ell) = n\left[\beta_\ell\left(\frac{k}{n}\right) + O\left(\frac{1}{n}\right)\right], \tag{56}$$

where $\ell, k \in \mathbb{Z}^d$, and β_ℓ is a given collection of rate functions.

In Section 5.1 of [2], Aldous observes that the results from Chapter 11 of Ethier and Kurtz [11] can be used to prove the weak convergence of a Marcus-Lushnikov process to the solutions of Smoluchowski system of equations in the case when the Marcus-Lushnikov process can be formulated as a finite dimensional density dependent population process. Specifically, the Marcus-Lushnikov processes corresponding to the multiplicative and Kingman coalescent with the monodisperse initial conditions (n singletons) can be represented as finite dimensional density dependent population processes defined above.

Define $F(x) = \sum\limits_\ell \ell\beta_\ell(x)$. Then, Theorem 2.1 in Chapter 11 of [11] (Theorem 8.1 in [25]) states the following law of large numbers. Let $\hat{X}_n(t)$ be the Markov process with the intensities $q^{(n)}(k, k+\ell)$ given in (56), and let $X_n(t) = n^{-1}\hat{X}_n(t)$. Finally, let $|x| = \sqrt{\sum x_i^2}$ denote the Euclidean norm in $\mathbb{R}^d$.

Theorem 5.1. *Suppose for all compact $\mathcal{K} \subset \mathbb{R}^d$,*

$$\sum_\ell |\ell| \sup_{x\in\mathcal{K}} \beta_\ell(\bar{x}) < \infty,$$

and there exists $M_\mathcal{K} > 0$ such that

$$|F(x) - F(y)| \le M_\mathcal{K}|x - y|, \qquad \textit{for all } x, y \in \mathcal{K}. \tag{57}$$

Suppose $\lim_{n\to\infty} X_n(0) = x_0$*, and* $X(t)$ *satisfies*

$$X(t) = X(0) + \int_0^t F(X(s))ds, \tag{58}$$

for all $T \geq 0$*. Then*

$$\lim_{n\to\infty} \sup_{s\in[0,T]} |X_n(s) - X(s)| = 0 \qquad a.s. \tag{59}$$

5.2. **Hydordynamic limit for multiplicative coalescent process.** Consider a multiplicative coalescent process with kernel $K(i,j) = ij$. Recall that in the definition of a coalescent process given in Sect. 2.2, a pair of clusters with masses i and j coalesces at the rate $K(i,j)/n$. Consider the corresponding Marcus-Lushnikov process

$$\mathbf{ML}_n(t) = \left(\zeta_1^{[n]}(t), \zeta_2^{[n]}(t), \ldots, \zeta_n^{[n]}(t), 0, 0, \ldots\right)$$

that keeps track for the numbers of clusters in each weight category. There, the initial conditions will be $\mathbf{ML}_n(0) = (n, 0, 0, \ldots) = ne_1$, where e_i denotes the i-th coordinate vector.

Next, for a fixed positive integer K, let $\hat{X}_n(t)$ be the restriction of process $\mathbf{ML}_n(t)$ to the first K dimensions, i.e.

$$\hat{X}_n(t) = \left(\zeta_1^{[n]}(t), \zeta_2^{[n]}(t), \ldots, \zeta_K^{[n]}(t)\right)$$

with the initial conditions $\hat{X}_n(0) = ne_1$. Apparently, $\hat{X}_n(t)$ is itself a (finite dimensional) Markov process with the following transition rates of $\hat{X}_n(t)$ stated as in (56). Let $x = (x_1, x_2, \ldots, x_K)$. Then, for any pair $1 \leq i < j \leq K$, the change vector $\ell = -e_i - e_j + e_{i+j}\mathbf{1}_{i+j\leq K}$ corresponding to a merger of clusters of respective sizes i and j is assigned the rate

$$q^{(n)}(x, x+\ell) = \frac{ij}{n} x_i x_j = n\beta_\ell\left(\frac{x}{n}\right),$$

where $\beta_\ell(x) = ijx_ix_j$.

For a given $1 \leq i \leq K$, the change vector $\ell = -2e_i + e_{2i}\mathbf{1}_{2i\leq K}$ corresponding to a merger of a pair of clusters of size i is assigned the rate

$$q^{(n)}(x, x+\ell) = \frac{1}{n}\left[\frac{i^2x_i^2}{2} - \frac{i^2x_i}{2}\right] = n\left[\beta_\ell\left(\frac{x}{n}\right) + O\left(\frac{1}{n}\right)\right],$$

where $\beta_\ell(x) = i^2\frac{x_i^2}{2}$.

For a given $1 \leq i \leq K$, the change vector $\ell = -e_i$ corresponding to a cluster of mass i merging with a cluster of mass greater than K is assigned the rate

$$q^{(n)}(x, x+\ell) = \frac{1}{n} i x_i \left[n - \sum_{j=1}^{K} jx_j\right] = n\beta_\ell\left(\frac{x}{n}\right),$$

where $\beta_\ell(x) = ix_i\left(1 - \sum_{j=1}^{K} jx_j\right)$.

Then, by Theorem 5.1, $X_n(t) = n^{-1}\hat{X}_n(t)$ converges to $X(t)$ as in (59), where $X(t)$ satisfies (58) with

$$\begin{aligned}
F(x) := \sum_{\ell} \ell\beta_\ell(x) &= \sum_{ij:\ 1\le i<j\le K} ijx_ix_j[-e_i - e_j + e_{i+j}\mathbf{1}_{i+j\le K}] \\
&\quad + \frac{1}{2}\sum_{i=1}^{K} i^2x_i^2[-2e_i + e_{2i}\mathbf{1}_{2i\le K}] - \sum_{i=1}^{K} ix_i\left(1 - \sum_{j=1}^{K} jx_j\right)e_i \\
&= \sum_{i=1}^{K}\left(-ix_i + \frac{1}{2}\sum_{\substack{1\le i_1,i_2\le K \\ i_1+i_2=i}} i_1i_2x_{i_1}x_{i_2}\right)e_i. \qquad (60)
\end{aligned}$$

Here, $F(x)$ is naturally satisfying the Lipschitz continuity conditions (57), and the initial conditions $X(0) = X_n(0) = e_1$.

Observe that the system of equations (58) with $F(x)$ as in (60) will yield the reduced system of Smoluckowski coagulation equations (9) also known as the Flory coagulation system [15]. Thus, for a given integer $K > 0$ and a fixed real $T > 0$,

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left| n^{-1}\zeta_k^{[n]}(s) - \zeta_k(s) \right| = 0 \qquad \text{a.s.} \qquad (61)$$

for $k = 1, 2, \ldots, K$.

Note that the above limit no longer requires a fixed K for each individual k in (61). However, we mainly use the following limit in our calculations,

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left| \sum_{k=1}^{K} n^{-1}\zeta_k^{[n]}(s) - \sum_{k=1}^{K} \zeta_k(s) \right| = 0 \qquad \text{a.s.} \qquad (62)$$

5.3. **Hydordynamic limit for cross-multiplicative coalescent processes.** Fix integers $K_1 > 0$ and $K_2 > 0$, and let $R := R(K_1, K_2) = \{1, 2, \ldots, K_1\} \times \{1, 2, \ldots, K_2\}$. Let $e_{\mathbf{i}}$ be the standard basis vectors in $\mathbb{R}^{K_1K_2}$, enumerated by $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in R$. Consider a restriction to $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in R$ of a Marcus-Lushnikov process $\zeta_{i_1,i_2}(t)$ with the cross-multiplicative kernel. Let

$$\hat{X}_n(t) = \left\{\zeta_{i_1,i_2}^{[n]}(t)\right\}_{\mathbf{i}\in R}$$

with the initial conditions $\hat{X}_n(0) = \alpha[n]e_{0'} + \beta[n]e_{0''}$, where $0' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $0'' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We observe the following transition rates of $\hat{X}_n(t)$ stated as in (56). Let $x = \sum\limits_{\mathbf{i}\in R} x_{\mathbf{i}} e_{\mathbf{i}}$. Then, for any $\mathbf{i}$ and $\mathbf{j}$ in R, the change vector $\ell = -e_{\mathbf{i}} - e_{\mathbf{j}} + \mathbf{1}_{\{\mathbf{i}+\mathbf{j}\in R\}} e_{\mathbf{i}+\mathbf{j}}$ corresponding to a merger of clusters of respective weights $\mathbf{i}$ and $\mathbf{j}$ is assigned the rate

$$q^{(n)}(x, x+\ell) = \frac{1}{n}(i_1 j_2 + i_2 j_1) x_{\mathbf{i}} x_{\mathbf{j}} = n\beta_\ell(x),$$

where $\beta_\ell(x) = (i_1 j_2 + i_2 j_1) x_{\mathbf{i}} x_{\mathbf{j}}$.

For a given $\mathbf{i} \in R$, the change vector $\ell = -e_{\mathbf{i}}$ corresponding to the merger of clusters whose weight vector is $\mathbf{i}$ with clusters whose weight vectors are not in R is assigned the rate

$$q^{(n)}(x, x+\ell) = \frac{1}{n}\left[i_1 x_{\mathbf{i}}\left(\beta[n] - \sum_{\mathbf{j}\in R} j_2 x_{\mathbf{j}}\right) + i_2 x_{\mathbf{i}}\left(\alpha[n] - \sum_{\mathbf{j}\in R} j_1 x_{\mathbf{j}}\right)\right] = n\left[\beta_\ell(x) + O\left(\frac{1}{n}\right)\right],$$

where $\beta_\ell(x) = i_1 x_{\mathbf{i}}\left(\beta - \sum_{\mathbf{j}\in R} j_2 x_{\mathbf{j}}\right) + i_2 x_{\mathbf{i}}\left(\alpha - \sum_{\mathbf{j}\in R} j_1 x_{\mathbf{j}}\right)$.

Thus, by Theorem 5.1, $X_n(t)$ converges to $X(t)$ as in (59), where $X(t)$ satisfies (58) with

$$\begin{aligned} F(x) := \sum_\ell \ell\beta_\ell(x) =& \frac{1}{2}\sum_{\mathbf{i},\mathbf{j}\in R}\left[-e_{\mathbf{i}} - e_{\mathbf{j}} + \mathbf{1}_{\{\mathbf{i}+\mathbf{j}\in R\}} e_{\mathbf{i}+\mathbf{j}}\right](i_1 j_2 + i_2 j_1) x_{\mathbf{i}} x_{\mathbf{j}} \\ & - \sum_{\mathbf{i}\in R} e_{\mathbf{i}} i_1 x_{\mathbf{i}}\left(\beta - \sum_{\mathbf{j}\in R} j_2 x_{\mathbf{j}}\right) - \sum_{\mathbf{i}\in R} e_{\mathbf{i}} i_2 x_{\mathbf{i}}\left(\alpha - \sum_{\mathbf{j}\in R} j_1 x_{\mathbf{j}}\right) \\ =& \sum_{\mathbf{i}\in R} e_{\mathbf{i}}\left(-(\beta i_1 + \alpha i_2) x_{\mathbf{i}} + \frac{1}{2}\sum_{\ell,\mathbf{k}:\ \ell+\mathbf{k}=\mathbf{i}} (\ell_1 k_2 + \ell_2 k_1) x_\ell x_{\mathbf{k}}\right) \end{aligned} \tag{63}$$

for a fixed $T > 0$. The system of equations (58) with $F(x)$ given in (63) will yield the reduced system of Smoluckowski coagulation equations (18). So, for a fixed a pair of positive integers K_1 and K_2, and a fixed real number $T > 0$,

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left| n^{-1}\zeta^{[n]}_{i_1,i_2}(s) - \zeta_{i_1,i_2}(s)\right| = 0 \qquad \text{a.s.} \tag{64}$$

for all $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in R$. Consequently,

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left| n^{-1} \sum_{\substack{1\le i_1\le K_1 \\ 1\le i_2\le K_2}} \zeta^{[n]}_{i_1,i_2}(s) - \sum_{\substack{1\le i_1\le K_1 \\ 1\le i_2\le K_2}} \zeta_{i_1,i_2}(s)\right| = 0 \qquad \text{a.s.} \tag{65}$$

5.4. **Central Limit Theorem and related results.** The usefulness of the framework set in [11, 25] for proving weak convergence is that the law of large numbers Theorem 5.1 is enhanced with the corresponding central limit theorem (see Theorem 5.2 below) and the large deviation theory [14]. The following central limit theorem is derived in Theorem 8.2 in [25] (and Theorem 2.3 in Chapter 11 of [11]).

Theorem 5.2. *Suppose for all compact $\mathcal{K} \subset \mathbb{R}^d$,*

$$\sum_{\ell} |\ell|^2 \sup_{x \in \mathcal{K}} \beta_\ell(x) < \infty \tag{66}$$

and that the β_ℓ and ∂F are continuous. Suppose X_n and X are as in Theorem 5.1, and suppose $V_n = \sqrt{n}(X_n - X)$ is such that $\lim_{n\to\infty} V_n(0) = V(0)$, where $V(0)$ is a constant. Then V_n converges in distribution to V, which is the solution of

$$V(t) = V(0) + U(t) + \int_0^t \partial F(X(s))V(s)ds, \tag{67}$$

where $U(t)$ is a Gaussian process and $\partial F(X(s)) = (\partial_j F_i(X(s)))_{i,j}$.

The proof of Theorem 5.2 is based on representing $V_n(t)$ as follows. Let Y_ℓ be independent Poisson processes with rate one. Then,

$$V_n(t) = V_n(0) + U_n(t) + \int_0^t \sqrt{n}\big(F(X_n(s)) - F(X(s))\big)ds, \tag{68}$$

where

$$U_n(t) = \sum_{\ell} \ell W_\ell^{(n)}\Big(\int_0^t \beta_\ell(X_n(s))ds\Big),$$

$W_\ell^{(n)}(u) = n^{-1/2}\hat{Y}_\ell(nu)$, and $\hat{Y}_\ell(u) := Y_\ell(u) - u$ are centralized Poisson processes.

Next, we will use formula (68) in order to derive an upper bound (69) on probability $P(|X_n(T) - X(T)| \geq \delta)$. Let us consider a simple case of a density dependent population process on $\mathbb{R}^d$ for which the following three conditions are satisfied.

- **i:** $V_n = \sqrt{n}(X_n - X)$ is such that $\lim_{n\to\infty} V_n(0) = V(0)$.
- **ii:** Both $X_n(t)$ and $X(t)$ live on a compact set $\mathcal{K}$.
- **iii:** There are finitely many vectors $\ell \in \mathbb{R}^d$ such that $\beta_\ell(x) > 0$ for some $x \in \mathcal{K}$.

Notice that the above conditions are satisfied for the Marcus-Lushnikov processes considered here, with the general bilinear kernel as in Sect. 5.2 and with the cross-multiplicative kernel as in Sect. 5.3. Specifically, for a given $m > 0$, let

$$\mathcal{K}_m = \Big\{x \in \mathbb{R}_+^d \ : \ \sum_i x_i \leq m\Big\}.$$

Then, in Sect. 5.2, $X_n(t), X(t) \in \mathcal{K}_2$, and in Sect. 5.3, $X_n(t), X(t) \in \mathcal{K}_m$ for $m > \alpha + \beta$.

Proposition 5.3. *Assuming the above conditions* **i-iii** *are satisfied together with the Lipschitz continuity conditions* (57)*, we have*

$$P(|X_n(T)-X(T)|\geq\delta)=O(n^{-2}). \tag{69}$$

Proof. Here,

$$\sqrt{n}\big|F(X_n(s))-F(X(s))\big|\leq\sqrt{n}M_{\mathcal{K}}|X_n(s)-X(s)|=M_{\mathcal{K}}|V_n(s)|$$

and for a fixed $T>0$ and any $t\leq T$,

$$|V_n(0)+U_n(t)|\leq\varepsilon_n(T):=|V_n(0)|+\sum_\ell|\ell|\;\max\left\{\left|W_\ell^{(n)}(s)\right|\;:\;s\in\Big[0,T\sup_{x\in\mathcal{K}}\beta_\ell(x)\Big]\right\}.$$

Hence, for a fixed $T>0$, equation (68) implies the following inequality,

$$|V_n(t)|\leq\varepsilon_n(T)+M_{\mathcal{K}}\int_0^t|V_n(s)|ds\qquad\text{for all } t\in[0,T].$$

Then, by Grönwall's inequality (see Appendix 5 in [11]),

$$|V_n(t)|\leq\varepsilon_n(T)e^{M_{\mathcal{K}}t}. \tag{70}$$

In particular, we use equation (70) together with Markov inequality to obtain the following simple bound for any $\delta>0$,

$$P(|X_n(T)-X(T)|\geq\delta)\leq\frac{V_n^4(T)}{n^2\delta^4}\leq\frac{E[\varepsilon_n^4(T)]e^{4M_{\mathcal{K}}T}}{n^2\delta^4}. \tag{71}$$

Here, for any fixed real $S>0$, integer $r>0$, and any real $\lambda>0$, we have by Doob's martingale inequality,

$$P\left(\max_{s\in[0,S]}\left|W_\ell^{(n)}(s)\right|^r\geq\lambda\right)=P\left(\max_{s\in[0,S]}\left|W_\ell^{(n)}(s)\right|\geq\lambda^{1/r}\right)\leq\frac{E\left[\left(W_\ell^{(n)}(S)\right)^{2+2r}\right]}{\lambda^{2+2/r}}$$

as $\left|W_\ell^{(n)}(s)\right|$ is a non-negative sub-martingale. Therefore,

$$\begin{aligned}E\left[\max_{s\in[0,S]}\left|W_\ell^{(n)}(s)\right|^r\right]&\leq1+\int_1^\infty P\left(\max_{s\in[0,S]}\left|W_\ell^{(n)}(s)\right|^r\geq\lambda\right)d\lambda\\&\leq1+(1+2/r)E\left[\left(W_\ell^{(n)}(S)\right)^{2+2r}\right],\end{aligned}$$

where by the classical central limit theorem,

$$\lim_{n\to\infty}E\left[\left(W_\ell^{(n)}(S)\right)^{2+2r}\right]=S^{1+r}E[Z^{2+2r}],\qquad Z\text{ - standard normal r.v.}$$

Thus, $E[\varepsilon_n^4(T)]=O(1)$, and (69) follows from (71). $\square$

6. Discussion: generalizations and open problems.

In this paper we considered an important example of coagulation ODEs obtained as a hydrodynamic limit of a Marcus-Lushnikov process that tracks the merger history of a coalescent process with two dimensional weight vectors. The coagulation equations and gelation in the Marcus-Lushnikov dynamics for other coalescent processes with multidimensional weight vectors is on its own an interesting object of studies. As a natural next step, one may consider a generalization of the existing results [1, 22, 23, 30, 12, 18, 19] on gelation phenomenon for vector weighted processes.

An extension of the application to minimal spanning trees may come from an observation that the convergence rates in the hydrodynamic limit yield the central limit theorem for L_n on $K_{\alpha[n],\beta[n]}$ similar to the central limit theorem for L_n on K_n proved in Janson [21]. Specifically, we hope to apply Theorem 5.2 in the analysis. Moreover, similarly to [8], it is possible to examine the second and third order terms in L_n.

Finally, genetic recombination is one of the issues facing the use of coalescent processes in genetics as models of genetic drift viewed backwards in time. Distinct gene loci would follow different pathways of ancestry, resulting in different gene genealogies. As a biological application, it is compelling to consider a coalescent process with multidimensional weight vectors as a means of addressing the issue of genetic recombination, and possibly, the issue of biological compatibility.

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Appendix: Some alternative proofs

Alternative proof of Lemma 3.3. We will repeat the approach in (11). By Stirling's approximation and equation (23), we have

$$\sum_{i_1,i_2}(i_1+i_2)^2\zeta_{i_1,i_2}(t) = \sum_{i_1,i_2}(i_1+i_2)^2\frac{i_1^{i_2-1}i_2^{i_1-1}\alpha^{i_1}\beta^{i_2}}{i_1!i_2!}e^{-(\beta i_1+\alpha i_2)t}t^{i_1+i_2-1}$$

$$=t^{-1}\sum_{i_1,i_2}\frac{(i_1+i_2)^2}{i_1^{3/2}i_2^{3/2}}i_1^{i_2-i_1}i_2^{i_1-i_2}e^{-(\beta t-\ln(\alpha t)-1)i_1}e^{-(\alpha t-\ln(\beta t)-1)i_2}$$

$$\cdot\frac{1}{2\pi}\big(1+O(i_1^{-1})\big)\big(1+O(i_2^{-1})\big), \tag{72}$$

where

$$i_1^{i_2-i_1}i_2^{i_1-i_2} = e^{-(i_1-i_2)(\ln(i_1)-\ln(i_2))}.$$

Next, we plug in $i_1 = x$ and $i_2 = cx$ into the exponent in (72), obtaining

$$-(i_1-i_2)\big(\ln(i_1)-\ln(i_2)\big)-\big(\beta t-\ln(\alpha t)-1\big)i_1-\big(\alpha t-\ln(\beta t)-1\big)i_2$$

$$= -\Big[(c\alpha+\beta)t-\ln(\alpha t)-c\ln(\beta t)+(c-1)\ln c-(c+1)\Big]x. \tag{73}$$

The maximal value of the exponent (73) is therefore achieved when

$$t = \frac{c+1}{c\alpha+\beta}. \tag{74}$$

We plug in the optimal value (74) into (73) with the exponent in (72) becoming equal to

$$\left[c\ln\left(\frac{(c+1)\beta}{c(c\alpha+\beta)}\right)+\ln\left(\frac{(c+1)c\alpha}{c\alpha+\beta}\right)\right]x. \tag{75}$$

Now, since $\ln(x)$ is a strictly concave function,

$$c\ln\left(\frac{(c+1)\beta}{c(c\alpha+\beta)}\right)+\ln\left(\frac{(c+1)c\alpha}{c\alpha+\beta}\right)\le(c+1)\ln 1 = 0 \tag{76}$$

with the equality obtained if and only if

$$c = \sqrt{\frac{\beta}{\alpha}}. \tag{77}$$

Thus, substituting (77) into (74) yields $t = \frac{1}{\sqrt{\alpha\beta}}$. Indeed, the exponent in (72) may equal zero if and only if $t = \frac{1}{\sqrt{\alpha\beta}}$. If $t < \frac{1}{\sqrt{\alpha\beta}}$, the series (72) converges. While taking $t = \frac{1}{\sqrt{\alpha\beta}}$, we have the portion of the series (72) corresponding to the indices satisfying $i_1\sqrt{\beta}-i_2\sqrt{\alpha} = o(i_1+i_2)$ diverging to infinity since $\sum\limits_i\frac{1}{i} = \infty$.

$\square$

Next, we give an alternative proof of Lemma 3.6 that uses differential equations approach in the style of [28].

Alternative proof of Lemma 3.6. Let $x = u\frac{\partial}{\partial u}s(u,v)$ and $y = v\frac{\partial}{\partial v}s(u,v)$ for all $u, v \geq 0$ for which the series converge. The expression for the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is obtained by first taking the partial derivatives of (25) and arriving with

$$u\frac{\partial^2 s}{\partial u^2} = \frac{xy}{u(1-y)} - \frac{1-x}{1-y}v\frac{\partial^2 s}{\partial u \partial v} \quad \text{and} \quad v\frac{\partial^2 s}{\partial v^2} = \frac{xy}{v(1-x)} - \frac{1-y}{1-x}u\frac{\partial^2 s}{\partial u \partial v}.$$

Next, we substitute the above into the determinant, arriving with

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \frac{xy}{uv} + y\frac{u}{v}\frac{\partial^2 s}{\partial u^2} + x\frac{v}{u}\frac{\partial^2 s}{\partial v^2} + uv\frac{\partial^2 s}{\partial u^2}\frac{\partial^2 s}{\partial v^2} - uv\left(\frac{\partial^2 s}{\partial u \partial v}\right)^2 \\
&= \frac{xy}{uv} + y\frac{u}{v}\frac{\partial^2 s}{\partial u^2} + x\frac{v}{u}\frac{\partial^2 s}{\partial v^2} + xy\frac{v}{u(1-y)}\frac{\partial^2 s}{\partial v^2} + xy\frac{u}{v(1-x)}\frac{\partial^2 s}{\partial u^2} \\
&= \frac{\partial s}{\partial u}\frac{\partial s}{\partial v}\left(1 + \frac{u^2}{x(1-x)}\frac{\partial^2 s}{\partial u^2} + \frac{v^2}{y(1-y)}\frac{\partial^2 s}{\partial v^2}\right). \qquad (78)
\end{aligned}$$

The above expression (78) implies $\frac{\partial(x,y)}{\partial(u,v)} > 0$ for $(u,v) \in \mathbb{R}^2_+$ in a small enough neighborhood of $(0,0)$, insuring $x, y < 1$.

Equation (25) rewrites as $s = x+y-xy$ with partial derivatives $\frac{\partial s}{\partial x} = 1-y$ and $\frac{\partial s}{\partial y} = 1-x$.

Next, let $\tilde{u} = xe^{-y}$ and $\tilde{v} = ye^{-x}$. The Jacobian $\frac{\partial(\tilde{u},\tilde{v})}{\partial(x,y)} = (1-xy)e^{-x}e^{-y} = 0$ if and only if $xy = 1$. Therefore, for $xy < 1$, we have

$$\begin{aligned}
1-y = \frac{\partial s}{\partial x} &= e^{-y}\frac{\partial s}{\partial \tilde{u}} - ye^{-x}\frac{\partial s}{\partial \tilde{v}} = \frac{1}{x}\tilde{u}\frac{\partial s}{\partial \tilde{u}} - \tilde{v}\frac{\partial s}{\partial \tilde{v}} \\
\text{and} \qquad 1-x = \frac{\partial s}{\partial y} &= -xe^{-y}\frac{\partial s}{\partial \tilde{u}} + e^{-x}\frac{\partial s}{\partial \tilde{v}} = -\tilde{u}\frac{\partial s}{\partial \tilde{u}} + \frac{1}{y}\tilde{v}\frac{\partial s}{\partial \tilde{v}},
\end{aligned}$$

yielding

$$\tilde{u}\frac{\partial s}{\partial \tilde{u}} = x = u\frac{\partial s}{\partial u} \quad \text{and} \quad \tilde{v}\frac{\partial s}{\partial \tilde{v}} = y = v\frac{\partial s}{\partial v}$$

with $s(u,v)\big|_{(\tilde{u},\tilde{v})=(0,0)} = s(0,0) = 0$. Hence, (u,v) and $(\tilde{u},\tilde{v})$ as functions of (x,y) will coincide in the whole domain

$$\{(x,y) \in \mathbb{R}^2_+ \,:\, xy < 1\},$$

and

$$x = \tilde{u}\frac{\partial}{\partial u}s(\tilde{u},\tilde{v}) \quad \text{and} \quad y = \tilde{v}\frac{\partial}{\partial v}s(\tilde{u},\tilde{v}).$$

Here, by Prop. 3.4, (x,y) is the smallest solution of $\tilde{u} = xe^{-y}$ and $\tilde{v} = ye^{-x}$. Equations (31) follow. □

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