Russo's formula for Lorentz Lattice Gas Model

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Abstract

We use a combinatorial approach to study the trajectory of a light ray constrained to Euclidian plane $\mathbb{R}^2$ with random reflecting obstacles placed throughout $\mathbb{R}^2$. For the 2D Lorentz lattice gas (LLG) model we derive an analogue of Russo's formula of increasing events in percolation.

Keywords: recurrence, lattice gas, localization

1 Introduction.

We consider a trajectory of a light ray moving in $\mathbb{R}^d$ but scattered according to reflecting obstacles randomly distributed about $\mathbb{R}^d$. Studying the trajectories was recognized to be extremely difficult even after restricting the light to a two dimensional square lattice, and limiting the number of possible types of obstacles. The first study of the above construction is usually attributed to Lorentz (see [5]).

Probably the best known version of the Lorentz gas model is that where the only possible obstacles are two-sided mirrors placed at the vertices of a square lattice. The model, sometimes called "Lorentz lattice gas model" or "the Lorentz mirror model", can be formulated as follows. We let $0 \leq p \leq 1$. At each vertex of the lattice we decide whether to place a two-sided North-East(NE) mirror, North-West(NW) mirror or nothing. We place a NE mirror and a NW mirror with probabilities $\frac{p}{2}$ each and place no mirror with probability $(1-p)$. From the origin, we shine a light ray ρ northward (so that the origin (0,0) and vertex (0,1) both belong to the ray ρ). We say that the light ray ρ is **localized** if it returns back to the origin. We notice that each ray, if continued in the opposite direction is actually a "cycle" of light, which might be infinite as well (contain an "infinity" point). We say that the cycle ρ is **localized** if it is of finite size (visits finitely many vertices). $P_p(\cdot)$ will denote the probability measure on the state space corresponding to the construction.

We let $\eta(p)$ to be the probability that the light illuminates infinitely many vertices (i.e. cycle ρ is not "localized"). Obviously, $\eta(0) = 1$. If $p = 1$, a beautiful percolation argument of Grimmett (see [3]) shows that the light is a.s. localized ($\eta(1) = 0$). See also Bunimovitch and Troubetzkoy ([1]) for a generalization of Grimmett's argument.

There are two important conjectures naturally arising here. One is that η is monotone in p. The other is that the light is a.s. localized ($\eta(p) = 1$) for all $p > 0$. We refer the reader to

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[2] for a detailed discussion of the conjectures above. For this version of LLG, the numerical simulations justifying some of the conjectures were done by E.G.D.Cohen and F.Wang. Another variant of the above mirror model, called "random walk in a random labyrinth" was studied in detail by Grimmett, Menshikov and Volkov (see [4]). Some interesting properties of the model are proved by A.Quas in [8].

Let D be a region containing the edge between $(0,0)$ and $(0,1)$. When restricting the model to D, we will use the following terminology: a cycle is *D-localized* when it lies entirely inside D. When it is not, we say that it contains "point" ∂D (write "$\partial D \in$ cycle" or "cycle $\leftrightarrow \partial D$"). We don't know what is happening outside D: the same cycle leaving D and then reappearing inside D is going to be treated as two different cycles, each containing ∂D.

2 Russo's Formula.

Russo's formula is one of the major tools used in percolation and related models in statistical mechanics. As an example we recall the Menshikov's proof of exponential cluster decay in subcritical phase of Bernoulli bond percolation (see [6], [7] and [3] for details). There, the Russo's formula of increasing events was used. In this paper we will prove an analogue of the percolation formula, where instead of a notion "increasing events" we will use a characteristic property of light cycles specific to non-localization events. Being one of the very few tools in this version of LLG model that deal with specific geometrical properties of light rays, the formula promises to play an important role in researching the field.

In this section we derive an elementary Russo's formula for LLG, which will be reinforced with localization-specific modification in the next section.

We consider a connected region $D \subset \mathbb{R}^2$ such that it contains all the edges connecting the vertices of the subgraph $V \equiv D \cap \mathbb{Z}^2$. We let $\Omega_V \equiv \{-1,0,1\}^V$ be the states space for the model: the set of all possible ways of positioning the mirrors at the vertices of V. Here "-1" corresponds to placing a NW mirror, "1" to placing a NE mirror and "0" to placing no mirror at a vertex. $P_p(\cdot)$ will be the probability measure on $\Omega_{\mathbb{Z}^2}$. Now, lets write $E \in \Omega_V$ if an event E depends entirely on the states of vertices of V, and therefore is a subset of Ω_V. We define the number of NE and NW mirrors for a configuration $\omega \in \Omega_V$ in the following way:

$$M_-(\omega) \equiv \sum_{v\in V} \omega(v)^- \text{ and } M_+(\omega) \equiv \sum_{v\in V} \omega(v)^+,$$

where $x^- = -\min(x,0)$ and $x^+ = \max(x,0)$ are the standard notations. Now, for $0 < p < 1$,

$$P_p(E) = \sum_{\omega\in\Omega_V} (\frac{p}{2})^{M_-(\omega)+M_+(\omega)}(1-p)^{|V|-(M_-(\omega)+M_+(\omega))}\mathbf{1}_E(\omega).$$

where $|V|$ is the cardinality of V. Then, differentiating the equation above, we get

$$\begin{aligned}\frac{d}{dp}P_p(E) &= \sum_{\omega\in\Omega_V}\left(\frac{M_-(\omega)+M_+(\omega)}{p}-\frac{|V|-(M_-(\omega)+M_+(\omega))}{1-p}\right)\mathbf{1}_E(\omega)P_p(\omega)\\ &= \frac{1}{p(1-p)}\mathbb{E}_p[\{M_+ + M_- - p|V|\}\mathbf{1}_E]\\ &= \frac{1}{p(1-p)}Cov_p(M_+ + M_-, \mathbf{1}_E). \end{aligned} \tag{1}$$

For the rest of the paper, the equation above will be called the general Russo's formula for the Lorentz lattice gas model.

We restrict ourselves to the region $D = B(n) \equiv \{(x, y) : |x| + |y| < n\}$. Let $V = V(n)$ be the set of vertices inside the box $D = B(n)$. Now, we let A_n be the event that the cycle ρ goes beyond $B(n)$:

$$A_n \equiv \{\rho \cap \partial B(n) \neq \emptyset\}.$$

We notice that A_n depends entirely on how the mirrors are placed inside $B(n)$. Lets define the function $\eta_n(p) \equiv P_p(A_n)$. Then

$$\eta_n(p) \downarrow \eta(p) \equiv P_p(A),$$

where A is the event that ρ visits an infinite number of sights ($A \equiv \{|\rho| = \infty\}$).

3 Pivotal and Indifferent Vertices. The Formula for Non-Localization.

We begin with a couple of definitions followed by an observation that leads us to proving the key formula of this paper. Here, as before, $\omega \in \Omega_V$. For a vertex $v \in V$, we let

$\omega_v^+(u) = \begin{cases}\omega(u) & \text{if} u \neq v,\\ 1 & \text{if} u = v;\end{cases}$ $\omega_v^-(u) = \begin{cases}\omega(u) & \text{if} u \neq v,\\ -1 & \text{if} u = v;\end{cases}$ and $\omega_v^0(u) = \begin{cases}\omega(u) & \text{if} u \neq v,\\ 0 & \text{if} u = v.\end{cases}$

Definition 1. We say that a vertex $v \in V$ is **pivotal** for an event $E \subset \Omega_V$ if

$$\begin{cases}\omega_v^+ \in E,\\ \omega_v^0 \notin E,\\ \omega_v^- \in E.\end{cases}$$

We also say a vertex $v \in V$ is **pivotal**$^+$ for the event $E \subset \Omega_V$ if

$$\begin{cases}\omega_v^+ \in E,\\ \omega_v^0 \in E,\\ \omega_v^- \notin E;\end{cases}$$

and **pivotal**$^-$ if

$$\begin{cases} \omega_v^+ \notin E, \\ \omega_v^0 \in E, \\ \omega_v^- \in E. \end{cases}$$

Definition 2. We say that a vertex $v \in V$ is **indifferent** for an event $E \subset \Omega_V$ if either is true:

$$\begin{cases} \omega_v^+ \in E, \\ \omega_v^0 \in E, \\ \omega_v^- \in E. \end{cases}$$

or

$$\begin{cases} \omega_v^+ \notin E, \\ \omega_v^0 \notin E, \\ \omega_v^- \notin E. \end{cases}$$

Important Observation: we notice that in case of the event A_n there can be only pivotal, pivotal$^+$, pivotal$^-$ and indifferent vertices. The observation above is crucial for the rest of the paper. The proof is very simple: in the event of A_n the cycle ρ will consist of two rays going from zero to the boundary of $D = B(n)$. One is the original ray ρ_u going from the origin up. The other ρ_d is the complementary ray, that we get by extending ρ_u to form a cycle. Now, all the vertices $v \in V$ that do not lie in the intersection $\rho_u \cap \rho_d$ of the two rays are automatically indifferent as both rays are leaving $D = B(n)$ and by changing the direction of one of them we are not stopping the other from leaving the region. Now, at every point of the intersection $\rho_u \cap \rho_d$ there is only ONE way to avoid A_n from happening, and that is if we place a mirror (or no mirror) so that to make a cycle out of the portions of rays ρ_u and ρ_d that connect the origin to the intersection vertex. The vertex therefore is either pivotal, pivotal$^-$ or pivotal$^+$. To complete the argument we notice that if we adjust the mirror at an intersection vertex in the way mentioned above, so that the event A_n is not happening, the vertex will still be pivotal, pivotal$^-$ or pivotal$^+$, depending on what it was before the adjustment.

In case when A_n is not happening ($\omega \in$ complement $A_n^c \subset \Omega_V$), each pivotal, pivotal$^+$ or pivotal$^-$ vertex is the one that belongs both to ρ and a cycle containing ∂D. The mirror placed at the vertex defines to which of the three categories the vertex belongs to.

Defining the event $J_v \equiv \{\omega(v) \neq 0\}$, we rewrite the covariance in 1 as follows

$$\mathbb{E}_p[\{(M_+ + M_-) - p|V|\}1_{A_n}] \;=\; \sum_{v \in V_n} \{P_p(A_n \cap J_v) - pP_p(A_n)\}, \tag{2}$$

where at a given vertex $v \in V = V(n)$,

$$\begin{aligned} P_p(A_n \cap J_v) - pP_p(A_n) \;&=\; P_p(J_v \cap \{\text{piv.}\}) + P_p(J_v \cap A_n \cap \{\text{not piv.}\}) \\ &-\; pP_p(A_n \cap \{\text{piv.}\}) - pP_p(A_n \cap \{\text{not piv.}\}) \\ &=\; p(1-p)P_p(\{\text{piv.}\}) + P_p(J_v \cap A_n \cap \{\text{not piv.}\}) - pP_p(A_n \cap \{\text{not piv.}\}) \end{aligned} \tag{3}$$

as $A_n \cap J_v \cap \{\text{piv.}\} = J_v \cap \{\text{piv.}\} = A_n \cap \{\text{piv.}\}$.

Now,

$$P_p(A_n \cap \{\text{not piv.}\}) = (1-\frac{p}{2})P_p(\{\text{not piv.}\} \cap \{\text{not indif.}\}) + P_p(\{\text{indif.}\} \cap A_n) \qquad (4)$$

as $\{\text{not piv.}\} \cap \{\text{not indif.}\} = \{\text{piv.}^-\} \cup \{\text{piv.}^+\}$ by the observation above and tehrefore

$$P_p(A_n \cap \{\text{not piv.}\} \cap \{\text{not indif.}\}) = (1-\frac{p}{2})P_p(\{\text{not piv.}\} \cap \{\text{not indif.}\}).$$

Also

$$P_p(J_v \cap A_n \cap \{\text{not piv.}\}) = \frac{p}{2}P_p(\{\text{not piv.}\} \cap \{\text{not indif.}\}) + pP_p(\{\text{indif.}\} \cap A_n) \qquad (5)$$

as $P_p(J_v \cap A_n \cap \{\text{not piv.}\} \cap \{\text{not indif.}\}) = \frac{p}{2}P_p(\{\text{not piv.}\} \cap \{\text{not indif.}\})$ and

$$P_p(J_v \cap A_n \cap \{\text{not piv.}\} \cap \{\text{indif.}\}) = P_p(J_v)P_p(A_n \cap \{\text{indif.}\}).$$

Thus substituting (4) and (5) into (3), we get

$$\begin{aligned} P_p(A_n \cap J_v) - pP_p(A_n) &= p(1-p)P_p(\{\text{piv.}\}) - \frac{p(1-p)}{2}P_p(\{\text{not piv.}\} \cap \{\text{not indif.}\}) \\ &= p(1-p)P_p(\{\text{piv.}\}) - \frac{p(1-p)}{2}P_p(\{\text{piv.}^+\} \cup \{\text{piv.}^-\}), \qquad (6) \end{aligned}$$

whence, by (2),

$$Cov_p(M_+ + M_-, 1_{A_n}) = p(1-p)\sum_{v\in V_n} P_p(\{v \text{ pivotal}\}) - p(1-p)\sum_{v\in V_n} P_p(\{v \text{ pivotal}^+\}).$$

Notice that $P_p(\text{piv.}^+) = P_p(\text{piv.}^-)$ due to the symmetry of $D = B(n)$ against y-axis. Thus, by general Russo's formula (1), we derive the following

Theorem 1. *For $0 < p < 1$, $\frac{d}{dp}P_p(A_n) = \sum_{v\in V_n} P_p(\{v \textit{ pivotal}\}) - \sum_{v\in V_n} P_p(\{v \textit{ pivotal}^+\})$.*

We will denote by $N(E)[\omega]$, $N^+(E)[\omega]$ and $N^-(E)[\omega]$ the number of pivotal, pivotal$^+$ and pivotal$^-$ vertices at a configuration $\omega \in \Omega_V$ corresponding to an event $E \subset \Omega_V$. Now we can rewrite the above formula as

$$\frac{d}{dp}P_p(A_n) = \mathbb{E}_p[N(A_n)] - \mathbb{E}_p[N^+(A_n)].$$

We notice that the theorem above holds for $p = 0$ and $p = 1$, and the proof works for all regions $D \subset \mathbb{R}^2$ symmetrical about y-axis, and all events with the property that there are only pivotal, pivotal$^+$, pivotal$^-$ and indifferent vertices inside D. When D is not symmetrical about y-axis, the analogue of (6) holds for the event $A_D \equiv \{\rho \leftrightarrow \partial D\}$ in Ω_V, and the corresponding formula can be written as follows

$$\frac{d}{dp}P_p(A_D) = \mathbb{E}_p[N(A_D)] - \frac{1}{2}\mathbb{E}_p[N^+(A_D) + N^-(A_D)].$$

4 Application to Recurrence Hypothesis.

Consider all possible configurations of mirrors under A_n^c. Let v be a vertex on the loop ρ. Suppose v is connected via another loop to $\partial B(n)$. Then if v contains no mirror, v must be pivotal for A_n. If v contains a NW mirror (NE mirror), then v must be a pivotal$^+$ (pivotal$^-$) vertex. Moreover, this classifies all possible pivotal, pivotal$^+$ and pivotal$^-$ vertices under A_n^c.

As a consequence, we can relate the expected number of pivotal vertices when A_n^c holds to the expected number of pivotal vertices when A_n does:

$$\begin{aligned}\mathbb{E}_p[N(A_n)\mathbf{1}_{A_n^c}] &= \sum_{v\in V_n} P_p[\{v\in\rho\}\{v\leftrightarrow\partial B(n)\}\{\omega(v)=0\}A_n^c]\\ &= \sum_{v\in V_n}\frac{1-p}{p}P_p[\{v \text{ is pivotal}\}A_n]\\ &= \frac{1-p}{p}\mathbb{E}_p[N(A_n)\mathbf{1}_{A_n}].\end{aligned}$$

The coefficient of $\frac{1-p}{p}$ appears when one removes a mirror from a vertex. Similarly we relate the number of pivotal$^+$ vertices when A_n holds, to the number of pivotal$^+$ vertices when A_n doesn't hold:

$$\begin{aligned}\mathbb{E}_p[N^+(A_n)\mathbf{1}_{A_n^c}] &= \sum_{v\in V_n} P_p[\{v\in\rho\}\{v\leftrightarrow\partial B(n)\}\{\omega(v)=-1\}A_n^c]\\ &= \sum_{v\in V_n}\frac{\frac{p}{2}}{1-\frac{p}{2}}P_p[\{v \text{ is pivotal}^+\}A_n]\\ &= \frac{p}{2-p}\mathbb{E}_p[N^+(A_n)\mathbf{1}_{A_n}],\end{aligned}$$

where the coefficient $\frac{1-\frac{p}{2}}{\frac{p}{2}}$ appears when one removes a NW mirror from a given vertex. So,

$$\mathbb{E}_p[N(A_n)\mathbf{1}_{A_n^c}] = \frac{1-p}{p}\mathbb{E}_p[N(A_n)\mathbf{1}_{A_n}],$$

and

$$\mathbb{E}_p[N^+(A_n)\mathbf{1}_{A_n^c}] = \frac{p}{2-p}\mathbb{E}_p[N^+(A_n)\mathbf{1}_{A_n}].$$

Now, from Theorem 1 it follows that

$$\frac{d}{dp}P_p(A_n) = (1+\frac{1-p}{p})\mathbb{E}_p[N(A_n)\mathbf{1}_{A_n}] - (1+\frac{p}{2-p})\mathbb{E}_p[N^+(A_n)\mathbf{1}_{A_n}].$$

Hence,

Theorem 2.

$$\frac{d}{dp}P_p[A_n] = \left(\frac{1}{p}\mathbb{E}_p[N(A_n)\mid A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)\mid A_n]\right)P_p[A_n].$$

Similarly,

Theorem 3. *For all $p > 0$,*

$$\frac{d}{dp}P_p[A_n] = \frac{1}{1-p}\mathbb{E}_p[N(A_n)\mathbf{1}_{A_n^c}] - \frac{2}{p}\mathbb{E}_p[N^+(A_n)\mathbf{1}_{A_n^c}].$$

5 More Generality.

In this section we will generalize the formula of the Theorem 1 to make it work for even greater class of LLG models than the ones studied in the preceding sections of this manuscript. The following modification of the original model adds an extra new parameter and an extra new dimension: we fix $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. We now assign the probabilities as follows. The probability of placing a NW mirror will be αp, a NE mirror will be βp and of placing no mirror will be the same $1-p$. The sample space, all the settings and all the definitions from the preceding sections continue to hold. The new probability measure is denoted by $P_p^{\alpha,\beta}(\cdot)$, and the non-localization probability is $\eta_n^{\alpha,\beta}(p) = P_p^{\alpha,\beta}(A_n)$ for all n. We immediately observe that $\eta_n^{1,0}(p) = \eta_n^{0,1}(p) = 1$ for all p and n. In these new settings the Theorem 1 is restated as follows:

Theorem 4. $\frac{d}{dp}\eta_n^{\alpha,\beta}(p) = \mathbb{E}_p[N(A_n)] - \alpha\mathbb{E}_p[N^+(A_n)] - \beta\mathbb{E}_p[N^-(A_n)]$.

Proof. We will modify the proof of Theorem 1. By analogy to (1),

$$\frac{d}{dp}P_p(E) \quad = \quad \frac{1}{p(1-p)}\mathbb{E}_p[\{M_+ + M_- - p|V|\}\mathbf{1}_E],$$

and identities (2) and (3) are valid unchanged. The equalities (4) and (5) can now be rewritten as

$$P_p(A_n \cap \{\text{not piv.}\}) = (1-\alpha p)P_p(\{\text{piv.}^+\}) + (1-\beta p)P_p(\{\text{piv.}^-\}) + P_p(\{\text{indif.}\} \cap A_n)$$

and

$$P_p(J_v \cap A_n \cap \{\text{not piv.}\}) = \beta p P_p(\{\text{piv.}^+\}) + \alpha p P_p(\{\text{piv.}^-\}) + p P_p(\{\text{indif.}\} \cap A_n)$$

The corresponding cancelations follow. □

6 Duality of pivotal points.

We recall that if A_n^c holds (i.e. $\omega \in A_n^c$), a vertex v is pivotal only when it contains no mirror ($\omega(v) = 0$), the cycle ρ goes through v ($v \in \rho$) and there exists a cycle γ leaving the region $D = B(n)$ ($\gamma \leftrightarrow \partial D$) such that $v \in \gamma$. So $v \in \rho \cap \gamma$. Now, since in that case $\omega \in A_n^c$, ρ must be contained inside the region D ($\rho \subset D$). Hence γ must cross the cycle ρ at an even number of vertices inside D containing no mirrors. Thus, the set of crossing intersections of the two cycles $\rho \cap \gamma \cap \{u : \omega(u) = 0\}$ inside D must contain an even number of vertices.

All of those vertices are also pivotal. We will call them **dual** to v. Similarly, vertices in $\rho \cap \gamma \cap \{u : \omega(u) \neq 0\}$ will be called **dual**$^{\pm}$.

We notice that if v and v^* are vertices dual to each other under A_n^c, then there must be a portion of γ contained inside D that connects v and v^*. Now, we can place the mirrors at v and v^* so that the portion of ρ connecting v to v^* (but avoiding the edge between (0,0) and (0,1)) is replaced by the above portion of γ cycle connecting v and v^*. After the replacement, each of the two pivotal vertices v and v^* becomes either pivotal$^-$ or pivotal$^+$. Moreover they will become dual$^{\pm}$. We use this property of dual vertices to relate $\mathbb{E}_p[N(A_n) \,|A_n^c]$ to $\mathbb{E}_p[N^+(A_n) \,|A_n^c]$ as follows. We have just shown that for a given dual couple of pivotal vertices v and v^* under A_n^c,

$$\left[\frac{p}{2(1-p)}\right]^2 P_p[v, v^* \text{ are dual } |A_n^c] = P_p[v, v^* \text{ are dual}^{\pm} \, |A_n^c].$$

Now, in case when A_n^c occurs, the two vertices v_1 and v_2 of ρ are dual or dual$^{\pm}$ if they belong to the same γ that starts with some edge e_1 that connects ∂D to D and ends with an edge $e_2 \neq e_1$ that connects D to ∂D. In this situation, we say that $\gamma = \gamma(e_1, e_2)$. Let $N_\gamma(A_n)$, $N_\gamma^-(A_n)$ and $N_\gamma^+(A_n)$ denote respectively the number of pivotal, pivotal$^-$ and pivotal$^+$ vertices that belong to γ. Also let $N_\gamma^{\pm}(A_n) = N_\gamma^-(A_n) + N_\gamma^+(A_n)$. Then the above trick of switching form two dual vertices to two dual$^{\pm}$ vertices works producing

$$\frac{p}{2(1-p)} E\Big[\binom{N_{\gamma(e_1,e_2)}(A_n)}{2} \; |A_n^c\Big] = E\Big[\binom{N^{\pm}_{\gamma(e_1,e_2)}(A_n)}{2} \; |A_n^c\Big].$$

Observe that if one denotes by $E(\partial D, D)$ all the edges that connect ∂D to D, then

$$\frac{1}{2} \sum_{e_1,e_2 \in E(\partial D, D):\ e_1 \neq e_2} N_{\gamma(e_1,e_2)}(A_n) = N(A_n),$$

and similarly

$$\frac{1}{2} \sum_{e_1,e_2 \in E(\partial D, D):\ e_1 \neq e_2} N^{\pm}_{\gamma(e_1,e_2)}(A_n) = N^{\pm}(A_n).$$

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