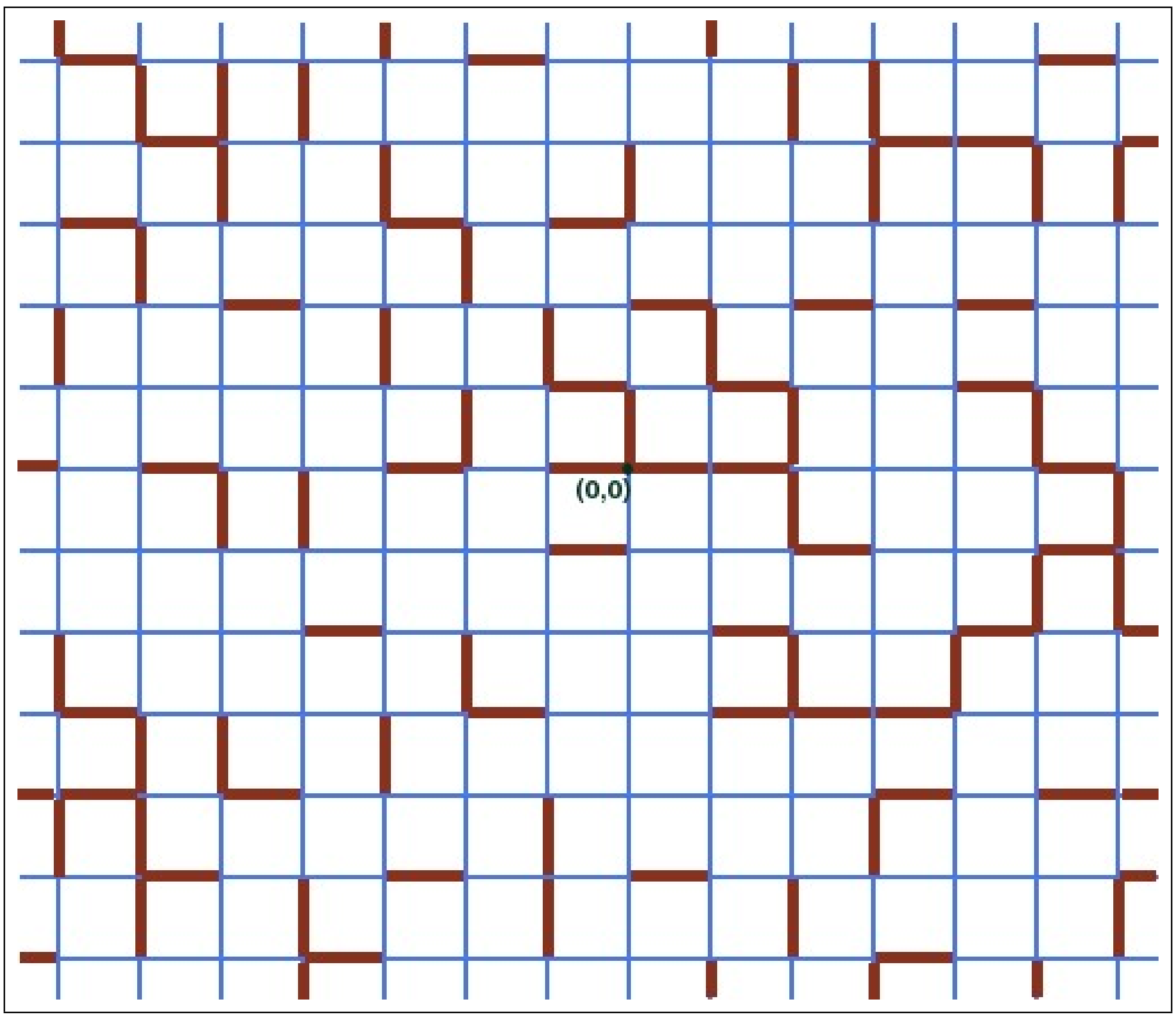
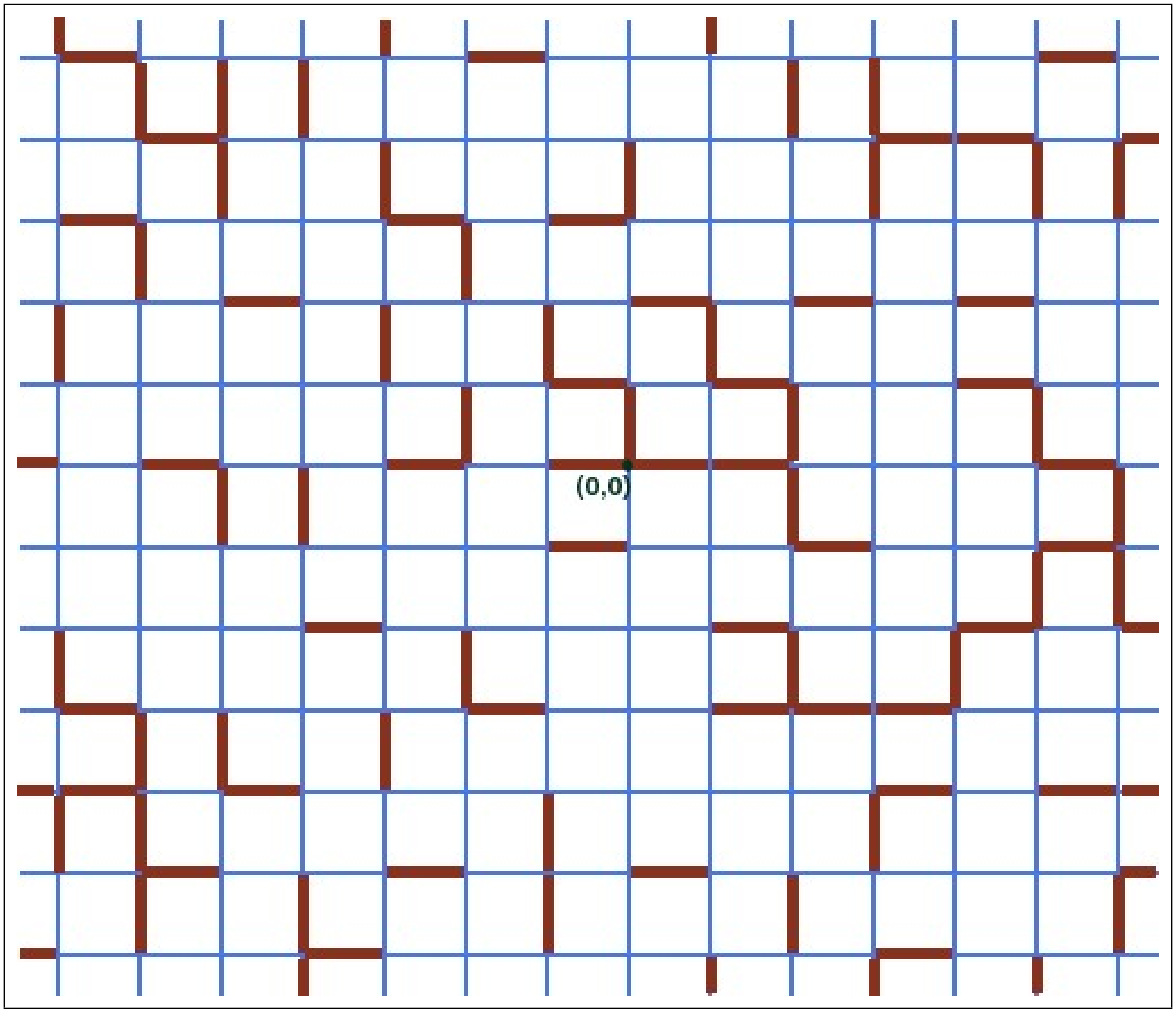
Critical percolation and Lorentz lattice gas model: an expository talk

Yevgeniy Kovchegov

Percolation For each edge of the d-dimensional square lattice $\mathbb{Z}^d$ in turn, we declare the edge *open* with probability p and *closed* with probability $1 - p$, independently of all other edges.





If we delete the closed edges, we are left with a random subgraph of $\mathbb{Z}^d$. A connected component of the subgraph is called a "cluster", and the number of edges in a cluster is the "size" of the cluster.

$$\theta(p) \equiv P_p[0 \leftrightarrow \infty]$$

Obviously $\theta(0) = 0$ and $\theta(1) = 1$.

$\exists$ critical $0 < p_c < 1$ such that

• $\theta(p) = 0$ if $p < p_c \Leftrightarrow$ *subcritical* model
and
• $\theta(p) > 0$ if $p > p_c \Leftrightarrow$ *supercritical* model

Standard reference:
• "*Percolation.*" by G.R.Grimmett (1999)

Increasing events.

Configurations: $\omega = \{\omega(e) : e \in \mathbb{E}^d\}$, where

$\omega(e) = 1 \Leftrightarrow e$ is open;

$\omega(e) = 0 \Leftrightarrow e$ is closed.

Sample space: $\Omega = \{0, 1\}^{\mathbb{E}^d}$.

Partial Order: we say $\omega_1 \leq \omega_2$ if and only if $\omega_1(e) \leq \omega_2(e)$ for all $e \in \mathbb{E}^d$.

Def. A random variable X is **increasing** if

$$X(\omega_1) \leq X(\omega_2) \text{ whenever } \omega_1 \leq \omega_2$$

Def. An event A is **increasing** if its indicator variable 1_A, given by $1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$, is increasing.

Russo's formula. Given a configuration $\omega \in \Omega$ and an event A, an edge e is **pivotal** if changing

$$\omega(e) \to 1 - \omega(e)$$

will determine weather the configuration is in A or A^C; i.e. either

$$\omega \in A, \quad \{\text{changed } \omega\} \in A^C$$

or

$$\omega \in A^C, \quad \{\text{changed } \omega\} \in A.$$

Suppose A depends on **finitely** many edges, and let $N_A :=$ # of pivotal edges .

Thm. (Russo's formula) If A is increasing, then

$$\boxed{\tfrac{d}{dp} P_p(A) = E_p[N_A]}$$

Exponential decay: M.V.Menshikov (1986), enhanced - M.Aizenman and D.J.Barsky (1987)

Thm. If $p < p_c$ then $\exists\ \psi(p) > 0$ such that

$$P_p[0 \leftrightarrow \partial B(n)] < e^{-n\psi(p)} \text{ for all } n.$$

Proof outline: let $K_n = \{0 \leftrightarrow \partial B(n)$, then by Russo's formula

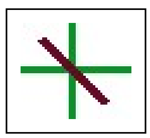
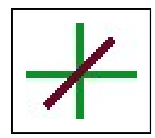
$$\frac{d}{dp} P_p(K_n) = E_p[N_{K_n}] = \frac{1}{p} E_p[N_{K_n}|K_n] P_p(K_n).$$

Integrating $\frac{d}{dp} P_p(K_n)/P_p(K_n)$, get

$$P_a(K_n) = P_b(K_n) e^{-\int_a^b \frac{1}{p} E_p[N_{K_n}|K_n]dp},$$

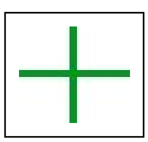
where $0 \le a < b \le 1$. It can be showed that $E_p[N_{K_n}|K_n]$ growth roughly linearly in n when $p < p_c$.

2D Lorentz Lattice Gas (LLG) model. We place two-sided mirrors on the vertices of $\mathbb{Z}^2$ according to the following law: for $0 \leq p \leq 1$, place

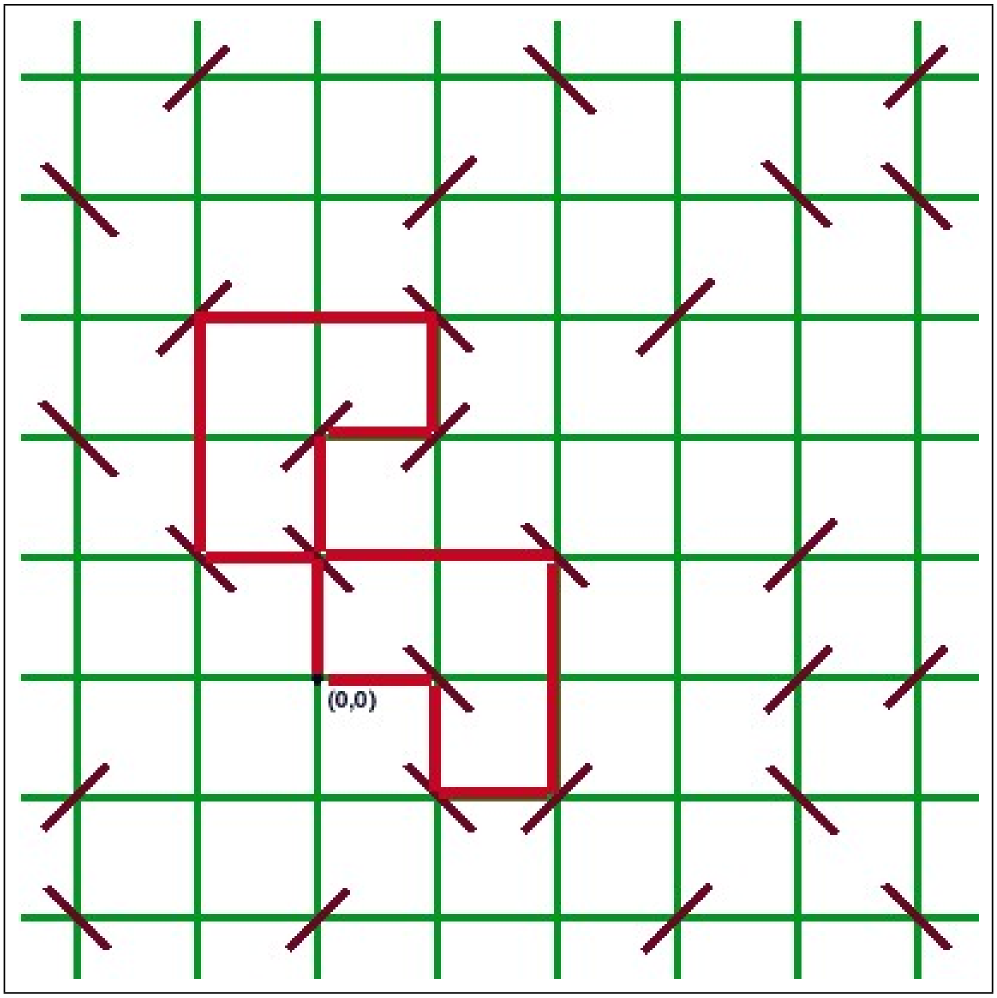
a NW mirror  or NE mirror 

with probability $= \frac{p}{2}$ each.

Place NO MIRROR with probability $1 - p$

 with probability $= 1 - p$.

$\eta(p) = P_p$(the light ray returns to origin).



$\eta(0) = 0$

Grimmett: $\eta(1) = 1$, using $p_c = \frac{1}{2}$ for 2D bond percolation model.

Russo's formula adapted for LLG model.

Here $A_n = \{$ light cycle reaches $\partial B(n)\}$ is not increasing.

Need: a substitute property for "increasing".

Consider $V = V(n)$ - the set of vertices inside the box $B(n)$.

Let $\Omega_V \equiv \{-1, 0, 1\}^V$ be the states space:

"-1" corresponds to NW mirror,

"1" to NE mirror

"0" to placing no mirror at a vertex.

For a vertex $v \in V$, we let

$$\omega_v^{+}(u) = \begin{cases} \omega(u) & \text{if } u \neq v, \\ 1 & \text{if } u = v; \end{cases}$$

$$\omega_v^{-}(u) = \begin{cases} \omega(u) & \text{if } u \neq v, \\ -1 & \text{if } u = v; \end{cases}$$

$$\omega_v^{0}(u) = \begin{cases} \omega(u) & \text{if } u \neq v, \\ 0 & \text{if } u = v. \end{cases}$$

Types of "pivotal" vertices: For an event $E \subset \Omega_V$, we say that a vertex $v \in V$ is

pivotal if
$$\begin{cases} \omega_v^+ \in E, \\ \omega_v^0 \notin E, \\ \omega_v^- \in E. \end{cases}$$

$\mathbf{pivotal}^+$ if
$$\begin{cases} \omega_v^+ \in E, \\ \omega_v^0 \in E, \\ \omega_v^- \notin E; \end{cases}$$

$\mathbf{pivotal}^-$ if
$$\begin{cases} \omega_v^+ \notin E, \\ \omega_v^0 \in E, \\ \omega_v^- \in E. \end{cases}$$

and $v \in V$ is **indifferent** if either

$$\begin{cases} \omega_v^+ \in E, \\ \omega_v^0 \in E, \\ \omega_v^- \in E. \end{cases} \quad \text{or} \quad \begin{cases} \omega_v^+ \notin E, \\ \omega_v^0 \notin E, \\ \omega_v^- \notin E. \end{cases}$$

Important Observation: we notice that in case of the event A_n there can be only pivotal, pivotal$^+$, pivotal$^-$ and indifferent vertices.

Thm. (K.) For $0 < p < 1$,

$$\frac{d}{dp}P_p(A_n) = \sum_{v \in V_n} P_p(\{v \text{ pivotal}\}) - \sum_{v \in V_n} P_p(\{v \text{ pivotal}^+\})$$

that is

$$\boxed{\frac{d}{dp}P_p(A_n) = \mathbb{E}_p[N(A_n)] - \mathbb{E}_p[N^+(A_n)].}$$