Multidimensional Lambert-Euler inversion and vector-multiplicative coalescent processes

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Lambert-Euler inversion.

In his 1783 work L. Euler considered the following transcendental equation entailed from 1758 work of J. H. Lambert

$$x^{\alpha} - x^{\beta} = (\alpha - \beta) v x^{\alpha+\beta}. \tag{1}$$

Letting $\alpha \to \beta$ in (1), Euler obtained

$$\ln x = v x^{\beta}. \tag{2}$$

Next, Euler set $y = x^{\beta}$ and $u = \beta v$ in (2), obtaining

$$\frac{\ln y}{y} = u. \tag{3}$$

Letting $y = e^{w}$, equation (3) yields

$$w e^{-w} = u. \tag{4}$$

Equation (4) gave rise to the Lambert W function, and in particular the function $W_0(x)$ for $-e^{-1} \leq x < 0$.

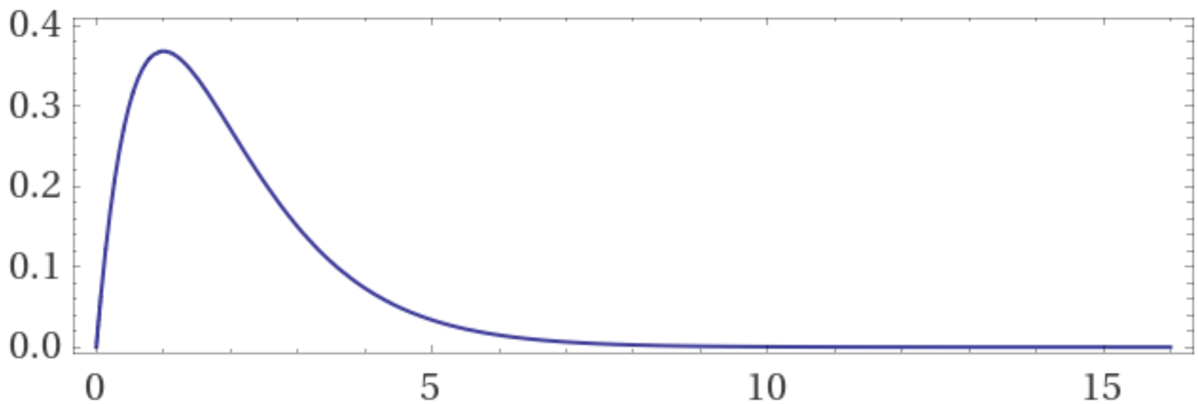
Lambert-Euler inversion.

$$we^{-w} = u$$

Denote $R_0 = (0,1)$, $\overline{R}_0 = (0,1]$, and $R_1 = (1,\infty)$.

For $0 < u < e^{-1}$ there are exactly two solutions: one in R_0 and one in R_1. For $u = e^{-1}$, $w = 1$ is the only solution.

For $0 < u \leq e^{-1}$ there is exactly one solution w in $\overline{R}_0$.

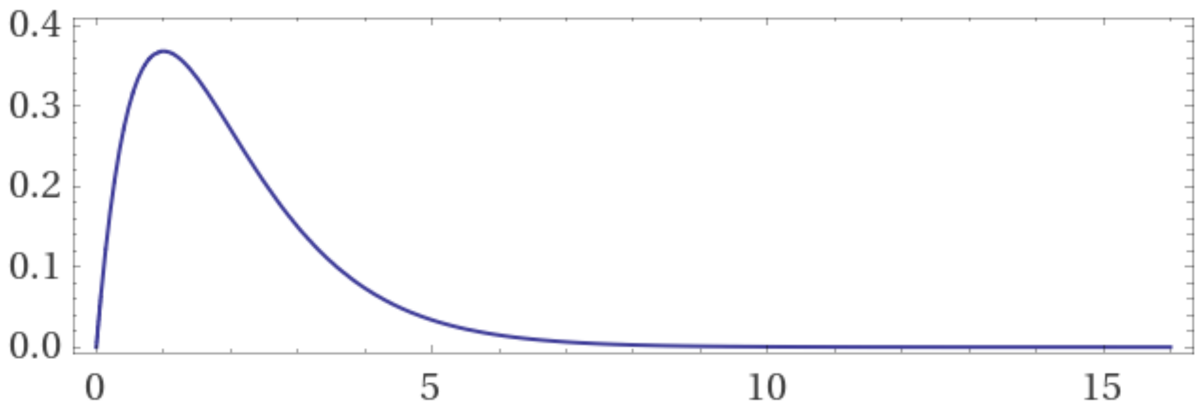


Lambert-Euler inversion yields

$$x(t) := \min\{x > 0 \ : \ xe^{-x} = te^{-t}\}, \quad t > 0, \quad \text{with range } \overline{R}_0 = (0, 1].$$

In 1960, $x(t)$ was used by P. Erdős and A. Rényi in random graphs, showing formation of giant cluster.

In 1962, $x(t)$ was used by J. B. McLeod in the analysis of Smoluchowski coagulation equations with multiplicative kernel, observing gelation phenomenon.



Multidimensional Lambert-Euler inversion.

For a given nonnegative irreducible symmetric matrix $V \in \mathbb{R}^{k\times k}$ and a vector $\mathbf{u} \in (0,\infty)^k$, we show that, if the systems

$$y_j \exp\left\{-\mathbf{e}_j^\top V\mathbf{y}\right\} = u_j \qquad j = 1,\ldots,k,$$

has at least one solution, it must have a minimal solution $\mathbf{y}^*$, where the minimum is achieved in all coordinates y_j simultaneously.

Moreover, such $\mathbf{y}^*$ is the unique solution satisfying $\rho\left(VD[y_j^*]\right) \le 1$, where $D[y_j^*] = \mathrm{diag}(y_j^*)$ is the diagonal matrix with entries y_j^* and ρ denotes the spectral radius.

Multidimensional Lambert-Euler inversion.

Notations:

- $V \in \mathbb{R}^{k \times k}$ is a nonnegative irreducible symmetric matrix;
- For $\mathbf{x} \in \mathbb{R}^k$ with coordinates x_j, $D[x_j] = \mathrm{diag}(x_j)$;
- $\rho(M)$ denotes the spectral radius of matrix M;
- For a given vector $\mathbf{z} \in (0, \infty)^k$, consider

$$R_0 = \left\{\mathbf{z} \in (0, \infty)^k : \rho\left(VD[z_j]\right) < 1\right\}$$

$$\overline{R}_0 = \left\{\mathbf{z} \in (0, \infty)^k : \rho\left(VD[z_j]\right) \leq 1\right\}$$

and

$$R_1 = \left\{\mathbf{z} \in (0, \infty)^k : \rho\left(VD[z_j]\right) > 1\right\}$$

Multidimensional Lambert-Euler inversion.

$$\overline{R}_0 = \left\{ \mathbf{z} \in (0,\infty)^k : \rho\left(VD[z_j]\right) \le 1 \right\}$$

$$R_1 = \left\{ \mathbf{z} \in (0,\infty)^k : \rho\left(VD[z_j]\right) > 1 \right\}$$

Theorem (YK and P. T. Otto, 2021). For any given $\mathbf{z} \in (0,\infty)^k$, there exists unique $\mathbf{y} \in \overline{R}_0$ such that

$$y_j \exp\left\{ -\mathbf{e}_j^\mathsf{T} V \mathbf{y} \right\} = z_j \exp\left\{ -\mathbf{e}_j^\mathsf{T} V \mathbf{z} \right\} \qquad j = 1, \ldots, k.$$

Moreover, if $\mathbf{z} \in \overline{R}_0$, then $\mathbf{y} = \mathbf{z}$. If $\mathbf{z} \in R_1$, then $\mathbf{y} < \mathbf{z}$ ($y_i < z_i \ \forall i$), i.e., $\mathbf{y}$ is the smallest solution.

Vector-Multiplicative Coalescent Process.

Consider k types of particles, $1, \ldots, k$.

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)^{\mathsf{T}}$ be a column vector, where $\alpha_i \geq 0$ are reals satisfying $\sum\limits_{i=1}^{k} \alpha_i > 0$.

Let matrix $V = \big(v_{i,j}\big)$ be nonnegative, irreducible, and symmetric.

Vector-multiplicative coalescent process:

- The process begins with $\sum\limits_{i=1}^{k} \alpha_i n + o(\sqrt{n})$ singletons distributed between the k types so that for each $i = 1, \ldots, k$, there are $\alpha_i n + o(\sqrt{n})$ particles of type i.

- A particle of type i **bonds** with a particle of type j with the intensity (rate) $v_{i,j}/n$. The bonds are formed independently.

Vector-Multiplicative Coalescent Process.

• Each cluster of weight $\mathbf{x}$ bonds together $x_1, \ldots, x_k$ particles of corresponding types $1, \ldots, k$.

• Clusters with weight vectors $\mathbf{x}$ and $\mathbf{y}$ would coalesce into a cluster of weight $\mathbf{x}+\mathbf{y}$ with rate $K(\mathbf{x},\mathbf{y})/n$, where

$$K(\mathbf{x},\mathbf{y}) = \mathbf{x}^{\mathsf{T}} V \mathbf{y}.$$

Hydrodynamic limit as $n \to \infty$:

For $\mathbf{x} = (x_1, x_2, \ldots, x_k)^{\mathsf{T}}$ let $\zeta_{\mathbf{x}}(t)$ denote the relative (to n) number of clusters bonding together particles into a cluster with x_i particles of type i for all i.

$$\frac{d}{dt}\zeta_{\mathbf{x}}(t) = -\zeta_{\mathbf{x}}(t)\big(\mathbf{x}^{\mathsf{T}} V \boldsymbol{\alpha}\big) + \frac{1}{2}\sum_{\mathbf{y},\mathbf{z}\,:\mathbf{y}+\mathbf{z}=\mathbf{x}} \big(\mathbf{y}^{\mathsf{T}} V \mathbf{z}\big)\zeta_{\mathbf{y}}(t)\,\zeta_{\mathbf{z}}(t), \quad \zeta_{\mathbf{x}}(0) = \sum_{i=1}^{k} \alpha_i \delta_{\mathbf{e}_i,\mathbf{x}}$$

Abel's binomial theorem: a variation.

$$y^{-1}(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^k (y+n-k)^{n-k-1}$$

Swap x with y, and k with $n-k$. Then,

$$x^{-1}(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-1} (y+n-k)^{n-k}$$

Add the two formulas together:

$$(x^{-1}+y^{-1})(x+y+n)^n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-1} (y+n-k)^{n-k-1} (x+y+n)$$

Thus,

$$(x^{-1}+y^{-1})(x+y+n)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-1} (y+n-k)^{n-k-1}$$

Abel's binomial theorem: a variation.

$$(x^{-1}+y^{-1})(x+y+n)^{n-1}=\sum_{k=0}^{n}\binom{n}{k}(x+k)^{k-1}(y+n-k)^{n-k-1}$$

Therefore,

$$(x^{-1}+y^{-1})(x+y+n)^{n-1}-x^{-1}(y+n)^{n-1}-y^{-1}(x+n)^{n-1}$$

$$=\sum_{k=1}^{n-1}\binom{n}{k}(x+k)^{k-1}(y+n-k)^{n-k-1}$$

Hence,

$$\sum_{k=1}^{n-1}\binom{n}{k}k^{k-1}(n-k)^{n-k-1}=\lim_{\substack{x\to 0\\ y\to 0}}\sum_{k=1}^{n-1}\binom{n}{k}(x+k)^{k-1}(y+n-k)^{n-k-1}$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x+y+n)^{n-1}-(y+n)^{n-1}}{x}+\lim_{\substack{x\to 0\\ y\to 0}}\frac{(x+y+n)^{n-1}-(x+n)^{n-1}}{y}$$

Abel's binomial theorem: a variation.

$$\sum_{k=1}^{n-1}\binom{n}{k}k^{k-1}(n-k)^{n-k-1} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(x+y+n)^{n-1}-(y+n)^{n-1}}{x}$$

$$+\lim_{\substack{x\to 0\\ y\to 0}} \frac{(x+y+n)^{n-1}-(x+n)^{n-1}}{y}$$

$$= 2\lim_{h\to 0}\frac{(h+n)^{n-1}-n^{n-1}}{h} = 2\frac{d}{dx}x^{n-1}\Big|_{x=n} = 2(n-1)n^{n-2}.$$

Identity $\sum_{k=1}^{n-1}\binom{n}{k}k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}$ rewrites as

$$n^{n-2} = \frac{1}{2(n-1)}\sum_{k=1}^{n-1}\binom{n}{k}k(n-k)\,k^{k-2}\,(n-k)^{n-k-2} \quad \text{which}\ldots$$

Minimal spanning trees.

Identity $\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}$ rewrites as

$$n^{n-2} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)\, k^{k-2}\, (n-k)^{n-k-2}$$

which has everything to do with spanning trees!

In particular,

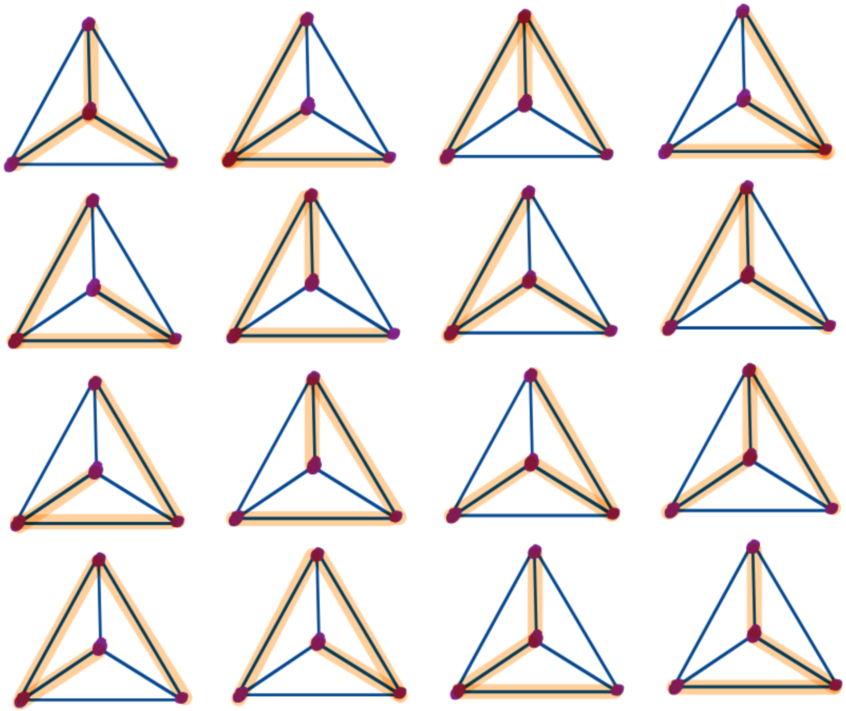
T_n = the number of spanning trees in a complete graph K_n satisfies the following recursion

$$T_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)\, T_k\, T_{n-k}$$

Thus, $T_n = n^{n-2}$

Minimal spanning trees.

$$T_n = n^{n-2} \quad \Rightarrow \quad T_4 = 4^2 = 16$$



Minimal spanning trees.

Abel's binomial theorem yields

$$n^{n-2} = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)\, k^{k-2}\, (n-k)^{n-k-2}$$

Equivalently,

$T_n =$ the number of spanning trees in a complete graph K_n satisfies

$$T_n = \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)\, T_k\, T_{n-k}$$

implying $T_n = n^{n-2}$.

Question: Is there a useful generalization?

Minimal spanning trees.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_k)^\mathsf{T}$ be a column vector, where $x_i \geq 0$ are integers satisfying $\sum_{i=1}^{k} x_i > 0$.

Consider a graph $K_{\mathbf{x}}(V)$ equipped with edge weights:

• $K_{\mathbf{x}}(V)$ is a complete graph with $\sum_{i=1}^{k} x_i$ vertices partitioned into k kinds;

• For each $i = 1, \ldots, k$, $K_{\mathbf{x}}(V)$ has x_i vertices of kind i;

• Let $v_{i,j} = v_{j,i}$ be the weight of each edge connecting a vertex i-th kind to a vertex of j-th kind, and let $V = \left(v_{i,j}\right)$ be the $k \times k$ matrix of weights.

Minimal spanning trees.

Consider a graph $K_{\mathrm{x}}(V)$ equipped with edge weights $V = (v_{i,j})$.

If $\mathcal{T}$ is a spanning tree of $K_{\mathrm{x}}(V)$, then the weight of $\mathcal{T}$ is the **product** of the weights of all of its edges.

Let T_{x} denote the weighted spanning tree enumerator of $K_{\mathrm{x}}(V)$, i.e., T_{x} is the sum of weights of all spanning trees of $K_{\mathrm{x}}(V)$.

Theorem (YK and P. T. Otto, 2021).

$$T_{\mathrm{x}} = \frac{1}{2\left(\sum_{i=1}^{k} x_i - 1\right)} \sum_{\mathrm{y},\mathrm{z}:\mathrm{y}+\mathrm{z}=\mathrm{x}} \binom{x_1}{y_1}\binom{x_2}{y_2}\cdots\binom{x_k}{y_k}\left(\mathbf{y}^{\mathsf{T}} V \mathbf{z}\right) T_{\mathrm{y}}\, T_{\mathrm{z}}$$

Minimal spanning trees.

Consider a graph $K_{\mathbf{x}}(V)$ equipped with edge weights $V = \left(v_{i,j}\right)$. Let $T_{\mathbf{x}}$ denote the weighted spanning tree enumerator of $K_{\mathbf{x}}(V)$.

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Example. Let $k = 1$, $\mathbf{x} = x_1 = n$, and $V = v_{1,1} = 1$. Then,

$T_n =$ the number of spanning trees in a complete graph K_n

and

$$T_n = \frac{1}{2(n-1)} \sum_{m=1}^{n-1} \binom{n}{m} m(n-m)\, T_m\, T_{n-m}$$

Question: Is there a general expression for $T_{\mathrm{x}} = T_{\mathrm{x}}(V)$?

Consider a complete graph K_k consisting of vertices $\{1, \ldots, k\}$ with weights $w_{i,j} = w_{j,i} \geq 0$ assigned to its edges $[i, j]$, let the weight $W(\mathcal{T})$ of a spanning tree $\mathcal{T}$ be the product of the weights of all of its edges.

Denote $\tau(K_k, w_{i,j}) = \sum_{\mathcal{T}} W(\mathcal{T})$ denote the weighted spanning tree enumerator, i.e., the sum of weights of all spanning trees in K_k.

Theorem (YK and P. T. Otto, 2021).

$$T_{\mathbf{x}}(V) = \frac{\tau(K_k, x_i x_j v_{i,j})}{\mathbf{x}^{\mathbf{1}}} (V\mathbf{x})^{\mathbf{x}-\mathbf{1}}$$

Notation: for vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^k$ let $\mathbf{a}^{\mathbf{b}} = a_1^{b_1} a_2^{b_2} \ldots a_k^{b_k}$

Also, $\mathbf{1} = (1, \ldots, 1)^{\mathsf{T}}$ and $\mathbf{x}^{\mathbf{1}} = \prod_{i=1}^{k} x_i$.

Vector-Multiplicative Coalescent Process.

$$\frac{d}{dt}\zeta_{\mathbf{x}}(t) = -\zeta_{\mathbf{x}}(t)\left(\mathbf{x}^{\mathsf{T}}V\boldsymbol{\alpha}\right) + \frac{1}{2}\sum_{\mathbf{y},\mathbf{z}\,:\mathbf{y}+\mathbf{z}=\mathbf{x}}\left(\mathbf{y}^{\mathsf{T}}V\mathbf{z}\right)\zeta_{\mathbf{y}}(t)\,\zeta_{\mathbf{z}}(t), \quad \zeta_{\mathbf{x}}(0) = \sum_{i=1}^{k}\alpha_i\delta_{\mathbf{e}_i,\mathbf{x}}$$

Solution: Plug in

$$\zeta_{\mathbf{x}}(t) = \boldsymbol{\alpha}^{\mathbf{x}}\frac{T_{\mathbf{x}}}{\mathbf{x}!}\,e^{-(\mathbf{x}^{\mathsf{T}}V\boldsymbol{\alpha})t}\;t^{x_1+\ldots+x_k-1}, \qquad \text{where } \mathbf{x}! = \prod_{i=1}^{k} x_i!$$

obtaining

$$T_{\mathbf{x}} = \frac{1}{2\left(\sum_{i=1}^{k} x_i - 1\right)}\sum_{\mathbf{y},\mathbf{z}:\mathbf{y}+\mathbf{z}=\mathbf{x}}\binom{x_1}{y_1}\binom{x_2}{y_2}\cdots\binom{x_k}{y_k}\left(\mathbf{y}^{\mathsf{T}}V\mathbf{z}\right)T_{\mathbf{y}}\,T_{\mathbf{z}}$$

Thus,

$$T_{\mathbf{x}}(V) = \frac{\tau(K_k, x_i x_j v_{i,j})}{\mathbf{x}^{\mathbf{1}}}(V\mathbf{x})^{\mathbf{x}-\mathbf{1}}$$

Vector-Multiplicative Coalescent Process.

$$\frac{d}{dt}\zeta_{\mathbf{x}}(t) = -\zeta_{\mathbf{x}}(t)\big(\mathbf{x}^{\mathsf{T}}V\boldsymbol{\alpha}\big) + \frac{1}{2}\sum_{\mathbf{y},\mathbf{z}\,:\mathbf{y}+\mathbf{z}=\mathbf{x}} \big(\mathbf{y}^{\mathsf{T}}V\mathbf{z}\big)\zeta_{\mathbf{y}}(t)\,\zeta_{\mathbf{z}}(t), \quad \zeta_{\mathbf{x}}(0) = \sum_{i=1}^{k}\alpha_i\delta_{\mathbf{e}_i,\mathbf{x}}$$

Theorem (YK and P. T. Otto, 2021).

$$\zeta_{\mathbf{x}}(t) = \frac{1}{\mathbf{x}!}\boldsymbol{\alpha}^{\mathbf{x}}\frac{\tau(K_k, x_i x_j v_{i,j})}{\mathbf{x}^{\mathbf{1}}}(V\mathbf{x})^{\mathbf{x}-\mathbf{1}}\,e^{-(\mathbf{x}^{\mathsf{T}}V\boldsymbol{\alpha})t}\;t^{x_1+\ldots+x_k-1},$$

where $\mathbf{x}! = \prod_{i=1}^{k} x_i!$ and $\mathbf{a}^{\mathbf{b}} = a_1^{b_1}a_2^{b_2}\ldots a_k^{b_k}$.

- $\tau(K_k, w_{i,j})$ is computed via Kirchhoff's **Weighted Matrix-Tree Theorem**.

Gelation.

Consider the hydrodynamic limit $\zeta_{\mathbf{x}}(t)$.

Initial total mass: $\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(0)\mathbf{x} = \boldsymbol{\alpha}$.

The gelation time T_{gel} is the time after which the total mass

$$\sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)\mathbf{x}$$

begins to dissipate, i.e.,

$$T_{gel} = \inf \Big\{ t > 0 \ : \ \sum_{\mathbf{x}} \zeta_{\mathbf{x}}(t)\mathbf{x} < \boldsymbol{\alpha} \Big\}.$$

Multidimensional Lambert-Euler inversion is used for establishing gelation and finding the value of the gelation time

$$T_{gel} = \frac{1}{\rho(VD[\alpha_i])}.$$

Application in minimal spanning trees.

Let matrix $V = \left(v_{i,j}\right)$ and vector $\boldsymbol{\alpha}$ be as before.

Let $K_{\boldsymbol{\alpha}[n]}$ denote a graph with vertices divided into k partitions of respective sizes

$$\alpha_1[n] = \alpha_1 n + o(\sqrt{n}), \ \ldots, \ \alpha_k[n] = \alpha_k n + o(\sqrt{n}),$$

where, each vertex in the i-th partition is connected with each vertex in the j-th partition by an edge if and only if $v_{i,j} = v_{j,i} > 0$. Even within an i-th partition.

We equip $K_{\boldsymbol{\alpha}[n]}$ with edge lengths: each edge e connecting a vertex in the i-th partition with a vertex in the j-th partition has an associated random variable $\ell_e \sim \mathrm{Beta}\left(1, v_{i,j}\right)$, distributed on $(0,1)$ via the beta probability density function

$$f_{i,j}(x) = v_{i,j}(1-x)^{v_{i,j}-1}, \qquad 0 < x < 1.$$

Random variables ℓ_e are sampled independently.

Application in minimal spanning trees.

Let random variable L_n denote the length of the minimal spanning tree of $K_{\boldsymbol{\alpha}[n]}$.

Theorem (YK, P. T. Otto, and A. Yambartsev).

$$\lim_{n\to\infty} \mathbb{E}[L_n] = \sum_{\mathbf{x}} \int_0^\infty \zeta_{\mathbf{x}}(t)\, dt.$$

Theorem (YK and P. T. Otto, 2021).

$$\lim_{n\to\infty} \mathbb{E}[L_n] = \sum_{\mathbf{x}} \frac{(\mathbf{x}^\mathsf{T}\mathbf{1}-1)!}{\mathbf{x}!}\, \boldsymbol{\alpha}^{\mathbf{x}}\, \frac{\tau(K_k, x_i x_j v_{i,j})}{\mathbf{x}^{\mathbf{1}}} (V\mathbf{x})^{\mathbf{x}-\mathbf{1}} \left(\mathbf{x}^\mathsf{T} V \boldsymbol{\alpha}\right)^{-\mathbf{x}^\mathsf{T}\mathbf{1}}.$$

The time of formation of a giant component in $G(n,p)$

$$p_c \sim \frac{1}{n\,\rho(VD[\alpha_i])}.$$