On tree-pruning and prune-invariances in random binary rooted trees

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joint work with
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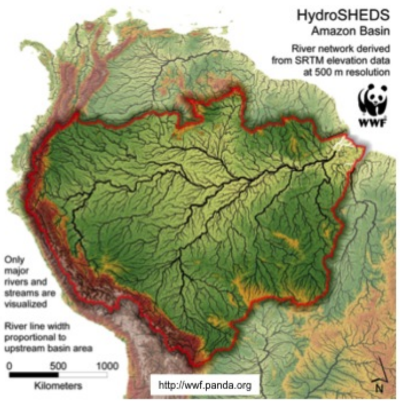
Horton-Strahler ordering.

The Horton-Strahler ordering assigns orders to the tree branches according to their relative importance in the hierarchy.



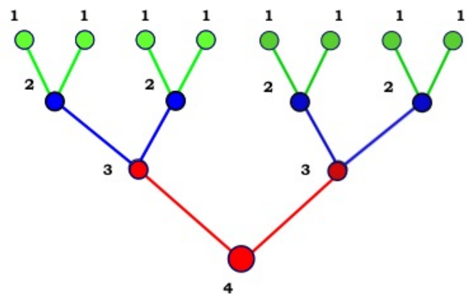
Introduced in hydrology in the 1950s to describe the dendritic structure of river networks (ranking river tributaries).

Horton-Strahler ordering.

Consider a **rooted tree** mod series reduction (removing degree two vertices).

- *Horton-Strahler orders* measure “importance” of tree branches within the hierarchy

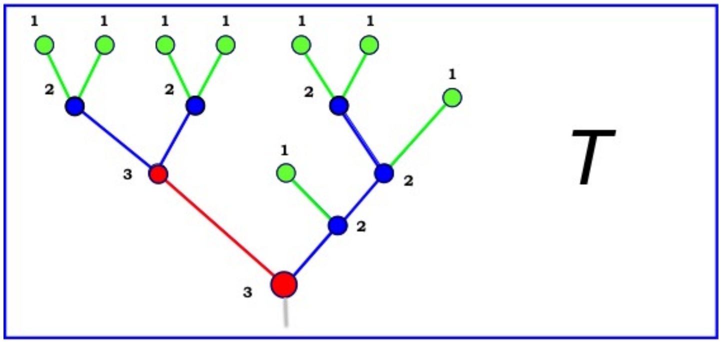
- In a perfect binary tree (all leaves having the same depth) the orders are proportional to depth



- How to assign orders in a non-perfect tree?

Horton-Strahler ordering via pruning.

- Pruning $\mathcal{R}(T)$ of a finite tree T cuts the leaves, followed by *series reduction*.
- A chain of the same order vertices with edges connecting to parent vertices is called branch.
- Branches cut at k-th pruning, $\mathcal{R}^{k-1}(T) \setminus \mathcal{R}^k(T)$, have order k, $k \geq 1$.
- N_k denotes the number of branches of order k in a finite tree T



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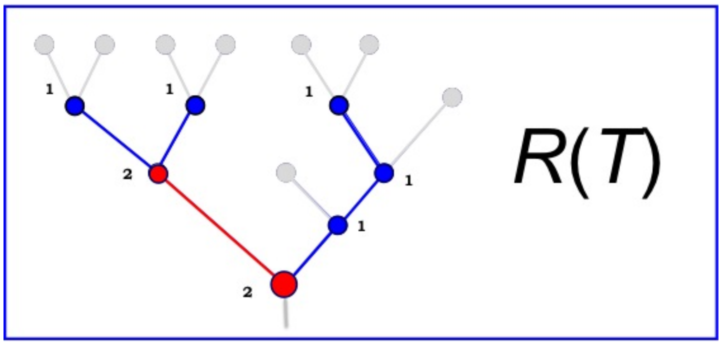

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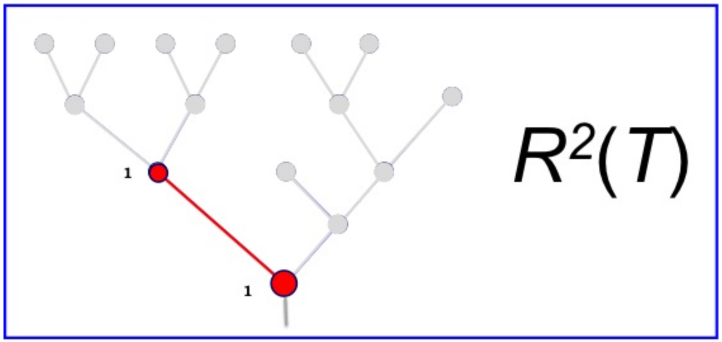

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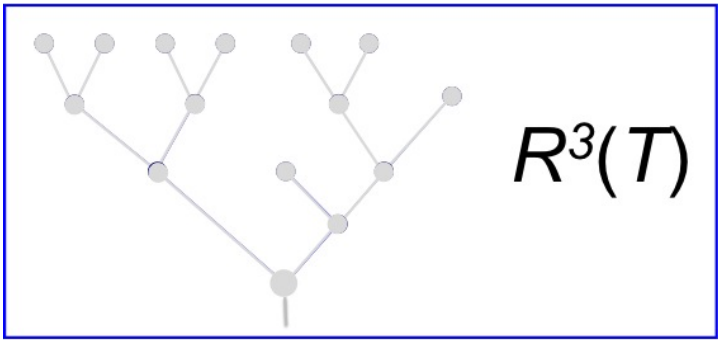

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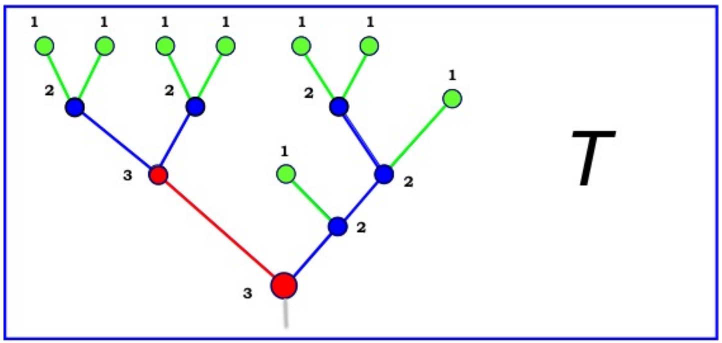
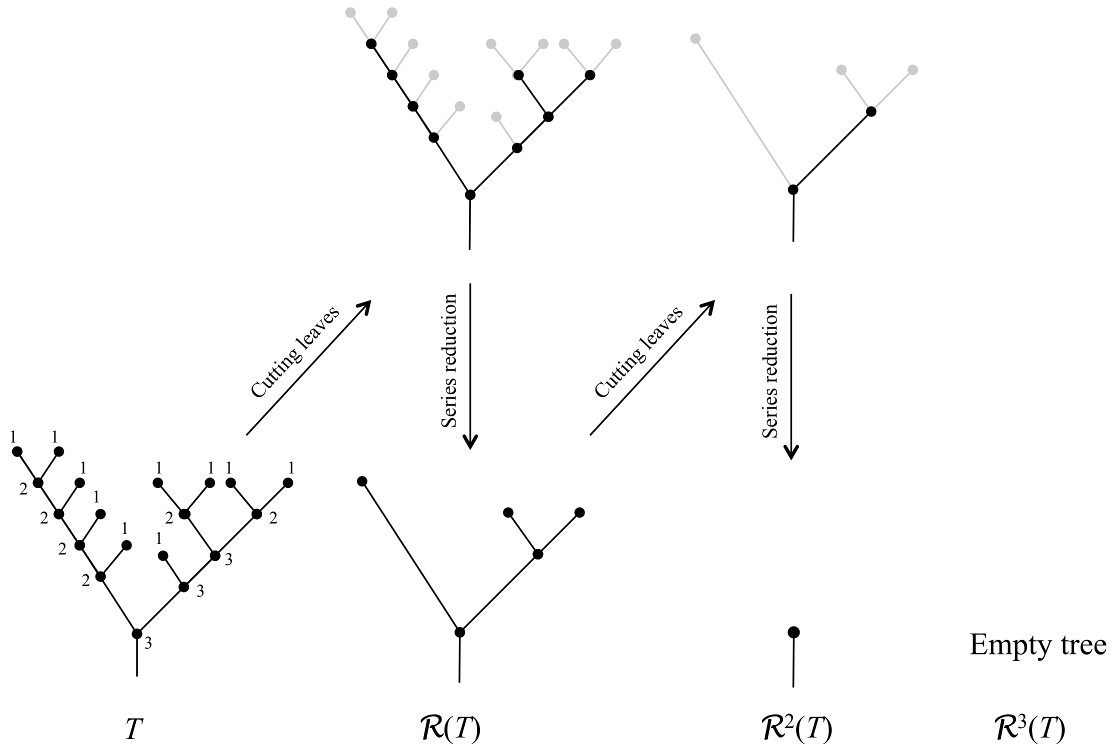
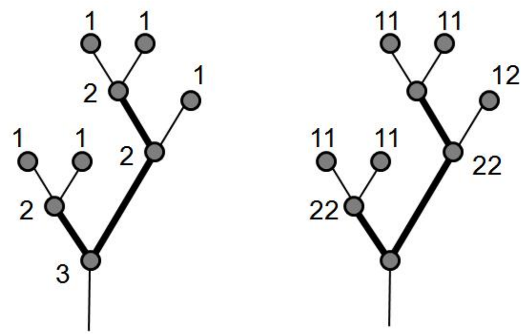


Pruning of a tree mod series reduction



Horton-Strahler ordering.



(a) Horton-Strahler orders (b)Tokunaga indices

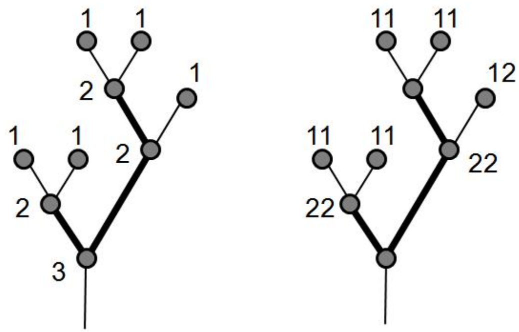
The **Horton-Strahler ordering** of the vertices of a finite rooted labeled binary tree is performed in a hierarchical fashion, from leaves to the root:

(i) each leaf has order $r(\text{leaf}) = 1$;

(ii) when both children, c_1, c_2, of a parent vertex p have the same order r, the vertex p is assigned order $r(p) = r + 1$;

(iii) when two children of vertex p have different orders, the vertex p is assigned the higher order of the two.

Horton-Strahler ordering and Tokunaga indexing.



(a) Horton-Strahler orders (b)Tokunaga indices

Example: (a) Horton-Strahler ordering

(b) Tokunaga indexing.

Two order-2 branches are depicted by heavy lines in both panels. The Horton-Strahler orders refer, interchangeably, to the tree nodes or to their parent links. The Tokunaga indices refer to entire branches, and not to individual vertices.

Tree self-similarity

Consider probability measures $\{\mu_K\}_{K\geq 1}$, each defined on the set $\mathcal{T}_K$ of finite binary trees of Horton-Strahler order K.

Define the average Horton numbers:

$$\mathcal{N}_k[K] = \mathsf{E}_K[N_k], \quad 1 \leq k \leq K, \quad K \geq 1.$$

Define the respective expectation

$$\mathcal{N}_{ij}[K] = \mathsf{E}_K[N_{ij}].$$

The Tokunaga coefficients $T_{ij}[K]$ for subspace $\mathcal{T}_K$ are defined as

$$T_{ij}[K] = \frac{\mathcal{N}_{ij}[K]}{\mathcal{N}_j[K]}, \quad 1 \leq i < j \leq K.$$

Tree self-similarity

Definition. A set of measures $\{\mu_K\}$ on $\{\mathcal{T}_K\}$ is called coordinated if $T_{ij} := T_{ij}[K]$ for all $K \geq 2$ and $1 \leq i < j \leq K$.

There, the Tokunaga matrix

$$\mathbb{T}_K = \begin{bmatrix} 0 & T_{1,2} & T_{1,3} & \dots & T_{1,K} \\ 0 & 0 & T_{2,3} & \dots & T_{2,K} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & T_{K-1,K} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

coincides with the restriction of $\mathbb{T}_M$, $M > K$, to the first K coordinates.

Definition. Coordinated probability measures $\{\mu_K\}$ are (mean) self-similar if $T_{ij} = T_{j-i}$ for some sequence $T_k \geq 0$ known as Tokunaga coefficients and any $K \geq 2$.

Tree self-similarity

Definition. Coordinated probability measures $\{\mu_K\}$ are (mean) self-similar if $T_{ij} = T_{j-i}$ for some sequence $T_k \geq 0$ known as Tokunaga coefficients and any $K \geq 2$.

There, the Tokunaga matrix

$$\mathbb{T}_K = \begin{bmatrix} 0 & T_1 & T_2 & \dots & T_{K-1} \\ 0 & 0 & T_1 & \dots & T_{K-2} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & T_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here **pruning** is equivalent to deleting the first row and first column.

Tokunaga self-similarity

Definition. A random self-similar tree is Tokunaga self-similar if

$$T_{k+1}/T_k = c \quad \Leftrightarrow \quad T_k = a\,c^{k-1} \quad a, c > 0,\ 1 \le k \le K-1.$$

There, the Tokunaga matrix

$$\mathbb{T}_K = \begin{bmatrix} 0 & a & ac & ac^2 & \dots & ac^{K-2} \\ 0 & 0 & a & ac & \dots & ac^{K-3} \\ 0 & 0 & 0 & a & \dots & ac^{K-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & a \end{bmatrix}.$$

McConnell and Gupta 2008 showed that Tokunaga self-similarity implies strong Horton law:

$$\lim_{K\to\infty} \frac{\mathcal{N}_k[K]}{\mathcal{N}_1[K]} = R^{1-k} < \infty \quad \text{for any } k \ge 1.$$

Tree self-similarity

Theorem (YK and Zaliapin, Fractals 2016).
Consider a sequence of coordinated self-similar probability measures $\{\mu_K\}$ such that

$$\limsup_{j\to\infty} T_j^{1/j} < \infty.$$

Then $\{\mu_K\}$ satisfy the strong Horton law for some $R > 0$: for each integer $j > 0$,

$$\lim_{K\to\infty} \frac{\mathcal{N}_j[K]}{\mathcal{N}_1[K]} = R^{1-j}.$$

Moreover, $1/R = w_0$ is the only real root of the function

$$\widehat{t}(z) = -1 + 2z + \sum_{j=1}^{\infty} z^j T_j$$

in the interval $\left(0, \frac{1}{2}\right]$.

Conversely, if $\limsup_{j\to\infty} T_j^{1/j} = \infty$, then the limit $\lim_{K\to\infty} \frac{\mathcal{N}_j[K]}{\mathcal{N}_1[K]}$ does not exist at least for some j.

Horton self-similarity: more generally.

A sequence of probability measures $\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ over binary trees has well-defined asymptotic Horton-Strahler orders if for each $k \in \mathbb{N}$, the following limit law is satisfied:

$$\frac{N_k^{(\mathcal{P}_N)}}{N} \longrightarrow \mathcal{N}_k \quad \text{in probability as} \quad N \to \infty,$$

where $\mathcal{N}_k$ is called the asymptotic ratio of the branches of order k.

Horton self-similarity: sequence $\mathcal{N}_k$ decreases in a regular geometric fashion with $k \to \infty$.

Informally,

$$\mathcal{N}_k \asymp N_0 \cdot R^{-k}$$

Horton self-similarity: more generally.

$$\mathcal{N}_k \asymp N_0 \cdot R^{-k}$$

A sequence $\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ with well-defined asymptotic Horton-Strahler orders obeys a Horton self-similarity law if and only if at least one of the following limits exists and is finite and positive:

$$\text{(a) root law :} \quad \lim_{k\to\infty}\left(\mathcal{N}_k\right)^{-\frac{1}{k}} = R > 0,$$

$$\text{(b) ratio law :} \quad \lim_{k\to\infty}\frac{\mathcal{N}_k}{\mathcal{N}_{k+1}} = R > 0,$$

$$\text{(c) geometric law :} \quad \lim_{k\to\infty}\mathcal{N}_k \cdot R^k = N_0 > 0.$$

The constant R is called the Horton exponent.

Galton-Watson tree.

• Critical binary Galton-Watson tree exhibits both Horton and Tokunaga self-similarities (Burd, Waymire, and Winn, 2000). This model has $R = 4$ and $(a, c) = (1, 2)$.

Theorem (Shreve, 1969; Burd et al., 2000). A critical binary Galton-Watson tree is Tokunaga self-similar with

$$(a, c) = (1, 2),$$

that is

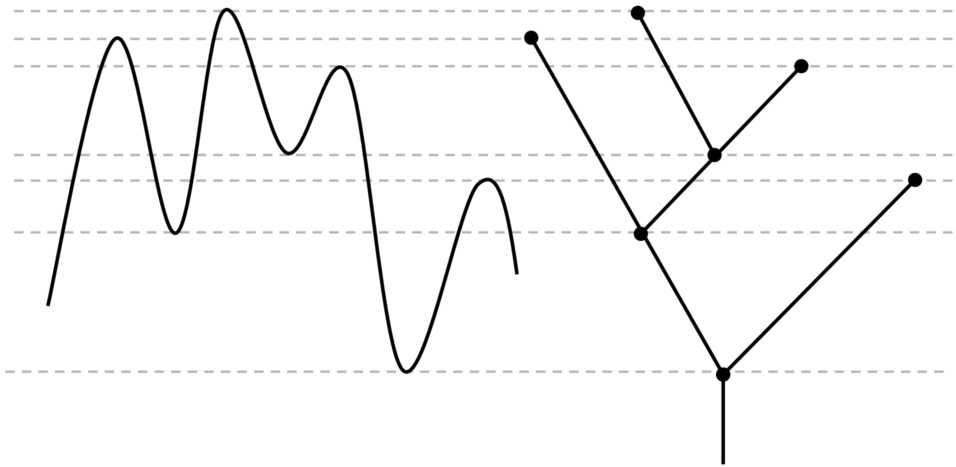
$$T_k = 2^{k-1} \quad \text{and} \quad R = 4.$$

Theorem (Burd et al., 2000).

1. Let $P_{\mathrm{GW}}(p_k)$ denote a Galton-Watson distribution on the space of finite trees with branching probabilities $\{p_k\}$. Then $P_{\mathrm{GW}}(p_k)$ is tree self-similar if and only if $\{p_k\}$ is the critical binary distribution $p_0 = p_2 = 1/2$.

2. Any critical Galton-Watson tree T, $\sum k\, p_k = 1$, converges to the binary critical tree under the operation of pruning, $\mathcal{R}^n(T)$, $n \to \infty$.

Pruning of a level-set tree of a function.

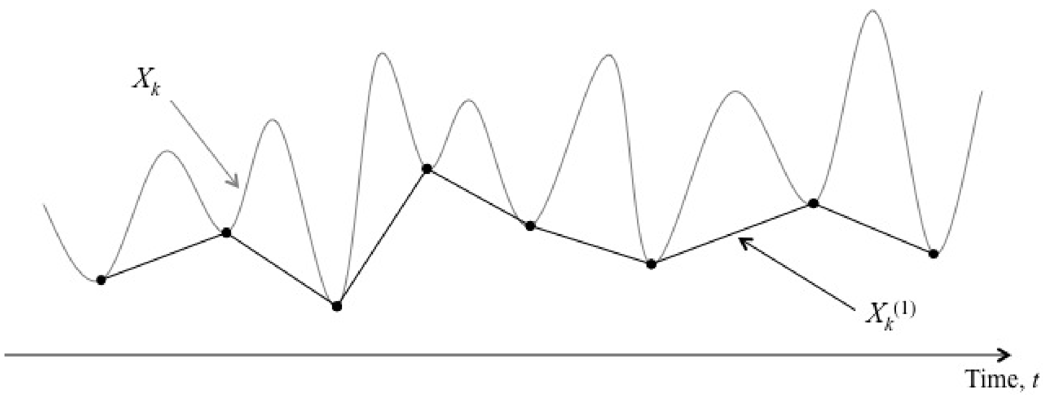


(a) Function X_t (b) Tree LEVEL(X)

Function X_t (panel a) with a finite number of local extrema and its level-set tree level(X) (panel b).

Pruning of time series

Proposition (Zaliapin and YK, CSF 2012). The transition from a time series X_k to the time series $X_k^{(1)}$ of its local minima corresponds to the pruning of the level-set tree level(X).



Horton and Tokunaga self-similarity: Markov chains

Theorem (Zaliapin and YK, CSF 2012).
Let X_k be a symmetric homogeneous random walk on $\mathbb{R}$, i.e. $p(x,y) = p(|x-y|)$, and

$$T = \text{shape}\,(\text{level}(X))$$

be the combinatorial level set tree of X_k.

Then, tree T is Tokunaga self-similar with parameters $(a,c) = (1,2)$:

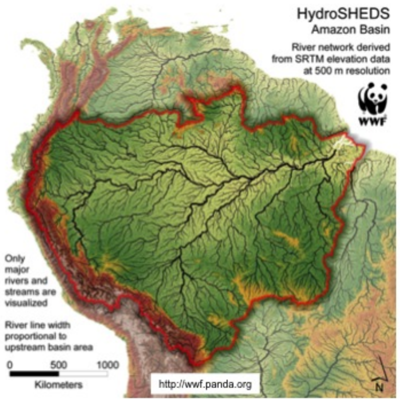
$$T_k = 2^{k-1},$$

and geometric-Horton self-similar, asymptotically in N, with $R = 4$.

River networks.

• Peckham'95 : high-precision extraction of river channels for Kentucky River, Kentucky and Powder River, Wyoming.

Reported Horton exponents and Tokunaga parameters: $R \approx 4.6$ and $(a, c) \approx (1.2, 2.5)$.



Sample citations: Shreve 1966, 1969; Tokunaga, 1978; Peckham, 1995; Zaliapin et al., 2009; Zanardo et al., 2013

Horton and Tokunaga self-similarity beyond river networks observed:

• Billiards / cluster dynamics (Zaliapin, Sinai, Gabrielov, Keilis-Borok)

• Diffusion limited aggregation (Yekutieli, Mandelbrot, Kaufman; Ossadnik; Masek, Turcotte)

• A general hierarchical coagulation model of Gabrielov introduced in the framework of self-organized criticality

• Phylogenetic trees (in collaboration with Zaliapin and Tree of Life biologists)

• Other: designing optimal computer codes, blood, botanical leaves, two dimensional site percolation, nearest-neighbor clustering in Euclidean spaces, etc.

Root-Horton law for the Kingman's coalescent

YK & Zaliapin (AIHP 2016):

• Established the root-Horton law for the Kingman's coalescent.

• Showed that the tree for Kingman's coalescent is combinatorially equivalent to the level-set tree of iid time series (the two measures are **one pruning apart**).

• Numerical experiments that suggest stronger Horton laws: ratio, geometric.

Coalescent tree.

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Root-Horton law for the Kingman's coalescent.

In **YK & Zaliapin (AIHP 2016)**, we prove the limit law (in probability) for the asymptotics of the number N_k of branches of Horton-Strahler order k in Kingman's N-coalescent process with constant collision kernel:

$$\mathcal{N}_k = \lim_{N\to\infty} N_k/N$$

We show that

$$\mathcal{N}_k = \frac{1}{2}\int_0^\infty g_k^2(x)\,dx,$$

where the sequence $g_k(x)$ solves:

$$g'_{k+1}(x) - \frac{g_k^2(x)}{2} + g_k(x)g_{k+1}(x) = 0, \quad x \geq 0$$

with $g_1(x) = 2/(x+2)$, $g_k(0) = 0$ for $k \geq 2$.

Root-Horton law for the Kingman's coalescent.

Theorem (YK & Zaliapin, AIHP 2016). The asymptotic Horton ratios $\mathcal{N}_k$ exist and finite and satisfy the convergence $\lim_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R$ with $2 \leq R \leq 4$.

Conjecture. The tree associated with Kingman's coalescent process is Horton self-similar with

$$\lim_{k\to\infty} \frac{\mathcal{N}_k}{\mathcal{N}_{k+1}} = \lim_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R \quad \text{and} \quad \lim_{k\to\infty} (\mathcal{N}_k R^k) = const.,$$

where $R = 3.043827\ldots$ and Tokunaga self-similar, asymptotically in k:

$$\lim_{i\to\infty} T_{i,i+k} =: T_k \quad \text{and} \quad \lim_{k\to\infty} \frac{T_k}{c^{k-1}} = a$$

for some positive a and c.

Root-Horton law for the Kingman's coalescent.

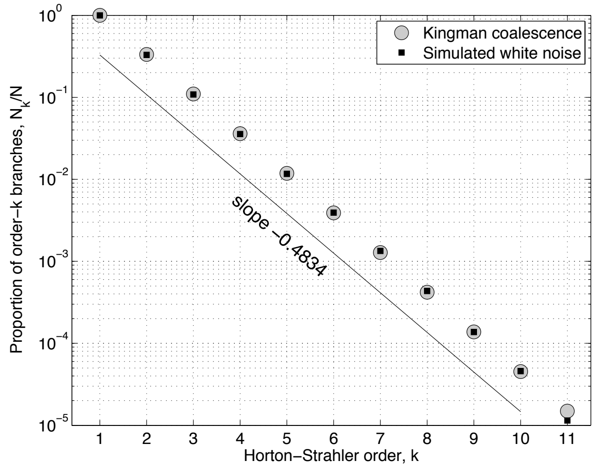
- **Theorem (YK & Zaliapin, AIHP 2016).** The asymptotic Horton ratios $\mathcal{N}_k$ exist and finite and satisfy the convergence $\lim_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R$ with $2 \leq R \leq 4$.

- Consider now a time series X with N local maxima separated by $N-1$ internal local minima such that the latter form a discrete white noise; we call X an *extended discrete white noise*.

Theorem (YK & Zaliapin, AIHP 2016). The combinatorial level set tree of the extended discrete white noise has the same distribution on $\mathcal{T}_N$ as the combinatorial tree generated by Kingman's N-coalescent.

- **Corollary (YK & Zaliapin, AIHP 2016).** The combinatorial level set tree of iid time series is root-Horton self similar with the same Horton exponent R as for Kingman's coalescent.

Root-Horton law for the Kingman's coalescent.



Filled circles: The asymptotic ratio $\mathcal{N}_k$ of the number N_k of branches of order k to N in Kingman's coalescent, as $N \to \infty$. Black squares: The empirical ratio N_k/N_1 in a level-set tree for a single trajectory of a iid time series of length $N = 2^{18}$.

Hierarchical Branching Processes.

Consider a multi-type branching process originating from a root of hierarchical order K with probability p_K.

• Each branch of order j branches out an offspring of order $i < j$ with rate $\lambda_j T_{j-i}$.

• The branch of order j terminates with rate λ_j, at which moment,

(i) the branch of order $j \geq 2$ splits into two branches, each of order $j - 1$

(ii) the branch of order $j = 1$ terminates without leaving offsprings.

The branching history of the process creates a random planar binary tree, with $T_{i,j} = T_{j-i}$.

Hierarchical Branching Processes.

YK and Zaliapin, 2016 *arXiv:1608.05032*

Generalized notions of self-similarity under pruning:

- **Coordination**
- **Prune-invariance**
- **Distributional self-similarity:** coordination plus prune invariance
- **Mean coordination** and **mean self-similarity**
- Self-similarity of trees with **edge lengths**

The hierarchical branching processes are self-similar.

Hierarchical Branching Processes.

Suppose $L = \limsup_{k\to\infty} T_k^{1/k} < \infty$.

Let the coordinates of $x(s)$represent the frequency of branches of respective orders at time s in a tree.

Initial distribution is $x(0) = \pi := \sum_{K=1}^{\infty} p_K e_K$, and

$$x(s) = e^{\mathbb{G}\Lambda s}\pi, \qquad \text{where} \quad \Lambda = \mathrm{diag}\Big\{\lambda_1, \lambda_2, \dots\Big\}$$

and

$$\mathbb{G} := \begin{bmatrix} -1 & T_1+2 & T_2 & T_3 & \dots \\ 0 & -1 & T_1+2 & T_2 & \dots \\ 0 & 0 & -1 & T_1+2 & \ddots \\ 0 & 0 & 0 & -1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Hierarchical Branching Processes.

Consider the width function at time $s \geq 0$

$$C(s) = \langle \mathbf{1}, x(s) \rangle = \langle \mathbf{1}, e^{\mathbb{G}\Lambda s}\pi \rangle.$$

For μ to be "distributionally" self-similar under pruning, need

- $\{p_K\}$ to be geometric: $p_K = p(1-p)^{K-1}$
- the sequence λ_j to be geometric: $\lambda_j = \gamma\, c^{-j}$

Hierarchical Branching Processes.

Recall:

$$\hat{t}(z) = -1 + 2z + \sum_{j=1}^{\infty} z^j T_j$$

and $w_0 = 1/R$ is the only real root within the radius of convergence.

Suppose $\{p_K\}$ is geometric with parameter p and $\lambda_j = \gamma\, c^{-j}$, then

$$x(s) = e^{\mathbb{G}\Lambda s}\pi = \pi + \sum_{m=1}^{\infty} s^m \left[\prod_{j=1}^{m} \hat{t}\big(c^{-j}(1-p)\big)\right] \Lambda^m \pi.$$

The convergence requirement here is that $c \geq 1$.

Criticality: $p_c = 1 - \frac{c}{R}$, i.e. $C(s) = \langle \mathbf{1}, x(s) \rangle = 1$.

Hierarchical Branching Processes.

The following two conditions are equivalent.

- The process is colorblue critical, i.e.,

$$C(t) = \langle \mathbf{1}, x(s) \rangle = 1 \qquad t \geq 0.$$

- The process has the forest invariance property at criticality: the frequencies of trees in the forest produced by the process dynamics are time-invariant

$$x(t) = \exp\{\mathbb{G}\Lambda\, t\}\, \pi = \pi, \qquad \text{where } x(0) = \pi.$$

Hierarchical Branching Processes.

$$p_c = 1 - \frac{c}{R}$$

Observe that for a hierarchical branching process with $\lambda_j = \gamma 2^{2-j}$ and $T_k = 2^{k-1}$, the critical probability is

$$p_c = \frac{1}{2}.$$

Therefore, $R = \frac{c}{1-p_c} = 4$.

Recall that the critical binary Galton-Watson tree exhibits both Horton and Tokunaga self-similarities (Burd, Waymire, and Winn, 2000) with parameters $R = 4,\ \ (a, c) = (1, 2)$ and

$$T_k = a \cdot c^{k-1} = 2^{k-1}.$$

Critical binary Galton-Watson tree.

Theorem (YK and Zaliapin, 2016). The tree of a hierarchical branching process with parameters

$$\lambda_j = \gamma 2^{2-j}, \quad p_K = 2^{-K}, \qquad \text{and} \quad T_k = 2^{k-1}$$

for any $\gamma > 0$ is equivalent to the critical binary Galton-Watson tree with independent edge lengths that have exponential pdf $f(x) = 2\gamma\, e^{-2\gamma\, x}$.

Other properties:

• Invariant under pruning (from the leafs).

• Satisfies the forest invariance property (when cut from below).

• It is also self-similar under various types of *continuous pruning*.

Conclusion.

- Generalized notion of prune invariance.
- Time series: extreme values.
- Burgers equations.