Invariant Galton-Watson measures

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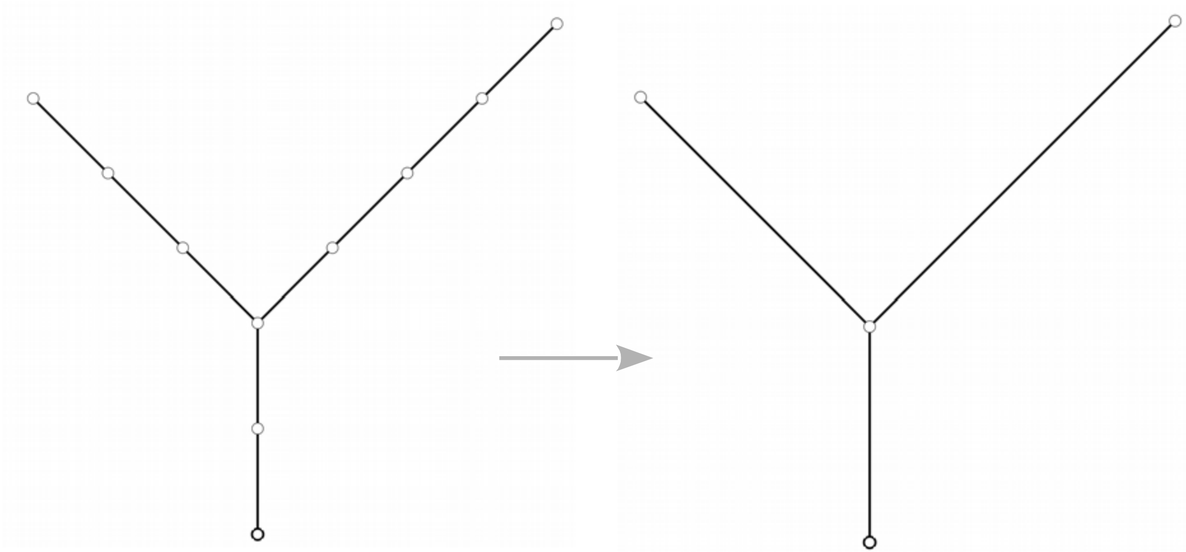
Combinatorial trees.

$\mathcal{T}$ - space of finite unlabeled rooted reduced trees.

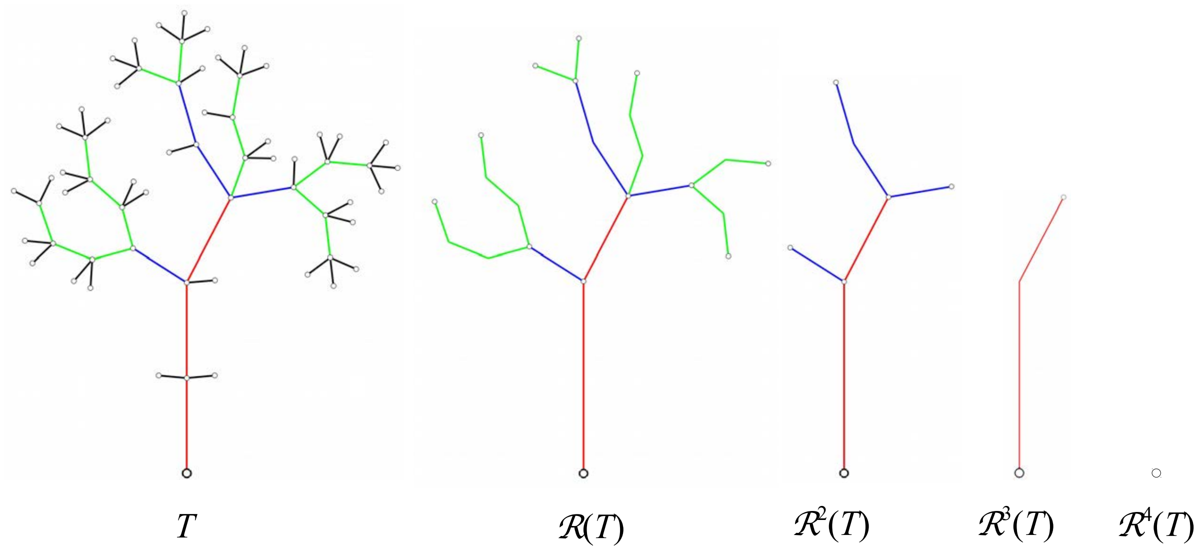
Empty tree $\phi = \{\rho\}$ comprised of a root vertex ρ and no edges.

$\mathcal{T}^{|}$ - subspace of $\mathcal{T}$ containing ϕ and all the trees in $\mathcal{T}$ with a stem (ρ has exactly one offspring).



(a) Original tree (b) Tree after series reduction

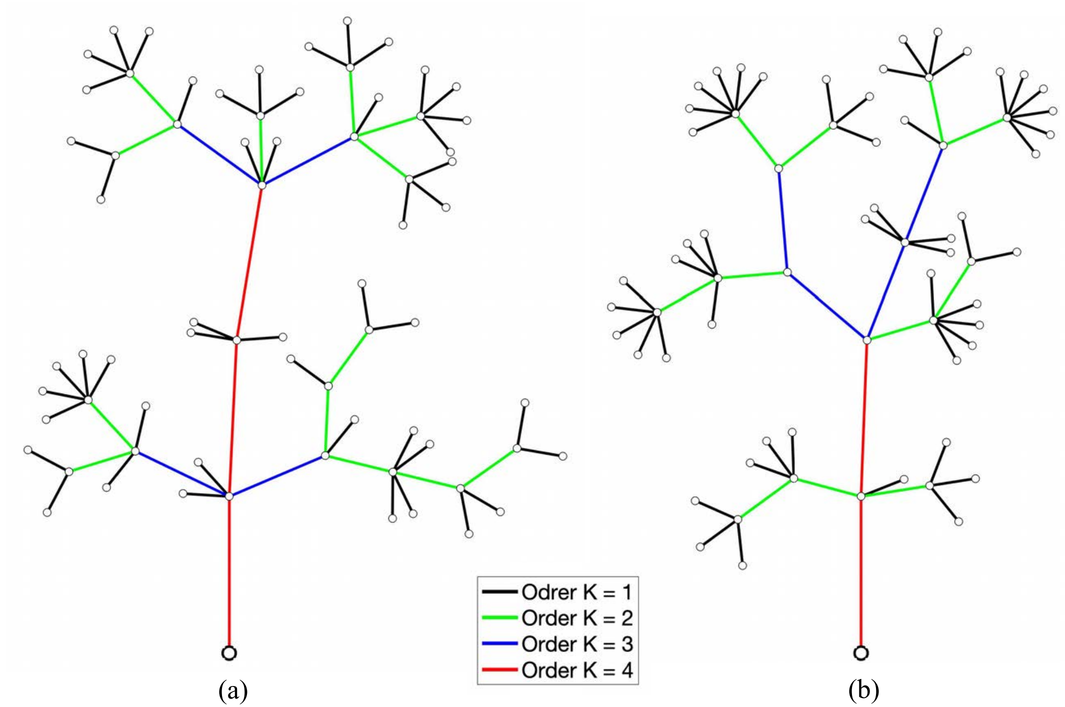
Horton pruning and Horton-Strahler order



Horton pruning $\mathcal{R} : \mathcal{T}^{|} \to \mathcal{T}^{|}$ is an onto function whose value $\mathcal{R}(T)$ for a tree $T \neq \phi$ is obtained by removing the leaves and their parental edges from T, followed by series reduction. We also set $\mathcal{R}(\phi) = \phi$.

Horton-Strahler order: $\mathsf{ord}(T) = \min\left\{k \geq 0 \ : \ \mathcal{R}^k(T) = \phi\right\}.$

Horton-Strahler order



Horton-Strahler order: $\text{ord}(T) = \min\left\{k \geq 0 \; : \; \mathcal{R}^k(T) = \phi\right\}$.

Horton prune-invariance

Consider a measure μ on $\mathcal{T}$ (or $\mathcal{T}^{|}$) such that $\mu(\phi) = 0$. Let ν be the pushforward measure, $\nu = \mathcal{R}_*(\mu)$, i.e.,

$$\nu(T) = \mu \circ \mathcal{R}^{-1}(T) = \mu\Big(\mathcal{R}^{-1}(T)\Big).$$

Measure μ is said to be Horton prune-invariant if for any tree $T \in \mathcal{T}$ (or $\mathcal{T}^{|}$) we have

$$\nu\,(T\,|T \neq \phi) = \mu(T).$$

Objective: finding and classifying Horton prune-invariant tree measures.

Attractors

For a tree measure ρ_0 let $\nu_k = \mathcal{R}_*^k(\rho_0)$ denote the pushforward probability measure induced by operator $\mathcal{R}^k$, i.e.,

$$\nu_k(T) = \rho_0 \circ \mathcal{R}^{-k}(T) = \rho_0\Big(\mathcal{R}^{-k}(T)\Big), \quad \text{and set} \quad \rho_k(T) = \nu_k\,(T\ |T \neq \phi).$$

If $\lim_{k\to\infty} \rho_k(T) = \rho^*(T)\ \ \forall T \in \mathcal{T}$, then measure ρ^* is an attractor.

Objective: finding and classifying attractors.

Pruning Galton-Watson trees

Consider a Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$.

Assume criticality or subcriticality, i.e., $\sum_{k=0}^{\infty} k q_k \leq 1$.

Theorem. [G. A. Burd, E. C. Waymire, R. D. Winn, *Bernoulli* (2000)]

• Assume finite second moment, i.e., $\sum_{k=0}^{\infty} k^2 q_k < \infty$.

Galton-Watson measure $\mathcal{GW}(\{q_k\})$ is Horton prune-invariant if and only if it is critical binary Galton-Watson $\mathcal{GW}(q_0 = q_2 = 1/2)$.

• Assume criticality and finite branching, i.e., $\Big|\{k \,:\, q_k > 0\}\Big| < \infty$.
Let $\rho_0 \equiv \mathcal{GW}(\{q_k\})$, $\nu_k = \mathcal{R}_*^k(\rho_0)$, and set $\rho_k(T) = \nu_k\left(T \;|T \neq \phi\right)$.
Then,

$$\lim_{k\to\infty} \rho_k(T) = \rho^*(T) \qquad \forall T \in \mathcal{T},$$

where $\rho^* = \mathcal{GW}(q_0 = q_2 = 1/2)$ is critical binary Galton-Watson measure.

• If $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ is subcritical, then $\rho_k(T)$ converges to a point mass measure, $\rho^* = \mathcal{GW}(q_0 = 1)$.

Pruning Galton-Watson trees

Consider a Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$.

Assume criticality or subcriticality, i.e., $\sum\limits_{k=0}^{\infty} k q_k \leq 1$.

Let $Q(z) = \sum\limits_{m=0}^{\infty} z^m q_m$ denote the generating function.

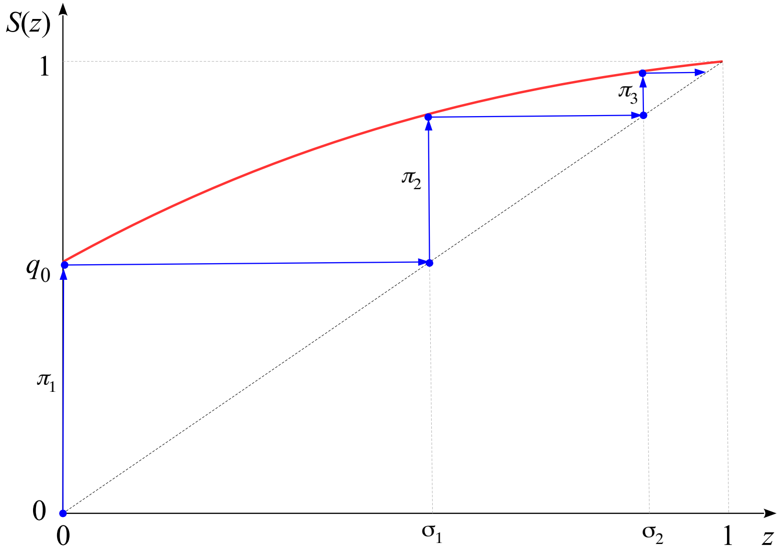
For $T \stackrel{d}{\sim} \mathcal{GW}(\{q_k\})$ let $\pi_j := P\big(\text{ord}(T) = j\big)$,

$$\sigma_0 = 0 \quad \text{and} \quad \sigma_j := \sum_{i=1}^{j} \pi_i \quad \forall j \geq 1$$

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)]

$$\sigma_j = \underbrace{S \circ \ldots \circ S}_{j \text{ times}}(0), \qquad \text{where} \qquad S(z) = \frac{Q(z) - zQ'(z)}{1 - Q'(z)}.$$

Pruning Galton-Watson trees



For $T \stackrel{d}{\sim} \mathcal{GW}(\{q_k\})$, let $\pi_j := P\big(\text{ord}(T) = j\big)$ and $\sigma_j := \sum_{i=1}^{j} \pi_i$

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)].

$$\sigma_j = \underbrace{S \circ \ldots \circ S}_{j \text{ times}}(0), \qquad \text{where} \qquad S(z) = \frac{Q(z) - zQ'(z)}{1 - Q'(z)}.$$

Regularity condition

Many of the results are proven under the following assumption.

Assumption 1. The following limit exists:

$$S'(1) = \lim_{x\to 1-} \frac{1-S(x)}{1-x} \quad \Leftrightarrow \quad \lim_{x\to 1-} \frac{Q(x)-x}{(1-x)\big(1-Q'(x)\big)} = 1 - S'(1)$$

Proposition. [YK and I. Zaliapin, *Bernoulli* (2021)]
If $\mathcal{GW}(\{q_k\})$ is a subcritical Galton-Watson measure with $q_1 = 0$, then Assumption 1 holds with $S'(1) = 0$.

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$. If

$$\mathsf{E}[X^2] = \sum_{k=0}^{\infty} k^2 q_k < \infty \qquad \text{where} \quad X \overset{d}{\sim} \{q_k\},$$

then Assumption 1 holds with $S'(1) = \frac{1}{2}$.

Regularity condition

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$ and infinite second moment, i.e., $\sum_{k=0}^{\infty} k^2 q_k = \infty$. Let $X \stackrel{d}{\sim} \{q_k\}$.
If the limit

$$\Lambda = \lim_{k\to\infty} \frac{k}{E[X \mid X \geq k]} = \lim_{k\to\infty} \frac{k \sum\limits_{m=k}^{\infty} q_m}{\sum\limits_{m=k}^{\infty} m q_m}$$

exists, then Assumption 1 holds with $S'(1) = \Lambda$.

Corollary. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson process $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$ and offspring distribution $\{q_k\}$ of Zipf type:

$$q_k \sim C k^{-(\alpha+1)} \quad \text{with} \quad \alpha \in (1, 2] \quad \text{and} \quad C > 0.$$

Then Assumption 1 holds with $S'(1) = \Lambda = \frac{\alpha-1}{\alpha}$.

Invariant Galton-Watson measures

For a given $q \in [1/2, 1)$, a critical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ is said to be the invariant Galton-Watson (IGW) measure with parameter q and denoted by $\mathcal{IGW}(q)$ if its generating function is given by

$$Q(z) = z + q(1 - z)^{1/q}.$$

Branching probabilities: $q_0 = q$, $q_1 = 0$, $q_2 = (1 - q)/2q$, and

$$q_k = \frac{1 - q}{k!\, q} \prod_{i=2}^{k-1} (i - 1/q) \quad (k \geq 3).$$

Here, if $q = 1/2$, then the distribution is critical binary, i.e., $\mathcal{GW}(q_0 = q_2 = 1/2)$.

If $q \in (1/2, 1)$, the distribution is of Zipf type with

$$q_k = \frac{(1 - q)\Gamma(k - 1/q)}{q\Gamma(2 - 1/q)\, k!} \sim C k^{-(1+q)/q}, \quad \text{where} \quad C = \frac{1 - q}{q\, \Gamma(2 - 1/q)}.$$

Invariant Galton-Watson measures

Recall

$$S(z) = \frac{Q(z) - zQ'(z)}{1 - Q'(z)}.$$

Assumption 1. The following limit exists:

$$S'(1) = \lim_{x \to 1-} \frac{1 - S(x)}{1 - x} \quad \Leftrightarrow \quad \lim_{x \to 1-} \frac{Q(x) - x}{(1 - x)\big(1 - Q'(x)\big)} = 1 - S'(1)$$

Horton prune-invariance: for $\nu(T) = \mu\Big(\mathcal{R}^{-1}(T)\Big)$,

$$\nu\left(T \,|T \neq \phi\right) = \mu(T).$$

Theorem. [YK and I. Zaliapin, *Bernoulli* (2021)] Consider a critical or subcritical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$ that satisfies Assumption 1. Then, measure $\mathcal{GW}(\{q_k\})$ is Horton prune-invariant if and only if it is $\mathcal{IGW}(q_0)$.

Attraction property of critical Galton-Watson trees

Theorem. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ with $q_1 = 0$. Let $\nu_k = \mathcal{R}_*^k(\rho_0)$ denote the pushforward probability measure induced by operator $\mathcal{R}^k$, i.e.,

$$\nu_k(T) = \rho_0 \circ \mathcal{R}^{-k}(T) = \rho_0\Big(\mathcal{R}^{-k}(T)\Big), \text{ and set } \rho_k(T) = \nu_k\left(T \mid T \neq \phi\right).$$

Suppose Assumption 1 is satisfied. Then,

$$\lim_{k\to\infty} \rho_k(T) = \rho^*(T),$$

where ρ^* denotes $\mathcal{IGW}(q)$ with $q = 1 - S'(1)$.

Finally, if $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ is subcritical, then $\rho_k(T)$ converges to a point mass measure, $\mathcal{GW}(q_0 = 1)$.

Attraction property of critical Galton-Watson trees

Corollary. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ with $q_1 = 0$, with offspring distribution q_k of Zipf type:

$$q_k \sim Ck^{-(\alpha+1)} \quad \text{with} \quad \alpha \in (1,2] \quad \text{and} \quad C > 0.$$

Let $\nu_k = \mathcal{R}_*^k(\rho_0)$ and $\rho_k(T) = \nu_k\,(T \,|T \neq \phi)$.

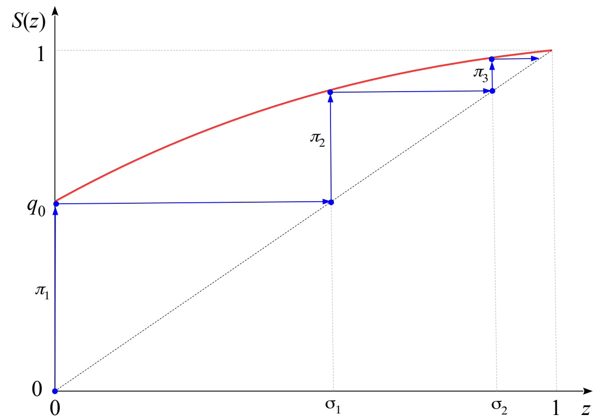
Then, $\lim_{k\to\infty} \rho_k(T) = \rho^*(T)$, where ρ^* is $\mathcal{IGW}(q)$ with $q = \frac{1}{\alpha}$.

Corollary. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ with $q_1 = 0$ such that $\sum_{k=2}^{\infty} k^2 q_k < \infty$.

Let $\nu_k = \mathcal{R}_*^k(\rho_0)$ and $\rho_k(T) = \nu_k\,(T \,|T \neq \phi)$.

Then, $\lim_{k\to\infty} \rho_k(T) = \rho^*(T)$, where ρ^* is $\mathcal{IGW}(1/2)$ (critical binary).

A gift from anonymous referee



For $\mathcal{IGW}(q_0)$, $S(z) = q_0 + (1 - q_0)z$

This is an example of Horton prune-invariant critical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ for which Assumption 1 does not hold.

Let $q_0 \in (1/2, 1)$, $q_1 = 0$, and

$$q_m = \frac{1}{m!A} \sum_{n \in \mathbb{Z}} B^n \rho^{nm} e^{-\rho^n} \qquad m = 2, 3, \ldots,$$

where $\rho = 1 - q_0$, parameter $B \in \left((1 - q_0)^{-1}, (1 - q_0)^{-2}\right)$ is found by solving

$$\sum_{n \in \mathbb{Z}} B^n \left(1 - \rho^{n+1} - (1 + \rho^n - \rho^{n+1})e^{-\rho^n}\right) = 0, \quad \text{and} \quad A = \sum_{n \in \mathbb{Z}} B^n \rho^n \left(1 - e^{-\rho^n}\right).$$

Metric trees.

$\mathcal{L}$ - space of finite unlabeled rooted reduced trees with edge lengths.

Empty tree $\phi = \{\rho\}$ comprised of a root vertex ρ and no edges.

$d(x, y)$: the length of the minimal path within T between x and y.

The length of a tree T is the sum of the lengths of its edges:

$$\text{length}(T) = \sum_{i=1}^{\#T} l_i.$$

The height of a tree T is the maximal distance between the root and a vertex:

$$\text{height}(T) = \max_{1 \leq i \leq \#T} d(v_i, \rho).$$

Metric invariant Galton-Watson measures

Denote

$$\text{shape}(T) = \text{ combinatorial shape of } T.$$

Continuous Galton-Watson measure: for a given p.m.f. $\{q_k\}$ and a parameter $\lambda > 0$, a metric T is distributed as

$$T \overset{d}{=} \mathcal{GW}(\{q_k\}, \lambda) \quad \text{if} \quad \text{shape}(T) \overset{d}{=} \mathcal{GW}(\{q_k\})$$

and, conditioned on shape(T), the edges of T are i.i.d. exponentially distributed with parameter λ.

Exponential critical binary Galton-Watson tree measure:

$$\text{GW}(\lambda) = \mathcal{GW}\Big(q_0 = q_2 = 1/2,\, \lambda\Big)$$

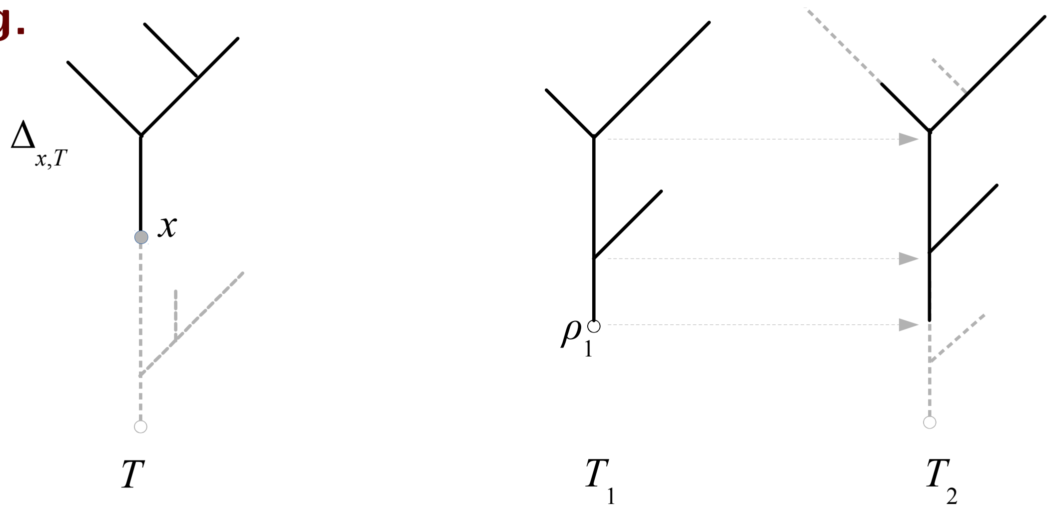
Exponential invariant Galton-Watson (IGW) tree measure: for a given $q \in [1/2, 1)$ and $\lambda > 0$, let p.m.f. $\{q_k\}$ be such that $\mathcal{GW}(\{q_k\}) = \mathcal{IGW}(q)$, then,

$$\mathcal{IGW}(q, \lambda) = \mathcal{GW}(\{q_k\}, \lambda)$$

is the IGW measure with parameters q and λ.

Notice that $\mathcal{IGW}(1/2, \lambda) = \text{GW}(\lambda)$.

Partial ordering.



(a) Descendant tree (b) Isometry

Consider $T \in \mathcal{L}$ and a point $x \in T$. Let $\Delta_{x,T}$ denote all points of T descendant to x, including x. Then $\Delta_{x,T}$ is itself a tree in $\mathcal{L}$ with root at x.

Let (T_1, d) and (T_2, d) be two metric rooted trees, and let ρ_1 denote the root of T_1. $f : (T_1, d) \to (T_2, d)$ is an isometry if $\text{Image}[f] \subseteq \Delta_{f(\rho_1),T_2}$ and $\forall x, y \in T_1$, $d\big(f(x), f(y)\big) = d(x, y)$.

Generalized dynamical pruning.

Partial order: $T_1 \preceq T_2$ if and only if $\exists$ an isometry $f : (T_1, d) \to (T_2, d)$.

Consider a monotone non-decreasing

$$\varphi : \mathcal{L} \to \mathbb{R}^+,$$

i.e., $\varphi(T_1) \le \varphi(T_2)$ whenever $T_1 \preceq T_2$.

Generalized dynamical pruning: for any $t \ge 0$, let

$$\mathcal{S}_t(\varphi, T) = \rho \cup \left\{ x \in T \setminus \rho \;:\; \varphi\Big(\Delta_{x,T}\Big) \ge t \right\}$$

Note that $\mathcal{S}_t(\varphi, T) : \mathcal{L} \to \mathcal{L}$ is an operator induced by φ.

It cuts all subtrees $\Delta_{x,T}$ for which the value of φ is below threshold t. Here,

$$S_s(T) \preceq S_t(T) \quad \text{whenever} \quad s \ge t.$$

Prune-invariance.

$$\mathcal{S}_t(\varphi, T) = \rho \cup \Big\{ x \in T \setminus \rho \ : \ \varphi\big(\Delta_{x,T}\big) \geq t \Big\}$$

Definition. Consider a probability measure μ on $\mathcal{L}$ such that $\mu(\phi) = 0$. Let ν be the pushforward measure induced by operator $\mathcal{S}_t$, i.e.,

$$\nu(T) = \mu \circ \mathcal{S}_t^{-1}(T) = \mu\big(\mathcal{S}_t^{-1}(T)\big).$$

A tree measure μ is called prune-invariant with respect to $\mathcal{S}_t$ if for any tree $T \in \mathcal{L}$ there $\exists\, \gamma_t > 0$ such that

$$\mu\Big(\text{shape}(T) \in A,\ \vec{\ell}(T) \in B\Big) = \nu\Big(\text{shape}(T) \in A,\ \gamma_t \vec{\ell}(T) \in B \,\Big|\, T \neq \phi\Big),$$

where

$$\vec{\ell}(T) = \text{ vector of edge-lengths.}$$

Objective: find and classify prune-invariant measures on $\mathcal{L}$.

See [YK and I. Zaliapin, *Probability Surveys* (2020)] for more on the topic.

Generalized dynamical pruning.

Example (Tree height). Let $\varphi(T) = \text{height}(T)$.

Continuous semigroup property: $\mathcal{S}_t \circ \mathcal{S}_s = \mathcal{S}_{t+s}$ for any $t, s \geq 0$.

It coincides with tree erasure in [J. Neveu, *Adv. Appl. Prob.* (1986)].

[J. Neveu, *Adv. Appl. Prob.* (1986)]: $\text{GW}(\lambda)$ is prune-invariant with respect to $\varphi(T) = \text{height}(T)$.

Example (Tree length). Let $\varphi(T) = \text{length}(T)$. No semigroup property.

It coincides with potential dynamics of 1D ballistic annihilation in [YK and I. Zaliapin, *JSP* (2020)].

Example (Horton pruning). Let $\varphi(T) = \text{ord}(T) - 1$, where $\text{ord}(T)$ denotes the Horton-Strahler order of T. Here, $\mathcal{S}_t = \mathcal{R}^{\lfloor t \rfloor}$.

Discrete semigroup property: $\mathcal{S}_t \circ \mathcal{S}_s = \mathcal{S}_{t+s}$ for any $t, s \in \mathbb{N}$.

[G. A. Burd, E. C. Waymire, and R. D. Winn, *Bernoulli* (2000)]: $\text{GW}(\lambda)$ is prune-invariant with respect to $\varphi(T) = \text{ord}(T) - 1$.

Prune-invariance.

Theorem. [YK and I. Zaliapin, *JSP* (2020)] Let $T \stackrel{d}{=} \mathsf{GW}(\lambda)$. Then, for any monotone non-decreasing function $\varphi : \mathcal{L} \to \mathbb{R}^+$,

$$T^t := \Big\{\mathcal{S}_t(\varphi,T) | \mathcal{S}_t(\varphi,T) \neq \phi\Big\} \stackrel{d}{=} \mathsf{GW}(\lambda p_t),$$

where $p_t = \mathsf{P}(\mathcal{S}_t(\varphi,T) \neq \phi)$.

That is, if $\mu \equiv \mathsf{GW}(\lambda)$, then, the pushforward measure ν induced by operator $\mathcal{S}_t$ satisfies

$$\nu\Big(\cdot \,\Big|\, \neq \phi\Big) \equiv \mathsf{GW}\Big(\mathcal{E}_t(\lambda,\varphi)\Big) \qquad \text{with} \quad \mathcal{E}_t(\lambda,\varphi) = \lambda p_t.$$

Theorem. [YK and I. Zaliapin, *JSP* (2020)]

(a) If $\varphi(T) = \mathsf{length}(T)$, then $\mathcal{E}_t(\lambda,\varphi) = \lambda e^{-\lambda t}\Big[I_0(\lambda t) + I_1(\lambda t)\Big]$.

(b) If $\varphi(T) = \mathsf{height}(T))$, then $\mathcal{E}_t(\lambda,\varphi) = \frac{2\lambda}{\lambda t+2}$.

(c) If $\varphi(T) = \mathsf{ord}(T) - 1$, then $\mathcal{E}_t(\lambda,\varphi) = \lambda 2^{-\lfloor t \rfloor}$.

Prune-invariance. Recall that $\mathcal{IGW}(1/2,\lambda) = \mathsf{GW}(\lambda)$.

Theorem. [YK, G. Xu, I. Zaliapin, *preprint* (2021)]

Let $T \stackrel{d}{=} \mathcal{IGW}(q,\lambda)$. Then, for any monotone non-decreasing function $\varphi : \mathcal{L} \to \mathbb{R}^+$,

$$T^t := \Big\{\mathcal{S}_t(\varphi,T) | \mathcal{S}_t(\varphi,T) \neq \phi\Big\} \stackrel{d}{=} \mathcal{IGW}\Big(q, \lambda p_t^{(1-q)/q}\Big),$$

where $p_t = \mathsf{P}(\mathcal{S}_t(\varphi,T) \neq \phi)$.

That is, if $\mu \equiv \mathcal{IGW}(q,\lambda)$, then, the pushforward measure ν induced by operator $\mathcal{S}_t$ satisfies

$$\nu\Big(\cdot \;\Big|\; \neq \phi\Big) \equiv \mathcal{IGW}\Big(q, \mathcal{E}_t(\lambda,\varphi)\Big) \quad \text{with} \quad \mathcal{E}_t(\lambda,\varphi) = \lambda p_t^{(1-q)/q}.$$

Theorem. [YK, G. Xu, I. Zaliapin, *preprint* (2021)]

(a) If $\varphi(T) = \mathrm{length}(T)$, then

$$p_t = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\Gamma(n/q+1)}{n!n!\Gamma(n/q-n+2)}(\lambda q)^n t^n.$$

(b) If $\varphi(T) = \mathrm{height}(T))$, then

$$p_t = \Big(\lambda(1-q)t+1\Big)^{-q/(1-q)} \quad \text{and} \quad \mathcal{E}_t(\lambda,\varphi) = \lambda p_t^{(1-q)/q} = \frac{\lambda}{\lambda(1-q)t+1}.$$

A related concept.

[T. Duquesne and M. Winkel, *SPA* (2019)] introduced concept of hereditary reduction.

The notion of hereditary reduction is a generalization of tree erasure in [J. Neveu, *Adv. Appl. Prob.* (1986)], similar to generalized dynamical pruning.

[T. Duquesne and M. Winkel, *SPA* (2019)]: GW(λ) is invariant with respect to hereditary reduction.

[YK, G. Xu, I. Zaliapin, *preprint* (2021)]: $\mathcal{IGW}(q,\lambda)$ is invariant with respect to hereditary reduction.

Tokunaga coefficients and Horton law.

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)]
For a given $q \in [1/2, 1)$, consider an invariant Galton-Watson measure $\mathcal{IGW}(q)$. Then, its Tokunaga coefficients are

$$T^o_{i,j} = \frac{\mathcal{N}^o_{i,j}[K]}{\mathcal{N}_j[K]} = T^o_{j-i}, \qquad \text{where} \quad T^o_k = c^{k-1} \ (k \geq 1) \quad \text{with} \quad c = \frac{1}{1-q}.$$

Additionally, $\pi_i = P\Big(\text{ord}(T) = j\Big) = q\,c^{1-i}$, and the strong Horton law $\lim\limits_{K\to\infty} \frac{\mathcal{N}_k[K]}{\mathcal{N}_1[K]} = R^{1-k}$ holds with Horton exponent

$$R = c^{c/(c-1)} = (1-q)^{-1/q}.$$

Critical binary: since $\mathcal{IGW}(1/2) = \mathcal{GW}\Big(q_0 = q_2 = 1/2\Big)$, for $q = 1/2$, we have

$$c = 2, \quad \pi_i = 2^{-i}, \quad T_k = 2^{k-1}, \quad \text{and} \quad R = 4.$$