Tokunaga self-similarity arises naturally from time invariance

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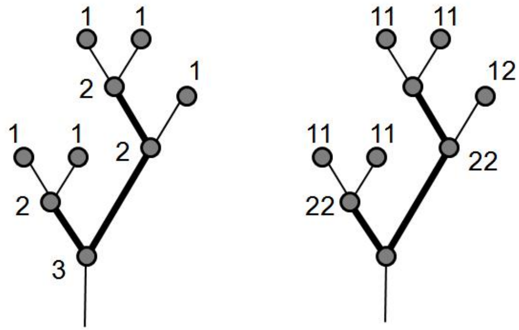
Horton-Strahler ordering.



(a) Horton-Strahler orders (b)Tokunaga indices

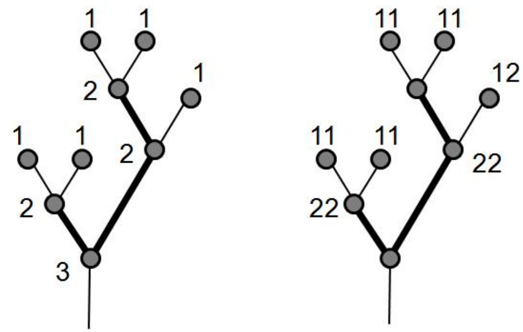
The **Horton-Strahler ordering** of the vertices of a finite rooted labeled binary tree is performed in a hierarchical fashion, from leaves to the root:

(i) each leaf has order $r(\text{leaf}) = 1$;

(ii) when both children, c_1, c_2, of a parent vertex p have orders i and j, the vertex p is assigned order

$$r = \left\lfloor \log_2(2^i + 2^j) \right\rfloor = \begin{cases} \max\{i,j\} & \text{if } i \neq j \\ i+1 & \text{if } i = j \end{cases}$$

Horton-Strahler ordering and Tokunaga indexing.



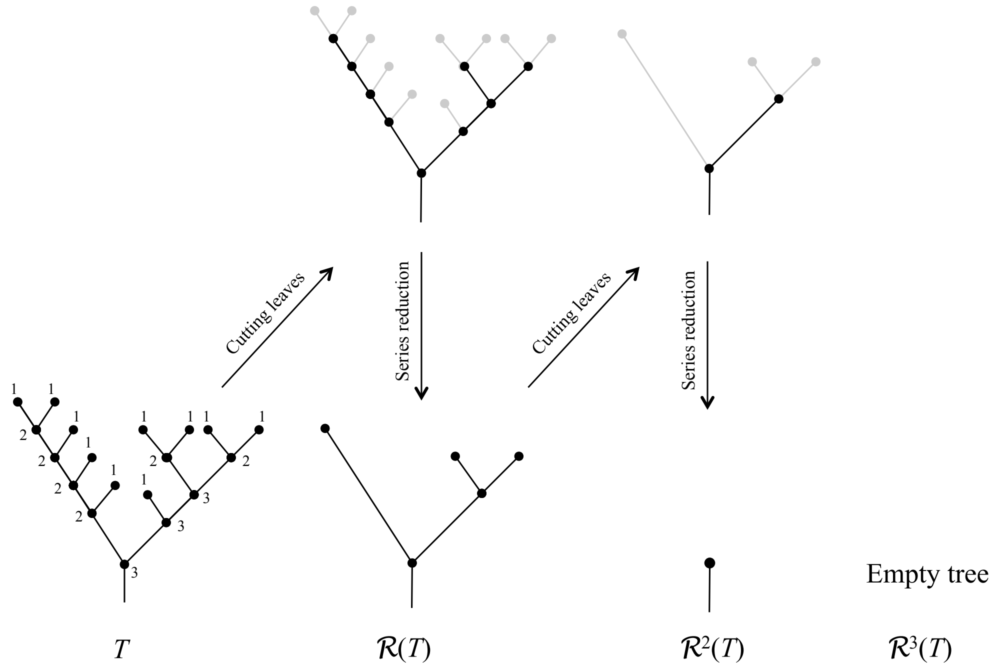
(a) Horton-Strahler orders (b)Tokunaga indices

Example: (a) Horton-Strahler ordering

(b) Tokunaga indexing.

Two order-2 branches are depicted by heavy lines in both panels. The Horton-Strahler orders refer, interchangeably, to the tree nodes or to their parent links. The Tokunaga indices refer to entire branches, and not to individual vertices.

Horton pruning of a tree mod series reduction



The order of the tree is $k(T) = 3$ with $N_1 = 10$, $N_2 = 3$, $N_3 = 1$, and $N_{1,2} = 3$, $N_{1,3} = 1$, $N_{2,3} = 1$.

Tree self-similarity

N_j – the number of branches of order j

N_{ij}, $i < j$ – the number of side branches of order $\{ij\}$, i.e. instances when an order-i branch merges with an order-j branch in a finite tree T.

The average number of branches of order i in a single branch of order j can be traced with

$$T_{ij} = \frac{\mathsf{E}[N_{ij}]}{\mathsf{E}[N_j]}$$

Tree self-similarity: $T_{ij} = T_{j-i}$ for a sequence $\{T_k\}_{k \geq 1}$.

Tokunaga self-similarity: $T_k = a\,c^{k-1}$, $k \geq 1$, $a, c > 0$.

Geometric branching process

$$\boxed{X \overset{d}{=} \mathsf{Geom}(r) \quad \text{if} \quad \mathsf{Prob}(X = k) = r(1-r)^k, \; k = 0, 1, \ldots}$$

Given a non-negative sequence $\{T_k\}_{k \geq 1}$. Let

$$S_K := 1 + T_1 + \cdots + T_K$$

for $K \geq 0$ by assuming $T_0 = 0$.

Geometric branching process:

• Markovian, where the numbers of side branches are independent.

• The number $m_{j,i}$ of side branches of order $i < j$ in a branch of order j is distributed as

$$m_{j,i} \overset{d}{=} \mathsf{Geom}\left([1 + T_{j-i}]^{-1} \right).$$

Thus, $\mathsf{E}[m_{j,i}] = T_{j-i}$.

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• Branches of order 1 have no side branches.

The independence of branches implies $\mathsf{E}\left[N_{ij}\right] = \mathsf{E}\left[N_j\right] \mathsf{E}\left[m_{j,i}\right]$ and hence

$$T_{ij} = \frac{\mathsf{E}[N_{ij}]}{\mathsf{E}[N_j]} = \frac{\mathsf{E}\left[N_j\right] \mathsf{E}\left[m_{j,i}\right]}{\mathsf{E}[N_j]} = T_{j-i}.$$

Geometric branching process: formally

(i) The process starts with $\mathsf{ord}(T) - 1 \stackrel{d}{=} \mathsf{Geom}(p)$.

(ii) At every time instant $s > 0$, each population member of order K terminates with probability S_{K-1}^{-1}, independently of other members.

At termination, a member of order $K > 1$ produces two offspring of order $(K-1)$; and a member of order $K = 1$ leaves no offsprings.

(iii) A population member of order K survives (with probability $1 - S_{K-1}^{-1}$, and produces a single offspring (side branch) of order i $(1 \le i < K)$ drawn from the distribution

$$p_{K,i} = \frac{T_{K-i}}{T_1 + \cdots + T_{K-1}}.$$

Geometric branching process

Properties:

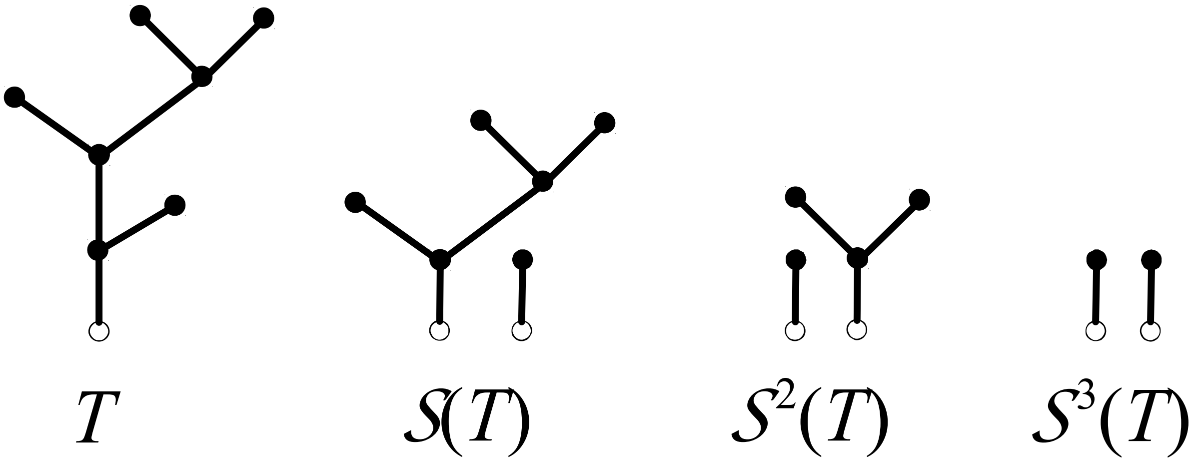
- The geometric branching process with $p = 1/2$ and $T_k = 2^{k-1}$ is the critical binary Galton-Watson tree.

- **Prune invariance:** Given an arbitrary sequence $\{T_k \geq 0\}_{k\geq 1}$ and $0 < p < 1$, the probability measure for the geometric branching process is invariant with respect to Horton pruning.

- **Easy to simulate.** Generation of geometric trees for arbitrary parameters $(p, \{T_k\})$ is easily implemented on a computer.

May facilitate the analysis in a range of simulation-heavy problems, from structure and transport on river networks to phylogenetic trees.

Time shift



Time shift operator $\mathcal{S}$ advances the process time by unity. It can be applied to individual trees and forests.

A consecutive applications of d time shifts to a tree T is equivalent to removing the vertices/edges at depth less than d from the root.

Difference equation for the state vector

Let $x_i(s)$, $i \geq 1$, denote the average number of vertices of order i at time s in a geometric branching process, and

$$\mathbf{x}(s) = (x_1(s), x_2(s), \dots)^T$$

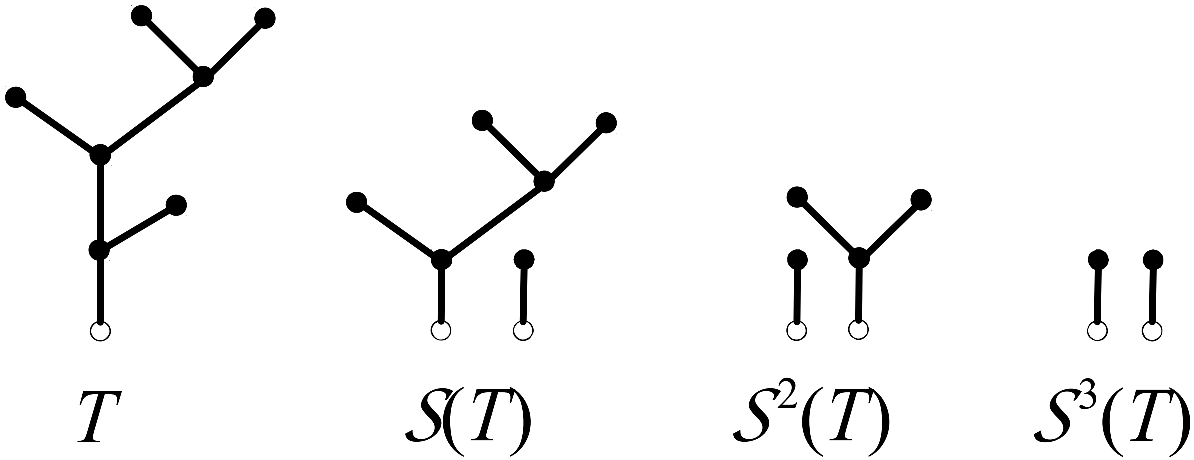
be the state vector. Then

$$\mathbf{x}(s+1) - \mathbf{x}(s) = \mathbb{G}\mathbb{S}^{-1}\mathbf{x}(s),$$

where $\mathbf{x}(0) = \pi := \sum_{K=1}^{\infty} p(1-p)^{K-1}\mathbf{e}_K$,

$$\mathbb{G} := \begin{bmatrix} -1 & T_1+2 & T_2 & T_3 & \dots \\ 0 & -1 & T_1+2 & T_2 & \dots \\ 0 & 0 & -1 & T_1+2 & \ddots \\ 0 & 0 & 0 & -1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \text{and} \quad \mathbb{S} = \mathrm{diag}\{S_0, S_1, \dots\}.$$

Time invariance



Geometric branching process is time invariant if and only if the state vector $\mathbf{x}(s)$ is invariant with respect to a unit time shift $\mathcal{S}$:

$$\mathbf{x}(s) = \mathbf{x}(0) \equiv \pi \quad \forall s \iff \mathbb{G}\mathbb{S}^{-1}\pi = \mathbf{0}.$$

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Theorem (YK and I. Zaliapin, Chaos 2018).
A geometric branching process is time invariant if and only if

$$p = 1/2 \quad \text{and} \quad T_k = (c-1)c^{k-1} \text{ for } c \geq 1.$$

We will call this a critical Tokunaga process, and the respective trees – critical Tokunaga trees.

Time invariance

Recall that $S_0 = 1$, and for $K \geq 1$,

$$S_K := 1 + T_1 + \cdots + T_K$$

First, we established the following.

Lemma 1. A geometric branching process is time invariant if and only if $p = 1/2$ and the sequence $\{T_k\}$ solves the following (nonlinear) system of equations:

$$\frac{S_0}{S_k} = \sum_{i=1}^{\infty} 2^{-i} \frac{S_i}{S_{k+i}} \quad \text{for all } k \geq 1.$$

Time invariance

Let $a_k = S_k/S_{k+1} \leq 1$ for all $k \geq 0$. Then, for any $i \geq 0$ and any $k > 0$ we have $S_i/S_{k+i} = a_i\, a_{i+1} \ldots a_{i+k-1}$. The system

$$\frac{S_0}{S_k} = \sum_{i=1}^{\infty} 2^{-i} \frac{S_i}{S_{k+i}} \quad \text{for all } k \geq 1$$

rewrites in terms of a_i as

$$\begin{aligned}
\frac{1}{2}a_1 + \frac{1}{4}a_2 + \frac{1}{8}a_3 + \ldots &= a_0, \\
\frac{1}{2}a_1a_2 + \frac{1}{4}a_2a_3 + \frac{1}{8}a_3a_4 + \ldots &= a_0a_1, \\
\frac{1}{2}a_1a_2a_3 + \frac{1}{4}a_2a_3a_4 + \frac{1}{8}a_3a_4a_5 + \ldots &= a_0a_1a_2,
\end{aligned}$$

and so on ...

Time invariance

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Lemma 2. The system

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \prod_{k=j}^{n+j-1} a_k = \prod_{k=0}^{n-1} a_k, \text{ for all } n \in \mathbb{N}$$

with $a_0 = \frac{1}{c}$ $(c > 0)$ has a unique solution $a_0 = a_1 = a_2 = \ldots = 1/c$.

Once established, Lemma 1 and Lemma 2 imply our main result (Theorem).

- **Zaliapin and YK (CSF 2012):** Extreme values and level-set trees of time series via Horton pruning.

- **YK and Zaliapin (Fractals 2016):** Tree self-similarity with $\limsup_{j\to\infty} T_j^{1/j} < \infty$ implies (strong) Horton law.

- **YK & Zaliapin (Annales Inst. H. Poincaré 2017):** Established a (weak) Horton law for the Kingman's coalescent.

- **YK and I. Zaliapin (2017) arXiv:1608.05032** A novel multi-type branching processes is considered: the Hierarchical Branching Processes.

- **M. Arnold, YK, I. Zaliapin (2018) arXiv:1707.01984** Generalized dynamical pruning with applications in continuum ballistic annihilation.

- **YK and I. Zaliapin (Chaos 2018):** This talk.