Critical Tokunaga model for river networks

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Critical Tokunaga model.

Critical Tokunaga Trees (aka Critical Tokunaga Model) is a one-parametric family of random trees that can be defined either as a continuous-time Multi-type Branching Process or as a Random Attachment Model (RAM).

In the RAM, we use Poisson attachment construction within exponential segments. This ensures that the link lengths have exponential distribution and the attachment of streams of lower orders to a given stream of a larger order is done in uniform random fashion.

The Critical Tokunaga Trees is a sub-family of a much larger family of random trees, called Hierarchical Branching Processes. Critical Tokunaga Model includes the celebrated Shreve's random topology model ($c = 2$).

The Random Attachment Model (RAM).

Next, we construct Critical Tokunaga model as a RAM.

For a given $c > 1$, consider a discrete time Markov tree process $\{\Upsilon_K\}_{K\in\mathbb{N}}$ such that each Υ_K is distributed as a tree of Horton-Strahler order K.

Let $X_K = N_1[\Upsilon_K]$ (number of leaves) and $Y_K = \text{length}(\Upsilon_K)$.

- Υ_1 is I-shaped tree of order one, with $X_1 = 1$ and $Y_1 \stackrel{d}{\sim} \text{Exp}(\gamma)$.

- Conditioned on Υ_K, tree Υ_{K+1} is obtained as follows:

(i) Attach new leaf edges to Υ_K at the points sampled with a homogeneous Poisson point process with intensity $\gamma(c-1)$ along the carrier space Υ_K.

(ii) Attach a pair of new leaf edges to each of the leaves in Υ_K.

The lengths of all the newly attached leaf edges are independent exponential random variables with parameter γc^K.

Horton's laws for stream numbers and magnitudes.

• Let $\mathcal{N}_i[K]$ denote the mean number of streams of order i in a basin of order K.

• Let M_i denote the mean magnitude (number of upstream sources) of a stream of order i.

In Critical Tokunaga Model, the following Horton's laws hold:

$$\lim_{K\to\infty} \frac{\mathcal{N}_i[K]}{\mathcal{N}_{i+1}[K]} = R_B \quad \text{for any } i, \text{ and } \lim_{i\to\infty} \frac{M_{i+1}}{M_i} = R_M,$$

where $R_B = R_M = 2c$.

Moreover,

$$\mathcal{N}_{K-j+1}[K] = \mathcal{N}_1[j] = M_j = \frac{R_B^j + R_B - 2}{2(R_B - 1)} \quad \text{for } 1 \le j \le K.$$

Other relevant Horton laws.

• Let L_j denote the mean length of a stream of order j.

Horton's law for the stream lengths:

$$L_j R_L^{-j} = \frac{1}{c\gamma} < \infty \quad \text{with} \quad R_L = c.$$

• Let Λ_k denote the mean length of the longest stream in a basin of order k.

Horton's law for the length of the longest stream:

$$\lim_{k\to\infty} \Lambda_k R_\Lambda^{-k} = \text{Const.} < \infty \quad \text{with} \quad R_\Lambda = c.$$

Fractal dimension and Hack's law.

Fractal dimension for Critical Tokunaga Tree: Consider the limiting tree in RAM. Then,

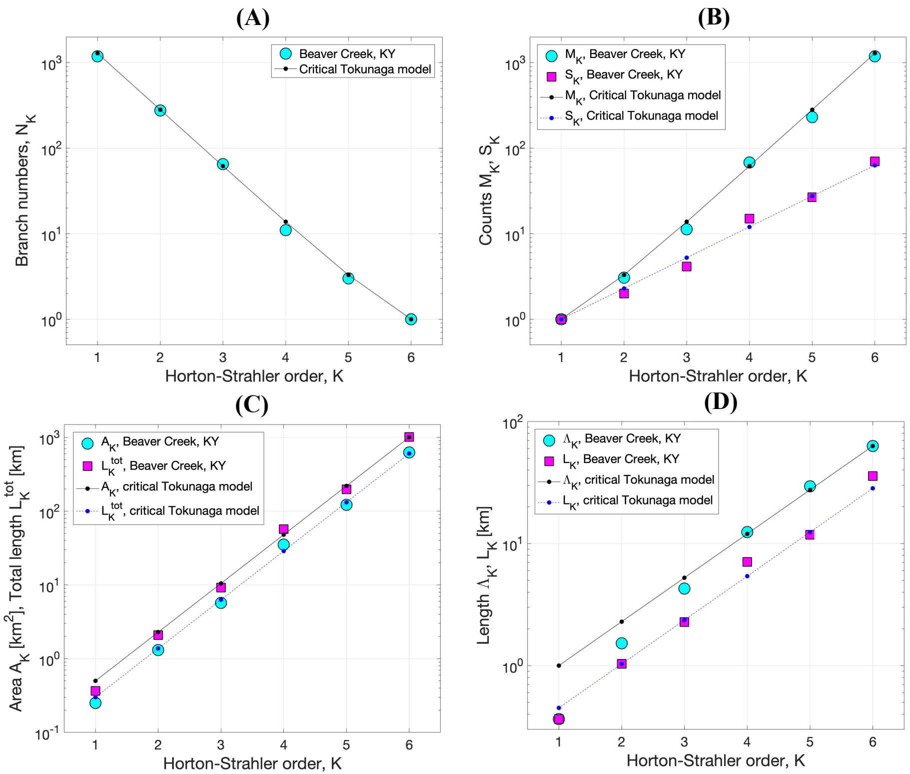
$$\mathbf{d} = \max\{1, \mathbf{d}_0\}, \quad \text{where} \quad \mathbf{d}_0 = \frac{\log R_B}{\log R_L} = \frac{\log(2c)}{\log c}.$$

• Assume the local contributing area A_i of a link of order i (area that contributes to the link directly, and not via its descendants) is a function of the link length.

Hack's law for the Critical Tokunaga Tree:

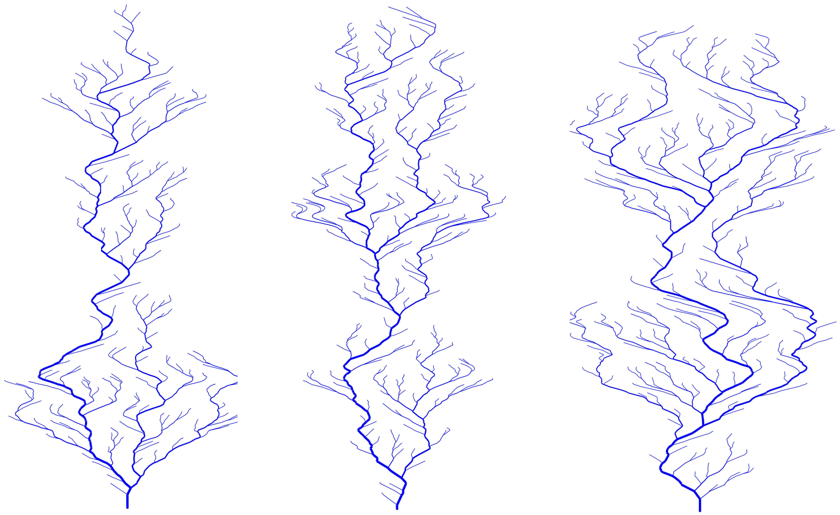
$$\Lambda_i \sim \text{Const.} \times \left(A_i\right)^{\mathbf{h}}, \text{ where } \mathbf{h} = \mathbf{d}^{-1} = \frac{\log R_L}{\log R_B} = \frac{\log c}{\log(2c)}.$$

Critical Tokunaga Model closely fits observations.



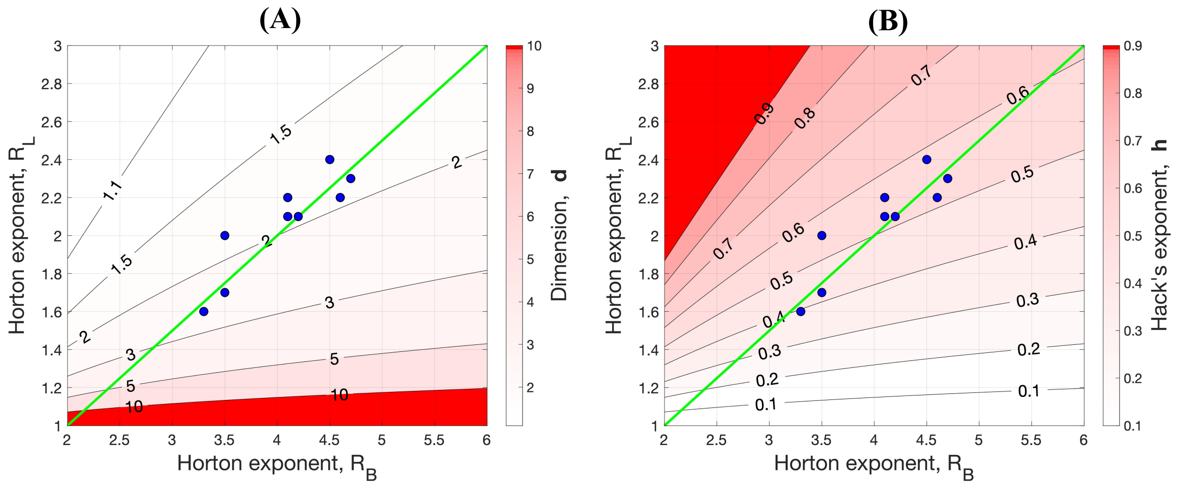
Critical Tokunaga ($c = 2.3$) fit to the Horton laws in Beaver creek, KY.

RAM for simulations.



The trees are generated by the critical Tokunaga process with $c = 2.3$ and Horton-Strahler order $K = 5$. The line width is proportional to the contributing area. The figure accurately represents the tree combinatorial structure; the edge lengths are scaled for a better planar embedding. Notice that the RAM generates trees with no planar embedding.

Fractal dimension and Hack's exponent.



Fractal dimension $\mathbf{d}$ (panel **A**) and Hack's exponent $\mathbf{h} = \mathbf{d}^{-1}$ (panel **B**) of a self-similar RAM (aka Hierarchical Branching Process) in the limit of infinite size as a function of the Horton's exponents R_B and R_L. Selected levels of $\mathbf{d}$ and $\mathbf{h}$ are shown by marked black lines. Green thick lines correspond to the Critical Tokunaga Model for which $R_B = 2R_L$. Blue dots depict the pairs (R_B, R_L) estimated in nine real river basins.

Other important properties of Critical Tokunaga Model.

Tokunaga self-similarity:

$$T_{i,j} = T_{j-i} \quad \text{with} \quad T_k = a\, c^{k-1}, \quad a = c - 1.$$

Horton prune-invariance, criticality, coordination, time-invariance, identically distributed link lengths, and identically distributed local contributing areas.

This class is surprisingly rich, extending from perfect binary trees ($c = 1$) to the famous Shreve's random topology model ($c = 2$) to the structures reminiscent of the observed river networks ($c \approx 2.3$) and beyond.

Critical Tokunaga model is merely a subclass of a much broader family of self-similar trees. However, the observed stream networks cluster around $R_B = 2R_L$.

Critical Tokunaga process.

Critical Tokunaga Processes satisfy a number of self-similarity and invariance properties as observed in multiple publications, e.g.

• Y. K., Ilya Zaliapin, and Efi Foufoula-Georgiou , "Critical Tokunaga model for river networks" Physical Review E Vol. 105 (2022), 014301

• Y. K., Ilya Zaliapin, and Efi Foufoula-Georgiou , "Random Self-Similar Trees: Emergence of Scaling Laws" Surveys in Geophysics Vol. 43 (2022), 353–421

• Y. K. and Ilya Zaliapin, "Random Self-Similar Trees: A mathematical theory of Horton laws" Probability Surveys Vol. 17 (2020), 1–213