Random Trees and Their Applications: Metric Trees

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collaboration with
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What is similar between the following two dynamics ?

Model #1. Ballistic annihilation: $A + A \longrightarrow 0$

- *Y. Elskens and H. L. Frisch, Phys. Rev. A (1985)*
- *E. Ben-Naim and S. Redner, PRL (1993)*
- *R. A. Blythe, M. R. Evans, and Y. Kafri, PRL (2000)*

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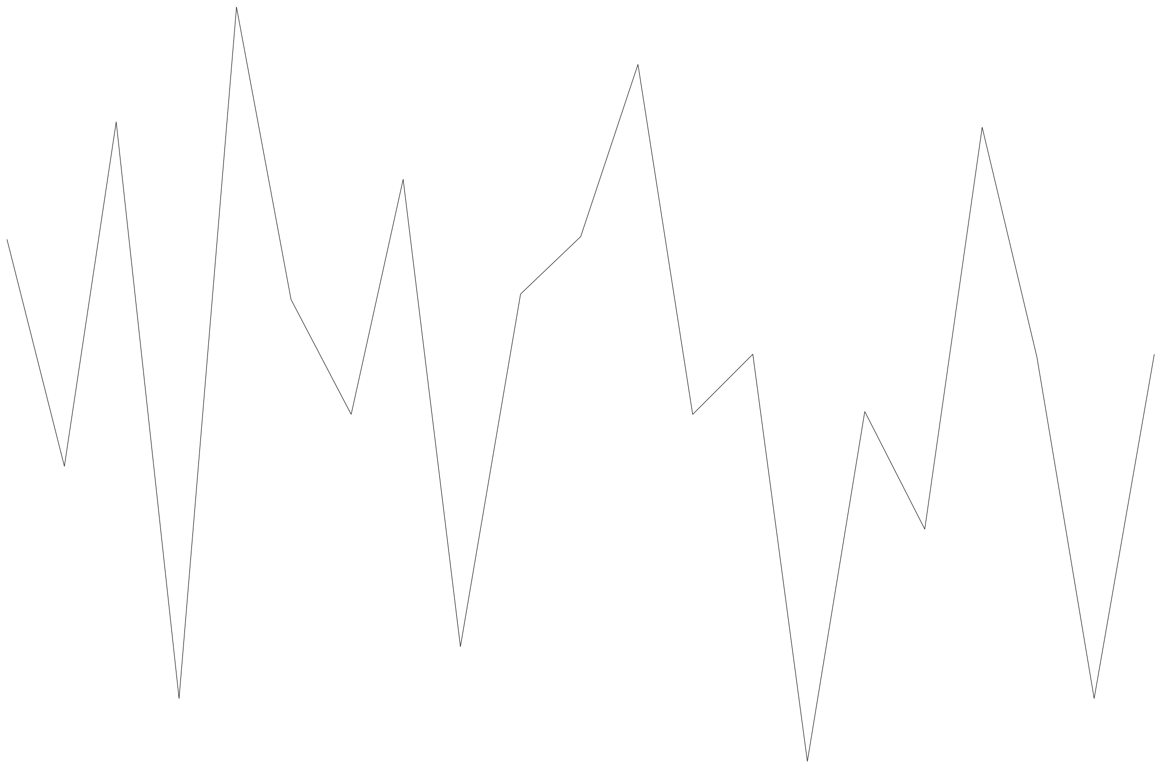
STOP!!!

Model #2. Extreme values.



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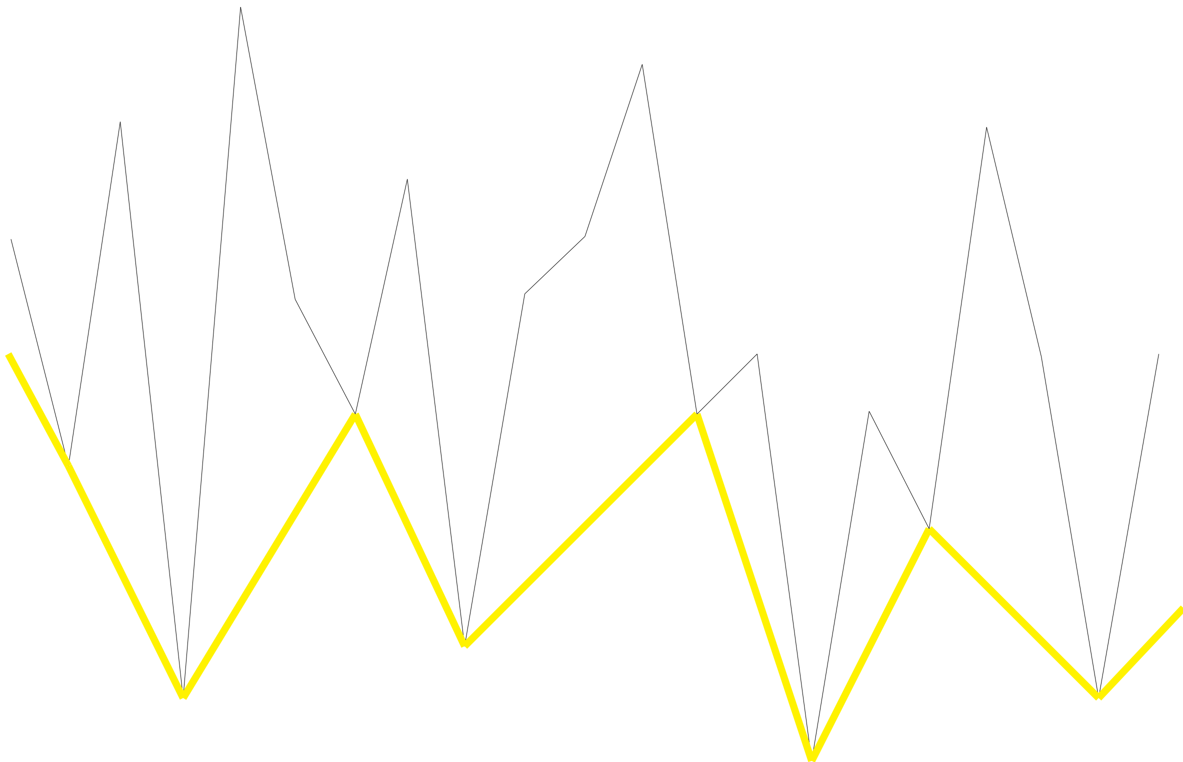

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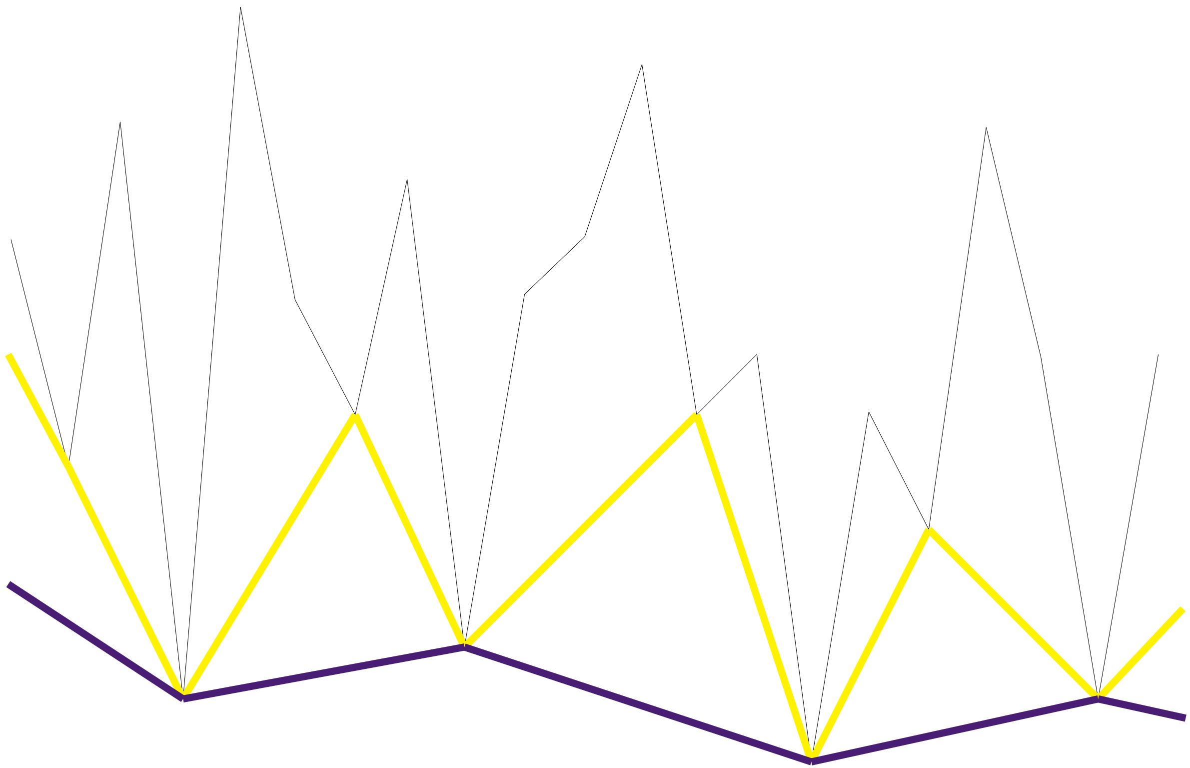

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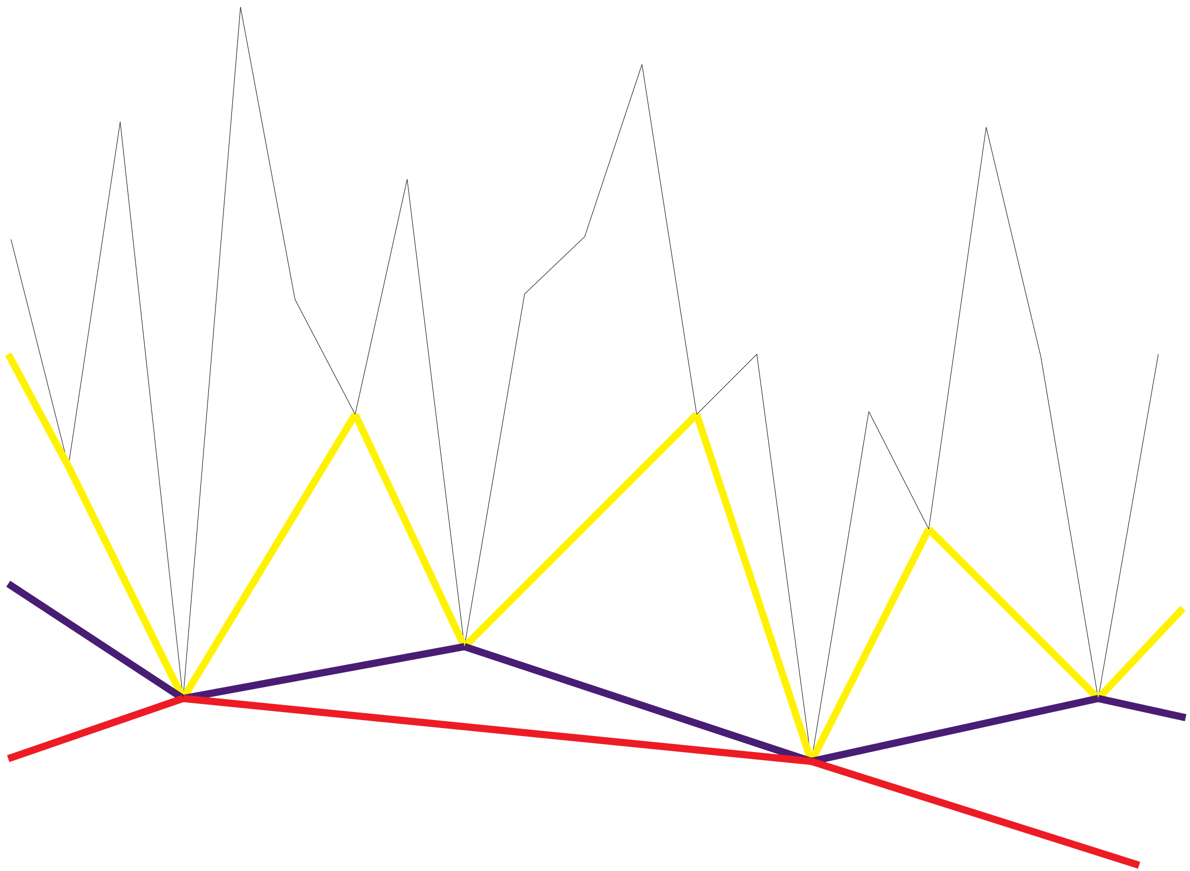

Tree pruning

Tree pruning: **YK and I. Zaliapin (2018)** *arXiv:1707.01984*

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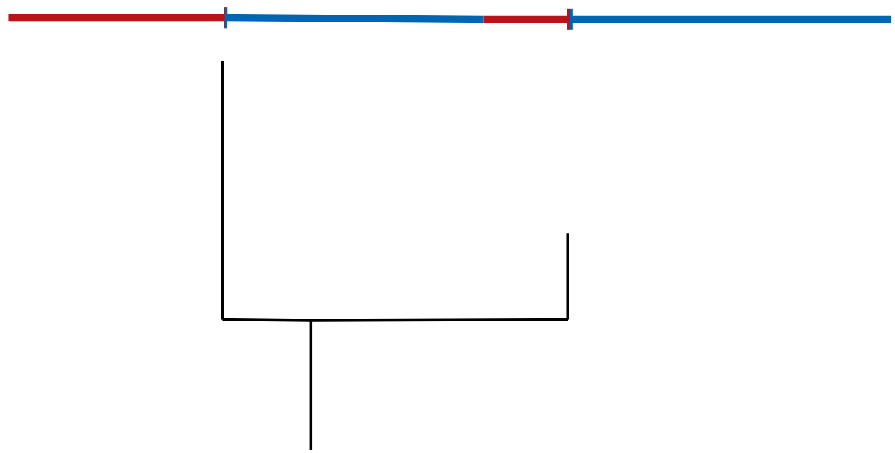

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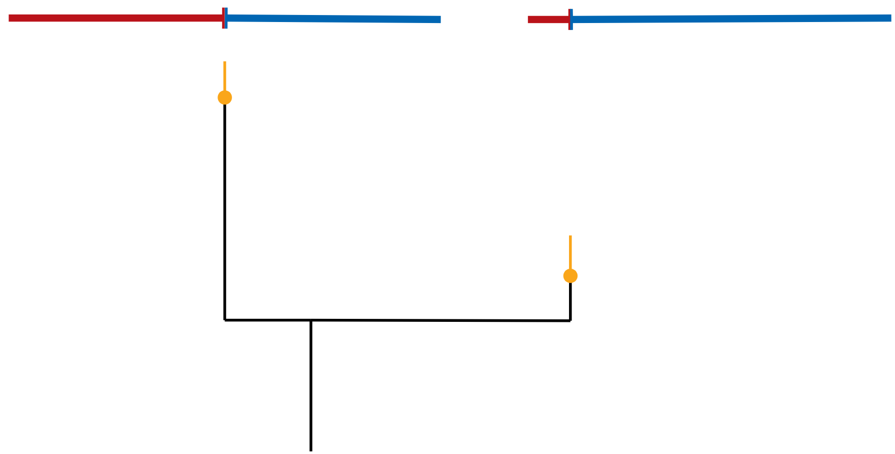

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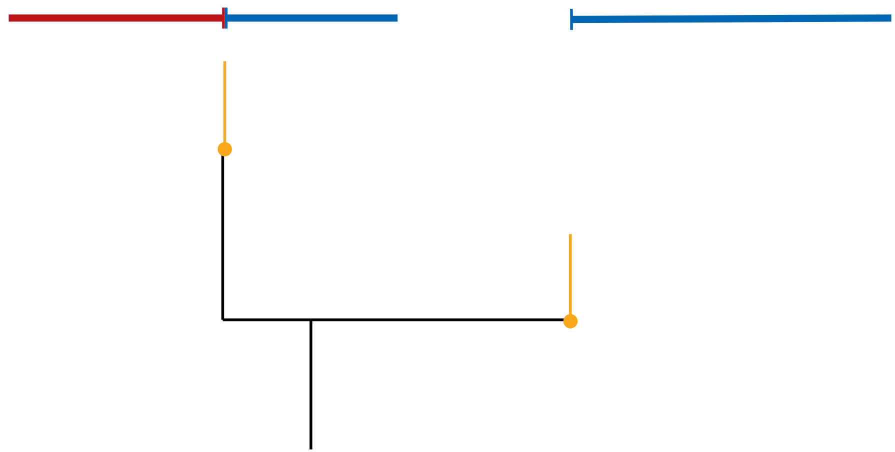

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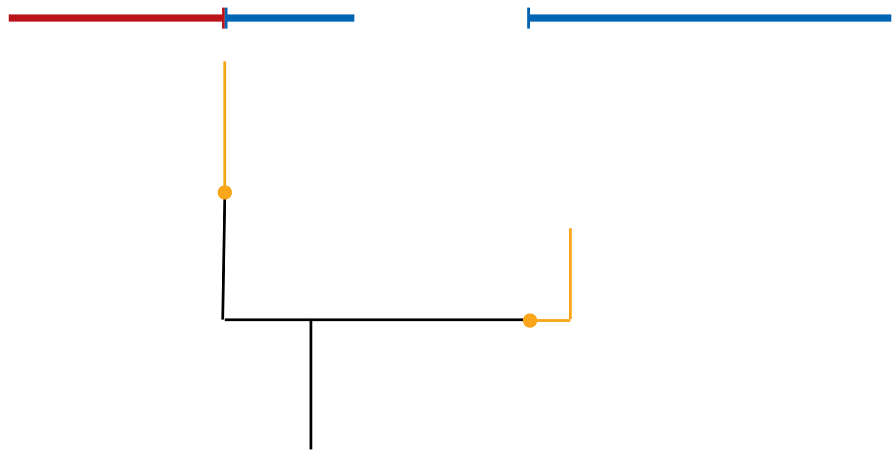

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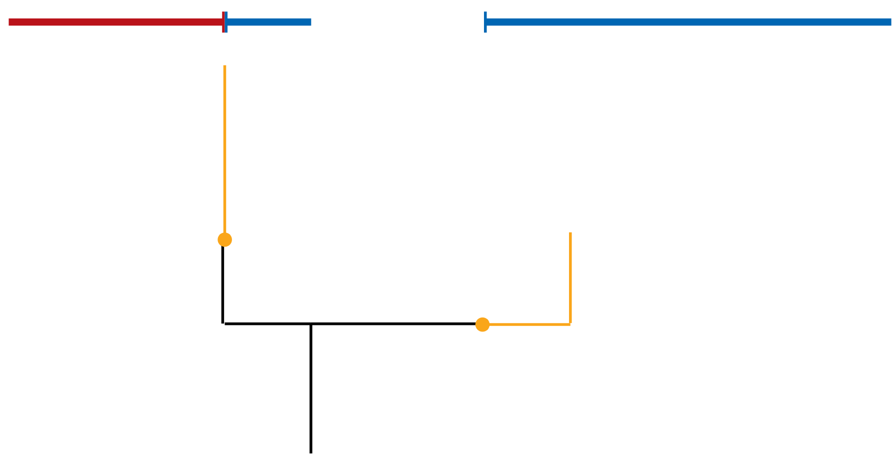

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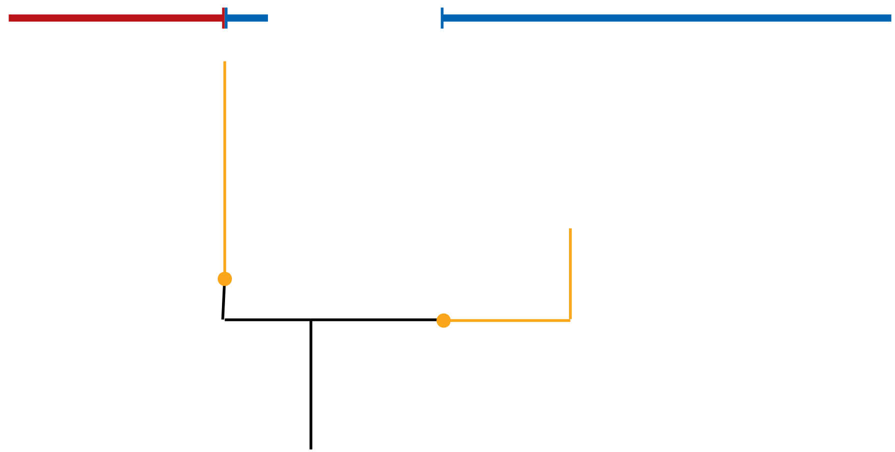

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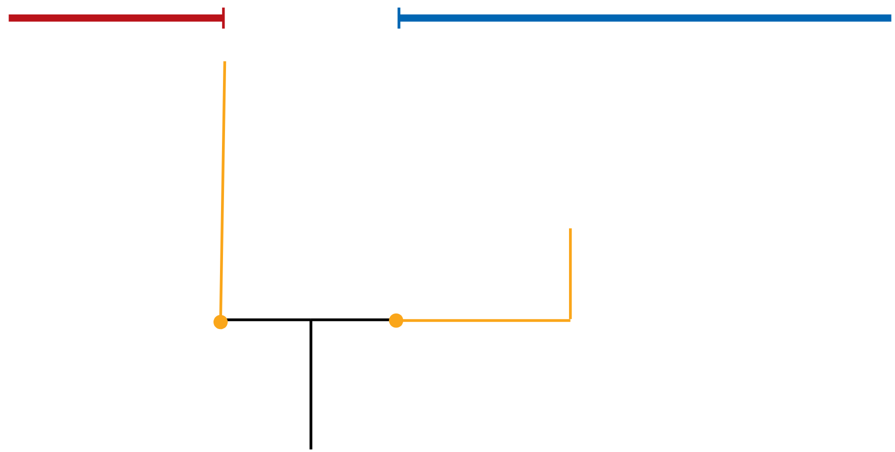

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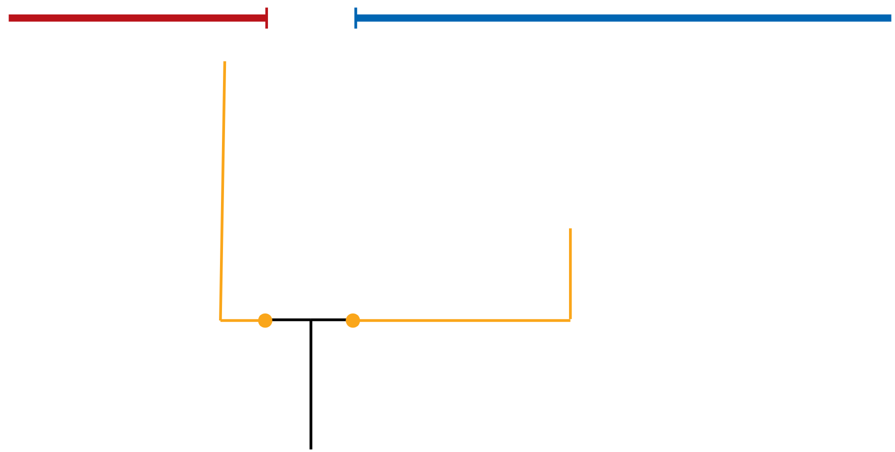

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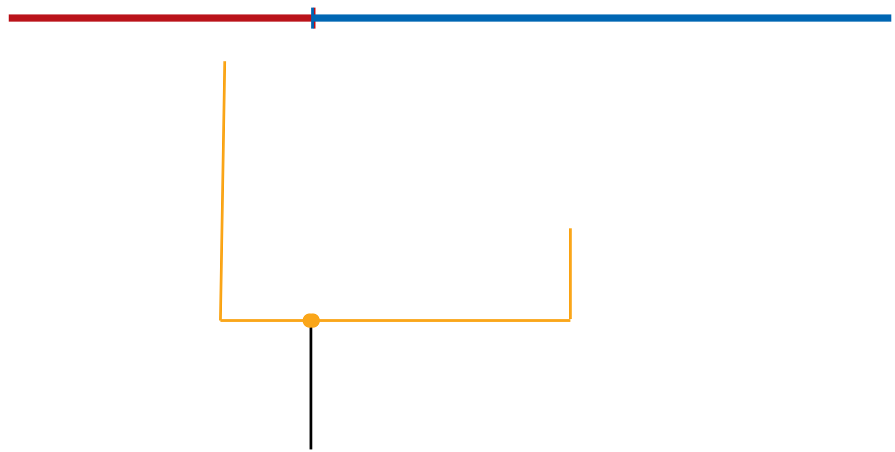

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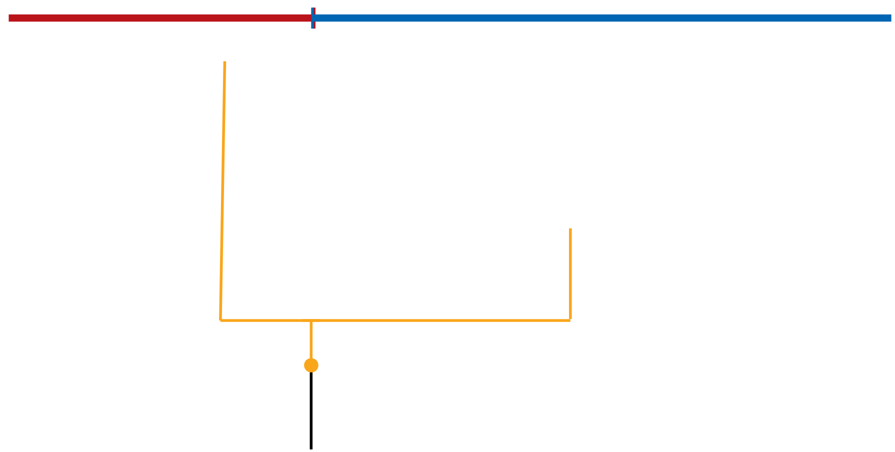

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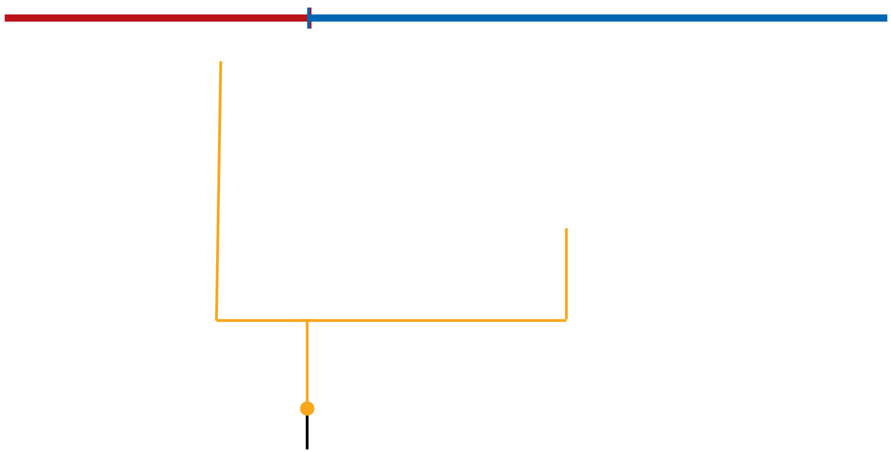
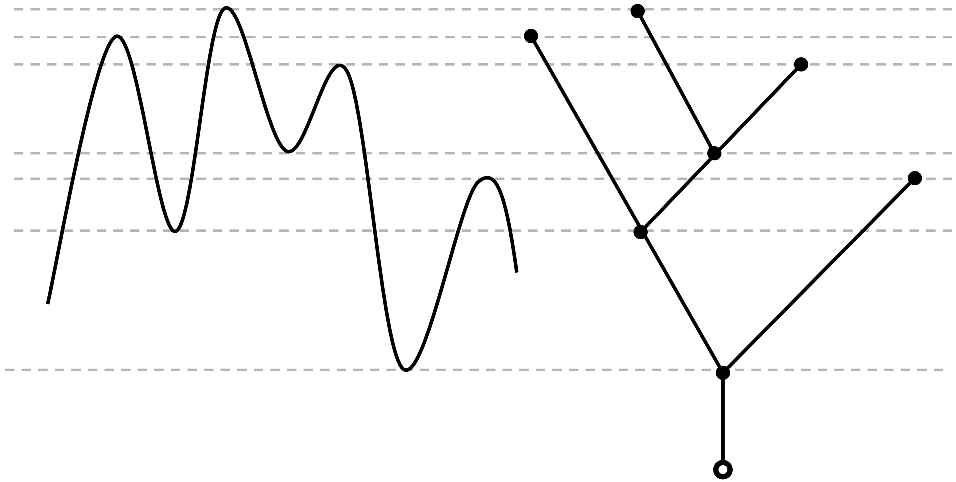


Level-set tree of a function.

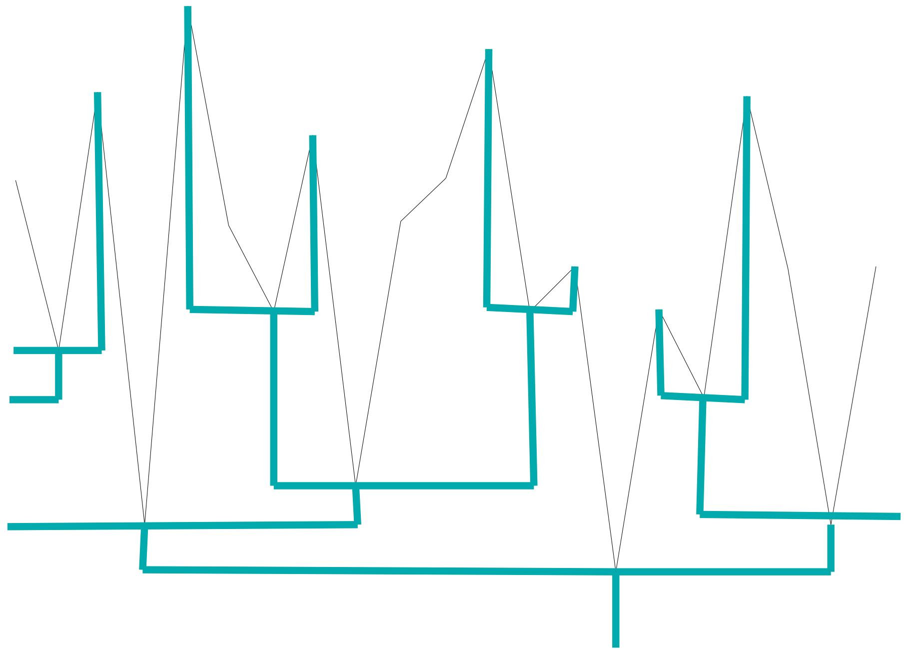


(a) Function X_t (b) Tree $\textsc{level}(X)$

Function X_t (panel a) with a finite number of local extrema and its level-set tree level(X) (panel b).

Tree pruning: **Zaliapin and YK, (2012)**

Model #2. Extreme values.



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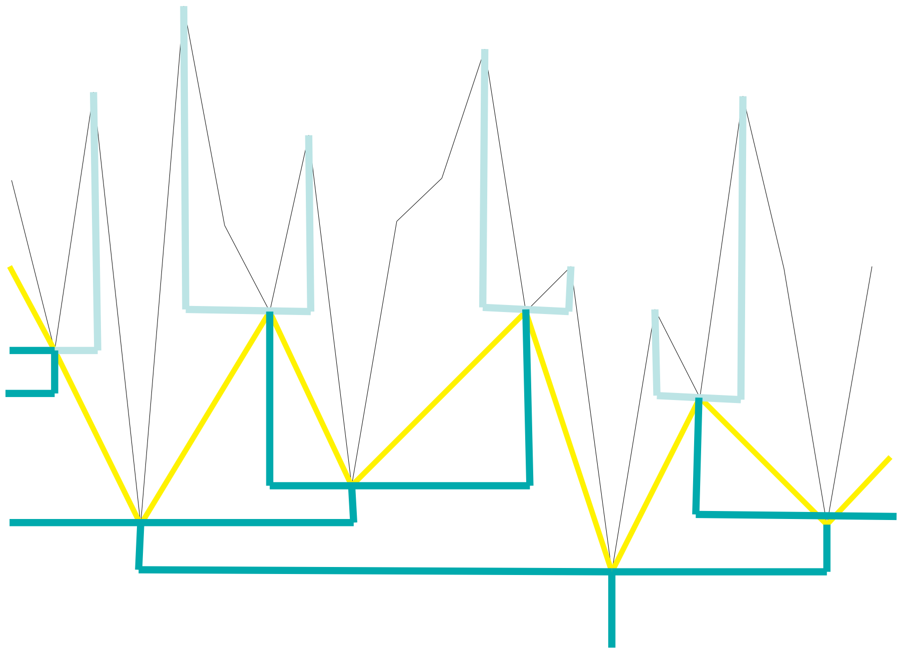

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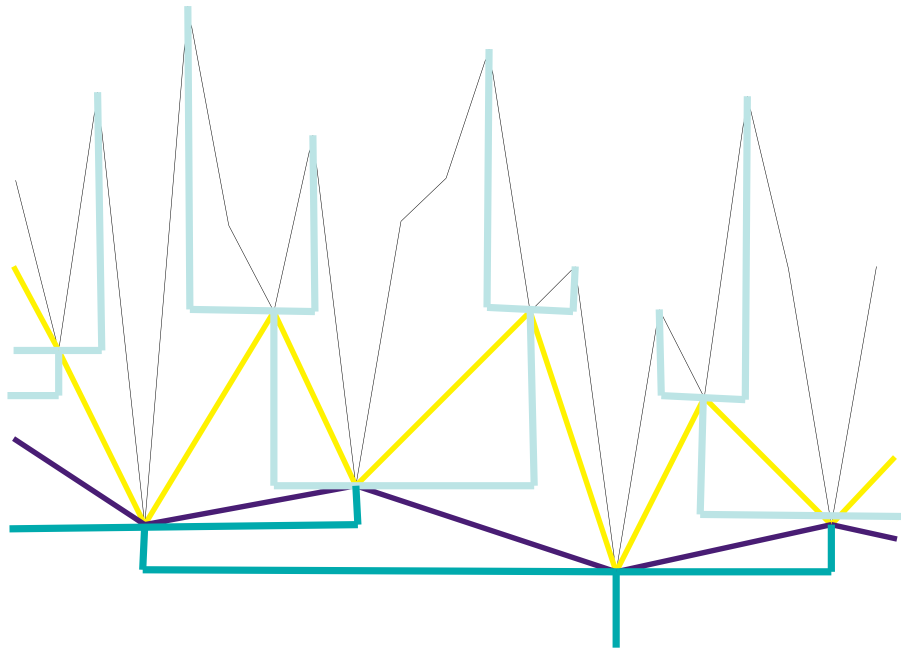

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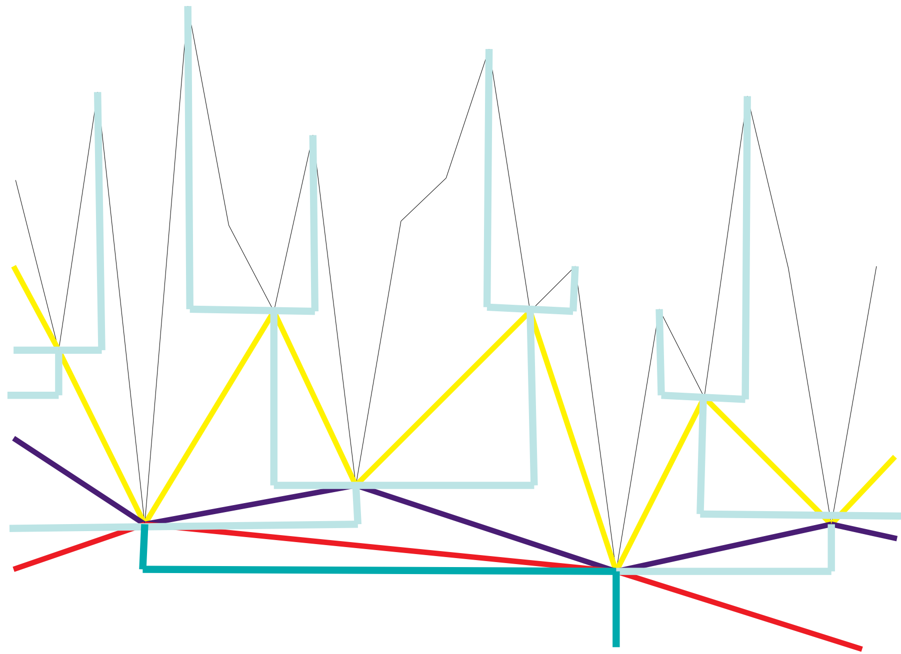


Trees.

$\mathcal{L}_{\text{plane}}$ - space of finite unlabeled rooted reduced binary trees with edge lengths and planar embedding.

The space $\mathcal{L}_{\text{plane}}$ includes the empty tree $\phi = \{\rho\}$ comprised of a root vertex ρ and no edges.

$d(x, y)$: the length of the minimal path within T between x and y.

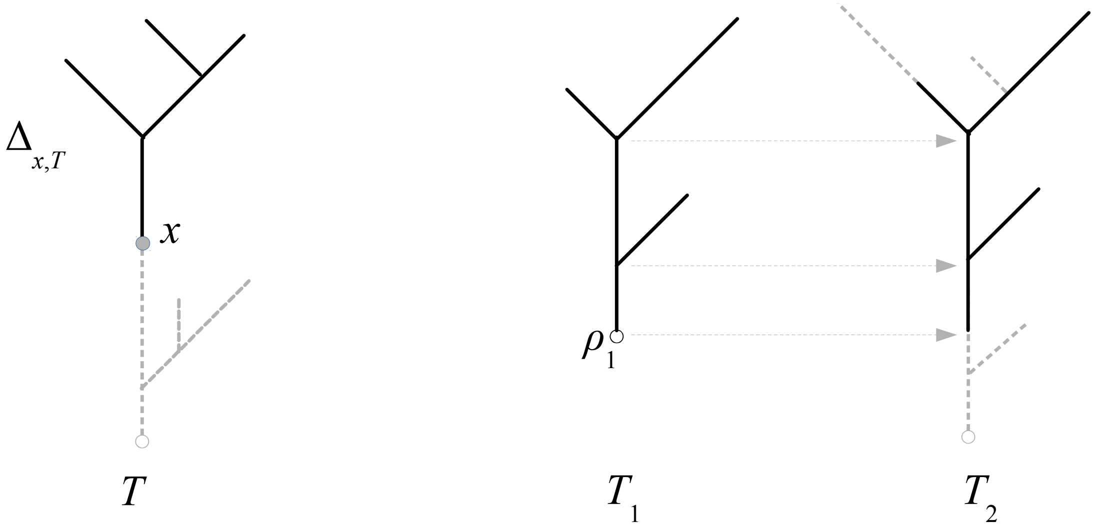
The length of a tree T is the sum of the lengths of its edges:

$$\text{length}(T) = \sum_{i=1}^{\#T} l_i.$$

The height of a tree T is the maximal distance between the root and a vertex:

$$\text{height}(T) = \max_{1 \le i \le \#T} d(v_i, \rho).$$

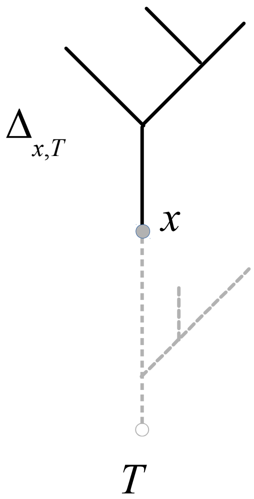
Partial ordering.



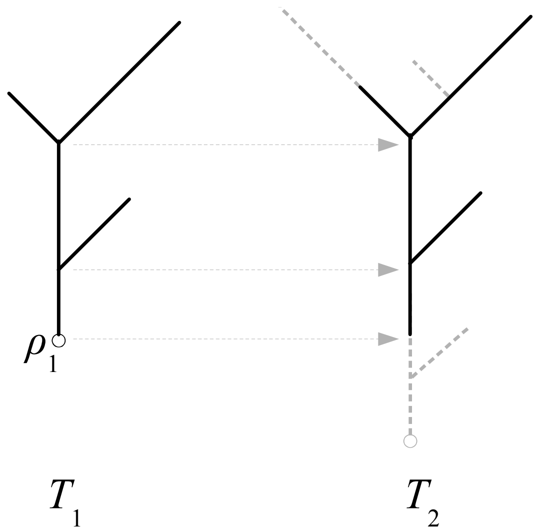
(a) Descendant tree (b) Isometry

Consider a tree $T \in \mathcal{L}_{\text{plane}}$ and a point $x \in T$. Let $\Delta_{x,T}$ denote all points of T descendant to x, including x. Then $\Delta_{x,T}$ is itself a tree in $\mathcal{L}_{\text{plane}}$ with root at x.

Partial ordering. Let (T_1, d) and (T_2, d) be two metric rooted trees, and let ρ_1 denote the root of T_1.



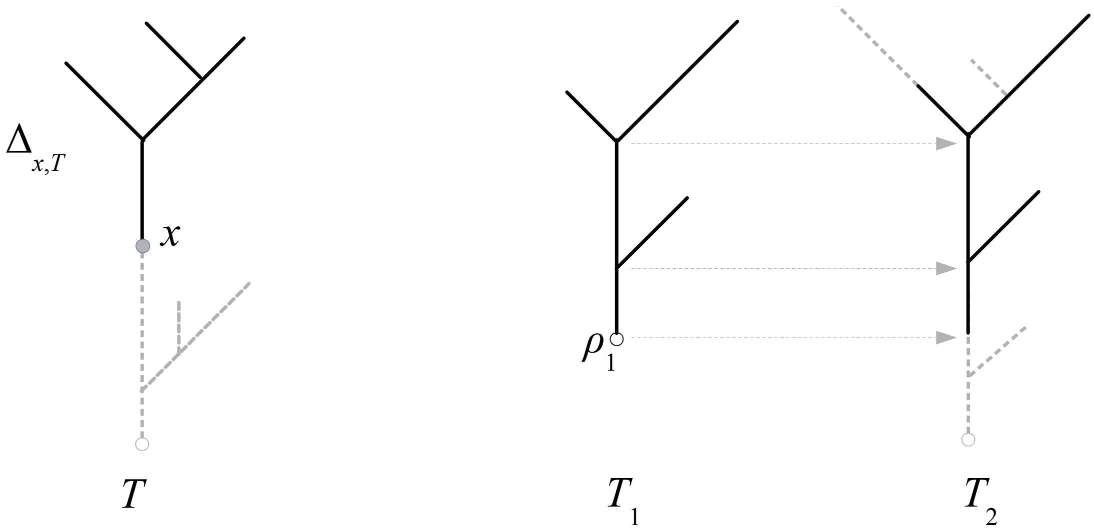
(a) Descendant tree



(b) Isometry

$f : (T_1, d) \to (T_2, d)$ is an isometry if Image$[f] \subseteq \Delta_{f(\rho_1),T_2}$ and $\forall x, y \in T_1$, $d\big(f(x), f(y)\big) = d(x, y)$.

Partial ordering.



(a) Descendant tree (b) Isometry

Partial order: $T_1 \preceq T_2$ if and only if $\exists$ an isometry $f : (T_1, d) \to (T_2, d)$.

Generalized dynamical pruning.

Consider a monotone non-decreasing

$$\varphi : \mathcal{L}_{\text{plane}} \to \mathbb{R}^+,$$

i.e. $\varphi(T_1) \le \varphi(T_2)$ whenever $T_1 \preceq T_2$.

Generalized dynamical pruning operator

$$\mathcal{S}_t(\varphi, T) : \mathcal{L}_{\text{plane}} \to \mathcal{L}_{\text{plane}}$$

induced by φ at any $t \ge 0$:

$$\mathcal{S}_t(\varphi, T) := \rho \cup \Big\{ x \in T \setminus \rho \ : \ \varphi\big(\Delta_{x,T}\big) \ge t \Big\}.$$

$\mathcal{S}_t$ cuts all subtrees $\Delta_{x,T}$ for which the value of φ is below threshold t. Here,

$$S_s(T) \preceq S_t(T)$$

whenever $s \ge t$.

Example: Tree height.

Recall: $$\mathcal{S}_t(\varphi, T) := \rho \cup \Big\{x \in T \setminus \rho \ : \ \varphi\big(\Delta_{x,T}\big) \geq t\Big\}.$$

Let the function $\varphi(T)$ equal the height of T:

$$\varphi(T) = \text{height}(T).$$

Continuous semigroup property: $\mathcal{S}_t \circ \mathcal{S}_s = \mathcal{S}_{t+s}$ for any $t, s \geq 0$.

It coincides with the tree erasure **Neveu (1986)**.

Neveu (1986): established invariance of a critical and sub-critical binary Galton-Watson processes with i.i.d. exponential edge lengths with respect to the tree erasure.

Example: Total tree length.

Recall: $$\mathcal{S}_t(\varphi, T) := \rho \cup \Big\{x \in T \setminus \rho \ : \ \varphi\big(\Delta_{x,T}\big) \geq t\Big\}.$$

Let the function $\varphi(T)$ equal the total lengths of T:

$$\varphi(T) = \text{length}(T).$$

No semigroup property!

In this case the pruning operator $\mathcal{S}_t$ coincides with the potential dynamics of 1D ballistic annihilation

YK and I. Zaliapin (2018) - arXiv:1707.01984

Example: Horton pruning.

Let

$$\varphi(T) = \mathsf{k}(T) - 1,$$

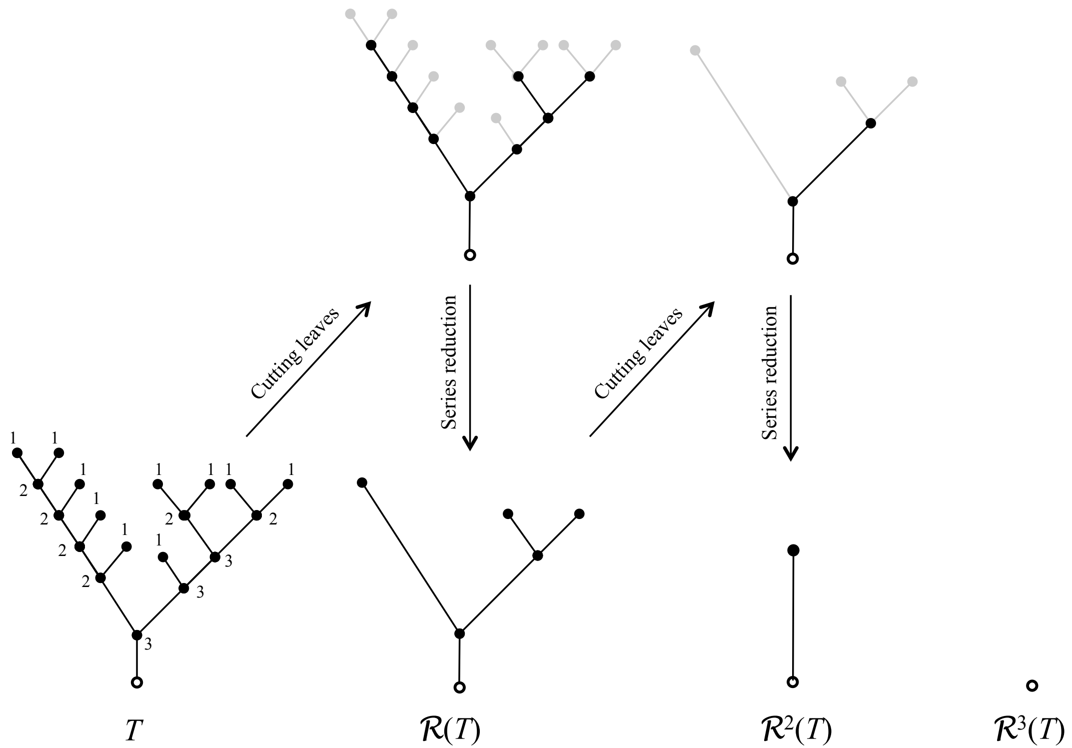
where the Horton-Strahler order $\mathsf{k}(T)$ is the minimal number of Horton prunings $\mathcal{R}$ (cutting the tree leaves and applying series reduction) necessary to eliminate all points in tree T except ρ.
Burd, Waymire, and Winn 2000.

Here,

$$\mathcal{S}_t = \mathcal{R}^{\lfloor t \rfloor}.$$

Discrete semigroup property: $\mathcal{S}_t \circ \mathcal{S}_s = \mathcal{S}_{t+s}$ for any $t, s \in \mathbb{N}$.

Pruning of a tree mod series reduction



Exponential critical binary Galton-Watson tree

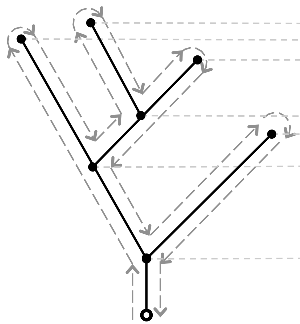
We say that a random tree $T \in \mathcal{L}_{\text{plane}}$ is an exponential critical binary Galton-Watson tree with parameter $\lambda > 0$, and write $T \stackrel{d}{=} \mathsf{GW}(\lambda)$, if

(i) shape(T) is a critical binary Galton-Watson tree;

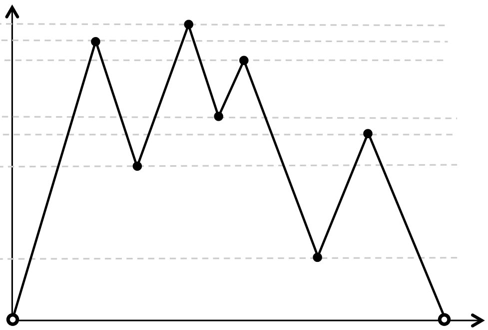
(ii) the orientation for every pair of siblings in T is uniformly random and symmetric;

(iii) given shape(T), the edges of T are sampled as independent exponential random variables with parameter λ.

Exponential critical binary Galton-Watson tree



(a) Tree T



(b) Harris path H_T

The level set tree level(X_t) is an exponential critical binary Galton-Watson tree GW(λ) if and only if the rises and falls of X_t, excluding the last fall, are distributed as independent exponential random variables with parameter $\lambda/2$.

J. Neveu and J. Pitman (1989), J. F. Le Gall (1993)

Invariance under pruning

Theorem. [YK and I. Zaliapin, 2018]

Let $T \stackrel{d}{=} \mathsf{GW}(\lambda)$ be an exponential critical binary Galton-Watson tree with parameter $\lambda > 0$.

Then, for any monotone non-decreasing function $\varphi : \mathcal{L}_{\text{plane}} \to \mathbb{R}^+$ we have

$$T^t := \big\{\mathcal{S}_t(\varphi, T) | \mathcal{S}_t(\varphi, T) \neq \phi\big\} \stackrel{d}{=} \mathsf{GW}\big(\lambda p_t\big(\lambda, \varphi\big)\big),$$

where $p_t(\lambda, \varphi) = \mathsf{P}(\mathcal{S}_t(\varphi, T) \neq \phi)$.

That is, the pruned tree T^t conditioned on surviving is an exponential critical binary Galton-Watson tree with parameter

$$\mathcal{E}_t(\lambda, \varphi) = \lambda p_t(\lambda, \varphi).$$

Invariance under pruning

Theorem. [YK and I. Zaliapin, 2018]

(a) If $\varphi(T)$ equals the total length of T ($\varphi = \text{length}(T)$), then

$$\mathcal{E}_t(\lambda,\varphi) = \lambda e^{-\lambda t}\Big[I_0(\lambda t) + I_1(\lambda t)\Big].$$

(b) If $\varphi(T)$ equals the height of T ($\varphi = \text{height}(T)$), then

$$\mathcal{E}_t(\lambda,\varphi) = \frac{2\lambda}{\lambda t + 2}.$$

(c) If $\varphi(T) + 1$ equals the Horton-Strahler order of the tree T, then

$$\mathcal{E}_t(\lambda,\varphi) = \lambda 2^{-\lfloor t \rfloor}.$$

Distributional prune-invariance

Definition. Consider a probability measure μ on $\mathcal{L}_{\text{plane}}$ such that $\mu(\phi) = 0$. Let

$$\nu(T) = \mu \circ \mathcal{S}_t^{-1}(T) = \mu\big(\mathcal{S}_t^{-1}(T)\big).$$

Measure μ is called invariant with respect to the pruning operator $\mathcal{S}_t(\varphi, T)$ if for any tree $T \in \mathcal{L}_{\text{plane}}$ we have

$$\mu(T) = \nu(T | T \neq \phi).$$

Also need the invariance of the distribution of edge lengths in the pruned tree $T_t := \mathcal{S}_t(\varphi, T)$.

YK and I. Zaliapin, SPA 2019

Open question: finding and classifying all the invariant probability measures μ on $\mathcal{L}_{\text{plane}}$.

Poisson distributed initial conditions.

Consider the following **initial conditions**:

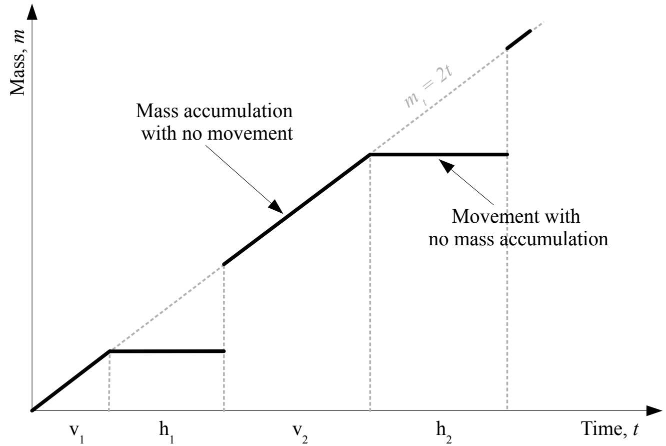
- a Poisson point process on $\mathbb{R}$ with rate $\lambda/2$;
- $v(x,0)$ alternates between the values ± 1.

Theorem. [YK and I. Zaliapin, 2018] The absorbed mass of a random sink at instant $t > 0$ has distribution with the p.d.f.

$$\mu_t(a) = \mathbf{1}_{(0,2t)}(a) \cdot \frac{\lambda}{2} e^{-\lambda t} \Big[I_0\big(\lambda(t - a/2)\big) + I_1\big(\lambda(t - a/2)\big) \Big] \cdot I_0(\lambda a/2) + e^{-\lambda t} I_0(\lambda t) \delta_{2t}(a),$$

where δ_{2t} denotes Dirac delta function (point mass) at $2t$.

Poisson distributed initial conditions.



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$$+ e^{-\lambda t} I_0(\lambda t) \delta_{2t}(a).$$

Other generalization of pruning.

- **T. Duquesne and M. Winkel (2012)** *arXiv:1211.2179*

There, the requirement for sets

$$A_t = \{T \ : \ \varphi(T) \geq t\}$$

to be Borel with respect to a topology induced by the Gromov-Hausdorff distance leads to semigroup property.

In our generalization, when $\varphi(T) = \text{length}(T)$, the semigroup is not satisfied.

Root-Horton law for the Kingman's coalescent

YK & Zaliapin, AIHP (2017):

• Established the root-Horton law for the Kingman's coalescent.

• Showed that the tree for Kingman's coalescent is combinatorially equivalent to the level-set tree of iid time series (the two measures are **one pruning apart**).

• Numerical experiments that suggest stronger Horton laws: ratio, geometric.

Root-Horton law for the Kingman's coalescent.

In **YK & Zaliapin, AIHP (2017)**, we prove the limit law (in probability) for the asymptotics of the number N_k of branches of Horton-Strahler order k in Kingman's N-coalescent process with constant collision kernel:

$$\mathcal{N}_k = \lim_{N\to\infty} N_k/N$$

We show that

$$\mathcal{N}_k = \frac{1}{2}\int_0^\infty g_k^2(x)\,dx,$$

where the sequence $g_k(x)$ solves:

$$g'_{k+1}(x) - \frac{g_k^2(x)}{2} + g_k(x)g_{k+1}(x) = 0, \quad x \geq 0$$

with $g_1(x) = 2/(x+2)$, $g_k(0) = 0$ for $k \geq 2$.

Root-Horton law for the Kingman's coalescent.

Theorem (YK & Zaliapin, AIHP 2017). The asymptotic Horton ratios $\mathcal{N}_k$ exist and finite and satisfy the convergence

$$\lim_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R$$

with $2 \leq R \leq 4$.

Conjecture. The tree associated with Kingman's coalescent process is Horton self-similar with

$$\lim_{k\to\infty} \frac{\mathcal{N}_k}{\mathcal{N}_{k+1}} = \lim_{k\to\infty} (\mathcal{N}_k)^{-\frac{1}{k}} = R \quad \text{and} \quad \lim_{k\to\infty} (\mathcal{N}_k R^k) = const.,$$

where $R = 3.043827\ldots$.

Hierarchical Branching Processes.

YK and Zaliapin, SPA (2019): Consider a multi-type branching process originating from a root of Horton-Strahler order K with probability p_K. For a given Tokunaga sequence $T_k \geq 0$, we have

• Each branch of order j branches out an offspring of order $i < j$ with rate $\lambda_j T_{j-i}$.

• The branch of order j terminates with rate λ_j, at which moment,

(i) the branch of order $j \geq 2$ splits into two branches, each of order $j - 1$

(ii) the branch of order $j = 1$ terminates without leaving offsprings.

Hierarchical Branching Processes.

Suppose $L = \limsup_{k\to\infty} T_k^{1/k} < \infty$.

Let the coordinates of $x(s)$represent the frequency of branches of respective orders at time s in a tree.

Initial distribution is $x(0) = \pi := \sum_{K=1}^{\infty} p_K e_K$, and

$$x(s) = e^{\mathbb{G}\Lambda s}\pi, \qquad \text{where} \quad \Lambda = \text{diag}\Big\{\lambda_1, \lambda_2, \ldots\Big\}$$

and

$$\mathbb{G} := \begin{bmatrix} -1 & T_1+2 & T_2 & T_3 & \ldots \\ 0 & -1 & T_1+2 & T_2 & \ldots \\ 0 & 0 & -1 & T_1+2 & \ddots \\ 0 & 0 & 0 & -1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Hierarchical Branching Processes.

Consider the width function at time $s \geq 0$

$$C(s) = \langle \mathbf{1}, x(s) \rangle = \langle \mathbf{1}, e^{\mathbb{G}\Lambda s} \pi \rangle.$$

For μ to be distributionally self-similar under pruning, need

- $\{p_K\}$ to be geometric: $p_K = p(1-p)^{K-1}$
- the sequence λ_j to be geometric: $\lambda_j = \gamma\, c^{-j}$

Hierarchical Branching Processes.

Recall:

$$\hat{t}(z) = -1 + 2z + \sum_{j=1}^{\infty} z^j T_j$$

and $w_0 = 1/R$ is the only real root within the radius of convergence.

Suppose $\{p_K\}$ is geometric with parameter p and $\lambda_j = \gamma\, c^{-j}$, then

$$x(s) = e^{\mathbb{G}\Lambda s}\pi = \pi + \sum_{m=1}^{\infty} s^m \left[\prod_{j=1}^{m} \hat{t}\big(c^{-j}(1-p)\big)\right] \Lambda^m \pi.$$

The convergence requirement here is that $c \geq 1$.

Criticality: $p_c = 1 - \frac{c}{R}$, i.e. $C(s) = \langle \mathbf{1}, x(s) \rangle = 1$.

Hierarchical Branching Processes.

The following two conditions are equivalent.

• The process is colorblue critical, i.e.,

$$C(t) = \langle \mathbf{1}, x(s) \rangle = 1 \qquad t \geq 0.$$

• The process has the time invariance property at criticality: the frequencies of trees in the forest produced by the process dynamics are time-invariant

$$x(t) = \exp\{\mathbb{G}\Lambda\, t\}\, \pi = \pi, \qquad \text{where } x(0) = \pi.$$

Hierarchical Branching Processes.

$$p_c = 1 - \frac{c}{R}$$

Observe that for a hierarchical branching process with $\lambda_j = \lambda 2^{1-j}$ and $T_k = 2^{k-1}$, the critical probability is

$$p_c = \frac{1}{2}.$$

Therefore, $R = \frac{c}{1-p_c} = 4$.

Recall that the critical binary Galton-Watson tree exhibits both Horton and Tokunaga self-similarities (**Burd, Waymire, and Winn, 2000**) with parameters $R = 4$, $(a, c) = (1, 2)$ and

$$T_k = a \cdot c^{k-1} = 2^{k-1}.$$

Critical binary Galton-Watson tree.

Theorem (YK and Zaliapin, SPA 2019). The tree of a hierarchical branching process with parameters

$$\lambda_j = \lambda 2^{1-j}, \quad p_K = 2^{-K}, \quad \text{and} \quad T_k = 2^{k-1}$$

for any $\lambda > 0$ is equivalent to the critical binary Galton-Watson tree $\mathsf{GW}(\lambda)$.

Hierarchical Branching Processes.

- Invariant under pruning (from the leafs).
- Satisfies the time invariance property at criticality.
- Is it self-similar under other types of *pruning*?