Subcritical percolation: cluster expansion and Brownian bridge asymptotics

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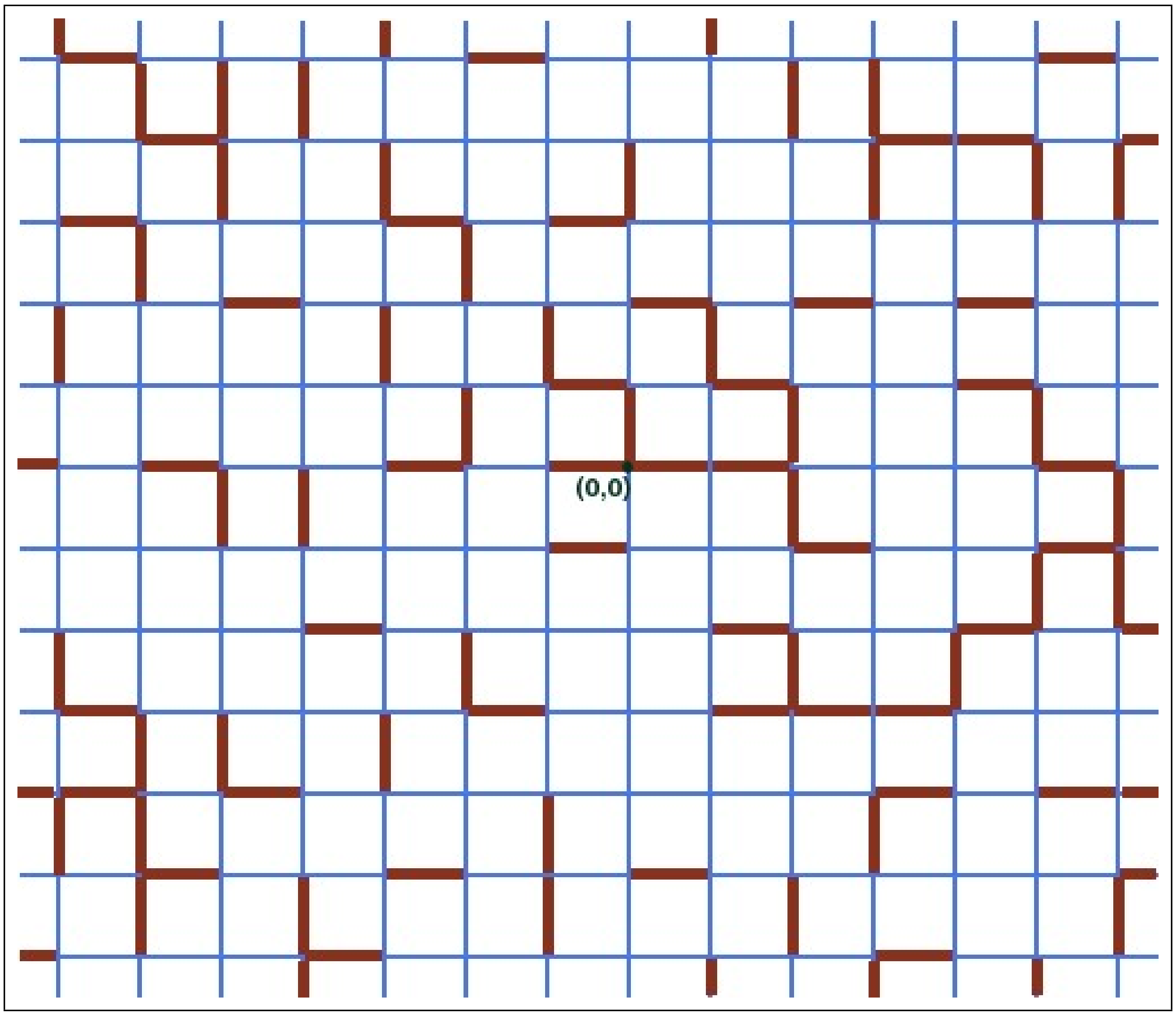
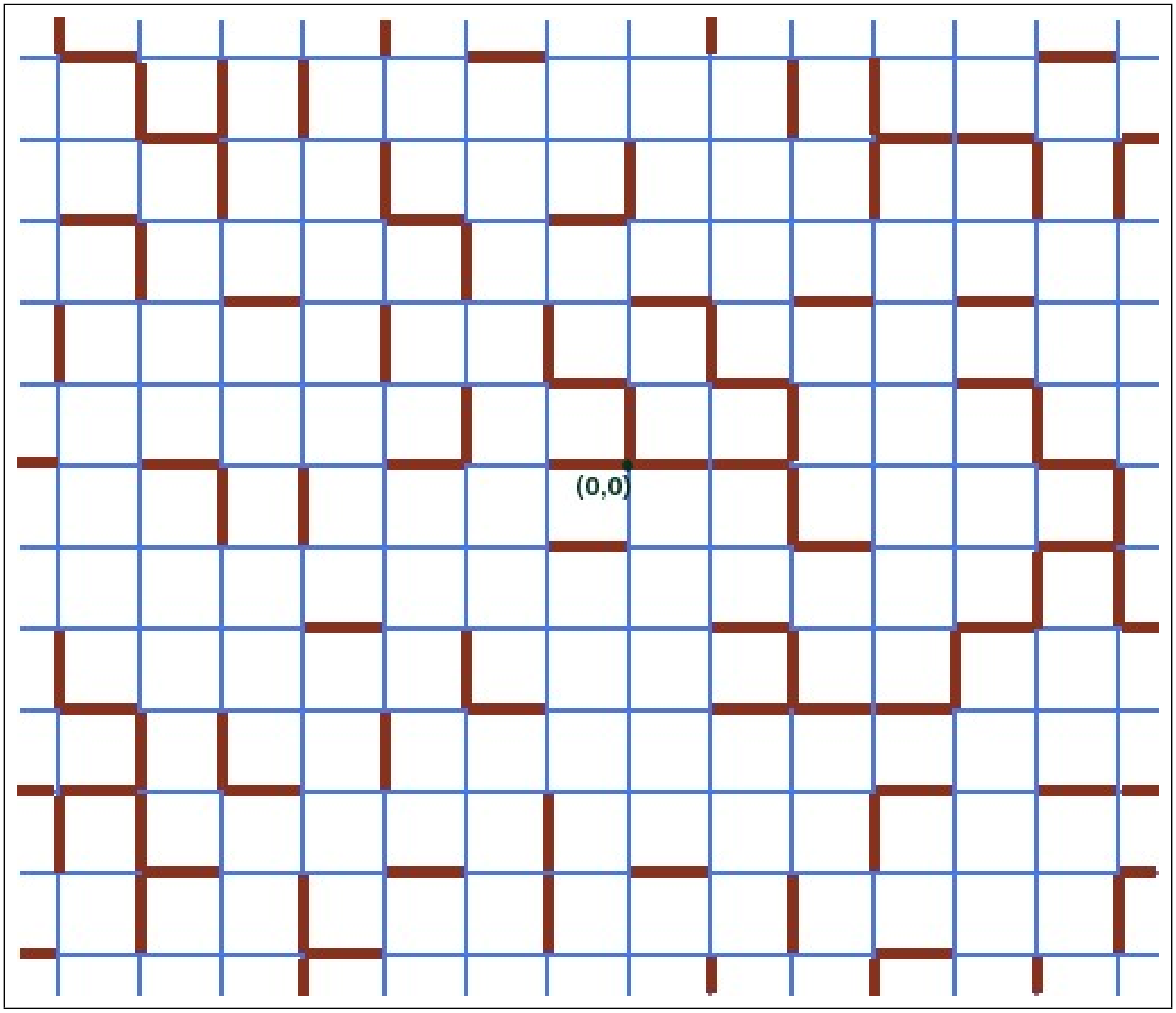
Percolation For each edge of the d-dimensional square lattice $\mathbb{Z}^d$ in turn, we declare the edge *open* with probability p and *closed* with probability $1 - p$, independently of all other edges.





If we delete the closed edges, we are left with a random subgraph of $\mathbb{Z}^d$. A connected component of the subgraph is called a "cluster", and the number of edges in a cluster is the "size" of the cluster.

$$\theta(p) \equiv P_p[0 \leftrightarrow \infty]$$

Obviously $\theta(0) = 0$ and $\theta(1) = 1$.

$\exists$ critical $0 < p_c < 1$ such that

- $\theta(p) = 0$ if $p < p_c \Leftrightarrow$ *subcritical* model

and

- $\theta(p) > 0$ if $p > p_c \Leftrightarrow$ *supercritical* model

Standard reference:

- "*Percolation.*" by G.R.Grimmett (1999)

Subcritical percolation ($p < p_c$): What to study?

- *1.Volume:* A.Pisztora (1996), $|C|$ -large.

- *2.Exponential decay:* M.V.Menshikov (1986), enhanced - M.Aizenman and D.J.Barsky (1987)

Theorem1. If $p < p_c$ then $\exists\ \psi(p) > 0$ such that

$$P_p[0 \leftrightarrow \partial S(n)] < e^{-n\psi(p)} \text{ for all } n.$$

- *3.Probability of reaching far away* ($P_p[0 \leftrightarrow [n\vec{\mathbf{x}}]]$). *Ornstein-Zernike Theory.*

- *4. K.* Geometry of the percolation cluster conditioned on reaching a far away point ($0 \leftrightarrow n\vec{\mathbf{a}}$ for a given $\vec{\mathbf{a}} \in \mathbb{Z}^d$ as $n \to \infty$).

Inverse correlation length $\xi_p(\vec{x})$:

$$\xi_p(\vec{x}) \equiv - \lim_{n \to \infty} \frac{1}{n} \log P_p(0 \leftrightarrow [n\vec{x}])$$

FKG property and Subadditivity Inequality ($x_{m+n} \leq x_m + x_n \Rightarrow \lim_{r \to \infty} \frac{x_r}{r}$ exists and equal to $\inf_{r \geq 1} \frac{x_r}{r}$).

- Definition of $\xi_p(\cdot) \Rightarrow P_p[0 \leftrightarrow [\vec{x}]] \asymp e^{-\xi_p(\vec{x})}$.

Ornstein-Zernike Theory:

- L.S.Ornstein and F.Zernike (1915)

- Correction factor: $\|\vec{x}\|^{-\frac{d-1}{2}}$.

Theorem2. $\exists\ A(\cdot,\cdot)$ on $(0,p_c)\times \mathbf{S}^{d-1}$ s. t.

$$P_p[0\leftrightarrow \vec{x}] = \frac{A(p,n(\vec{x}))}{\|\vec{x}\|^{\frac{d-1}{2}}} e^{-\xi_p(\vec{x})}(1+o(1))$$

uniformly in $\vec{x}\in\mathbb{Z}^d$, where $n(\vec{x})\equiv \frac{\vec{x}}{\|\vec{x}\|}$.

• M.Campanino, J.T.Chayes and L.Chayes (1991)
Case $\vec{x}=(n,0,...,0)$.
$\hookrightarrow$ The hitting distribution of the cluster in the intermediate planes, $x_1 = tn$, $0<t<1$ obeys a multidimensional local limit theorem.

• General Case. M.Campanino and D.Ioffe (1999)

Ornstein-Zernike for Self-Avoiding Walks:

• J.T.Chayes and L.Chayes (1986)
Case $\vec{x}=(n,0,...,0)$.

• General Case. D.Ioffe (1998)

B.B. Asymptotics for Percolation:
Given: d-dim. percolation model ($p < p_c$) and a point $\vec{\mathbf{a}}$ in $\mathbb{Z}^d$, conditioned on $\{0 \leftrightarrow n\vec{\mathbf{a}}\}$.

New basis: $f_1, f_2, ..., f_d$,
coord. $[\cdot,\cdot]_f \in \mathbb{R} \times \mathbb{R}^{d-1}$ s.t. $\vec{\mathbf{a}} = [\|\vec{\mathbf{a}}\|, 0]_f$

Theorem (K. 2002) The process corresponding to the last $d-1$ coordinates (in the new basis $\{\vec{g_1}, \vec{g_2}, ..., \vec{g_d}\}$) of the scaled $(\frac{1}{n\|\vec{\mathbf{a}}\|} \times \frac{1}{\sqrt{n}})$ interpolation of regeneration points of $\mathbf{C}_{0,n\vec{\mathbf{a}}}$ (where the first coordinate is time) conditioned on $0 \leftrightarrow n\vec{\mathbf{a}}$ converges weakly to $\sqrt{\frac{\|\vec{\mathbf{a}}\|}{\|\mu_a\|}}\mathbf{A}_{r_0,r}B^o$, where $B^o = \{B^o_t\}$ is the (d-1)-dimensional Brownian bridge.
($\mathbf{A}_{r_0,r}$ - certain linear transformation.)

- For a given $\epsilon > 0$, cond. probability that the scaled cluster is inside ϵ-neigbhd. of the interpolation "skeleton" converges to one.

“Skeleton” convergence: $\exists$ path that

- connects zero and $n\vec{\mathbf{a}}$
- is interpolation of ”regeneration points”

Scaling:

- $\frac{1}{n\|\vec{\mathbf{a}}\|}$ times along $\vec{\mathbf{a}}$
- $\frac{1}{\sqrt{n}}$ times along $<\vec{\mathbf{a}}>^{\perp}$

Converges $\rightarrow$ Time $\times$ $(d-1)$-dim. Br.Bridge as $n \rightarrow +\infty$.

Where scaled $\vec{\mathbf{a}} \rightsquigarrow$ Time.

$\xi_p(\vec{x})$ is the support function for

$$\mathbf{K}^p \equiv \bigcap_{\vec{n}\in\mathbf{S}^{d-1}} \{\vec{r} \in \mathbb{R}^d \,:\, \vec{r}\cdot\vec{n} \le \xi_p(\vec{n})\},$$

- compact convex set with non-empty interior int$\{\mathbf{K}^p\}$ containing zero.

$r \in \partial\mathbf{K}^p$, $\vec{e}$ -basis vec. s.t. $\vec{e}\cdot\vec{r}$ is maximal.

For $\vec{x}, \vec{y} \in \mathbb{Z}^d$ define

$$S^r_{x,y} \equiv \{\vec{z} \in R^d | \vec{r}\cdot\vec{x} \le \vec{r}\cdot\vec{z} \le \vec{r}\cdot\vec{y}\}.$$

$\mathbf{C}^r_{x,y}$: the corresponding common cluster of $\vec{x}$ and $\vec{y}$.

Definition 1. h_r*-connectivity* $\{\vec{x} \overset{h_r}{\longleftrightarrow} \vec{y}\}$*:*

1. $\vec{x}$ *and* $\vec{y}$ *are connected in the restriction of the percolation configuration to the slab* $S^r_{x,y}$*.*

2. If $\vec{x} \neq \vec{y}$*, then* $\mathbf{C}^r_{x,y} \bigcap S^r_{x,x+e} = \{\vec{x}, \vec{x} + \vec{e}\}$ *and* $\mathbf{C}^r_{x,y} \bigcap S^r_{y-e,y} = \{\vec{y} - \vec{e}, \vec{y}\}$*.*

Definition 2. *f_r-connectivity* $\{\vec{x} \longleftarrow^{f_r}\longrightarrow \vec{y}\}$*:*

1. $\vec{x} \neq \vec{y}$

2. $\vec{x} \longleftarrow^{h_r}\longrightarrow \vec{y}$ *.*

3. For no $\vec{z} \in \mathbb{Z}^d \setminus \{\vec{x}, \vec{y}\}$ *both*
$\{\vec{x} \longleftarrow^{h_r}\longrightarrow \vec{z}\}$ *and* $\{\vec{z} \longleftarrow^{h_r}\longrightarrow \vec{y}\}$ *take place.*

Definition 3. *Suppose* $0 \xleftrightarrow{h_r} \vec{x}$,
$\vec{z} \in \mathbb{Z}^d$ *is* **a regeneration point** *of* $\mathrm{C}^r_{0,x}$ *if*

1. $\vec{r} \cdot \vec{e} \le \vec{r} \cdot \vec{z} \le \vec{r} \cdot (\vec{y} - \vec{e})$

2. $S^r_{z-e,z+e} \bigcap \mathrm{C}^r_{0,x}$ *contains exactly three points:*

$\vec{z} - \vec{e}$, $\vec{z}$ *and* $\vec{z} + \vec{e}$.

Laplace Transforms Let

$$H_{r_0}(\vec{\mathbf{r}}) = \sum_{x \in \mathbb{Z}^d} h_{\vec{\mathbf{r}}_0}(x) e^{\vec{\mathbf{r}} \cdot \vec{x}}$$

and

$$F_{r_0}(\vec{\mathbf{r}}) = \sum_{x \in \mathbb{Z}^d} f_{\vec{\mathbf{r}}_0}(x) e^{\vec{\mathbf{r}} \cdot \vec{x}}.$$

Taking Laplace transform of the a renewal relation

$$h_r(\vec{x}) = \sum_{\vec{z} \in \mathbb{Z}^d} f_r(\vec{z}) h_r(\vec{x} - \vec{z}) \text{ with } h_r(0) = 1$$

get

$$H_{r_0}(\vec{\mathbf{r}}) = \frac{1}{1 - F_{r_0}(\vec{\mathbf{r}})}.$$

Mass gap: $\exists$ radius $\bar{\lambda}$, s.t. if $\vec{\mathbf{r}}$ is inside a Euclidean ball $\mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0)$,

$$f_{r_0}(x) = P_p[0 \xleftarrow{\;f_{r_0}\;}\!\!\rightarrow \vec{x}] \leq e^{-\vec{\mathbf{r}}\cdot x - c\|x\|},$$

$\forall \vec{\mathbf{r}} \in \mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0)$ and $\forall x$ in the "$\vec{\mathbf{r}}_0$-directional cone" and fixed $c > 0$. While $\forall \vec{\mathbf{r}} \in \partial \mathbf{K}^p$,

$$h_{r_0}(x) \simeq P_p[0 \leftrightarrow \vec{x}] \simeq e^{-\xi_p(\vec{x})} = e^{-\vec{\mathbf{r}}\cdot x}.$$

Again $\boxed{H_{r_0}(\vec{\mathbf{r}}) = \frac{1}{1 - F_{r_0}(\vec{\mathbf{r}})}.}$

"mass gap" implies

$$F_{r_0}(\vec{\mathbf{r}}) < \infty \text{ for } \vec{\mathbf{r}} \in \mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0)$$

and $H_{r_0}(\vec{\mathbf{r}})$ **diverges** for $\vec{\mathbf{r}} \in \mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0) \backslash \mathbf{K}^p$, where $\mathcal{B}_{\bar{\lambda}}(\cdot)$ denotes a Euclidean ball of radius $\bar{\lambda}$.

Thus $\boxed{F_{r_0}(\vec{\mathbf{r}}) = \sum_{x \in \mathbb{Z}^d} f_{\vec{\mathbf{r}}_0}(x) e^{\vec{\mathbf{r}}\cdot\vec{x}} = 1}$ whenever $\vec{\mathbf{r}} \in \mathcal{B}_{\bar{\lambda}}(\vec{\mathbf{r}}_0) \bigcap \partial \mathbf{K}^p$.

Probability measure $Q^r_{r_0}(x)$.

For a given $\vec{r}_0 \in \mathbf{K}^p$, $\exists \bar{\lambda} > 0$ s.t.

$$\sum_{\vec{x} \in Z^d} P_p[0 \xleftrightarrow{f_{r_0}} \vec{x}] e^{\vec{r} \cdot \vec{x}} = 1$$

whenever $\vec{r} \in B_{\bar{\lambda}}(r_0) \bigcap \partial \mathbf{K}^p$ and therefore

$$Q^r_{r_0}(\vec{x}) := P_p[0 \xleftrightarrow{f_{r_0}} \vec{x}] e^{\vec{r} \cdot \vec{x}}$$

is a **probability measure** on $\mathbb{Z}^d$.
(see M.Campanino and D.Ioffe)

Event: $0 \overset{h_{r_0}}{\longleftrightarrow} \vec{x}$ with exactly one regeneration point $\vec{x}_1$.

Probability of the event

$$= P[0 \overset{f_{r_0}}{\longleftrightarrow} \vec{x}_1] P[\vec{x}_1 \overset{f_{r_0}}{\longleftrightarrow} \vec{x}].$$

$$= e^{-\vec{r}\cdot\vec{x}} Q^r_{r_0}(\vec{x}_1) Q^r_{r_0}(\vec{x} - \vec{x}_1).$$

Important Observation.

Event: $0 \longleftarrow^{h_{r_0}}\!\!\longrightarrow \vec{x}$ with k-1 regeneration points $\vec{x}_1, \vec{x}_1 + \vec{x}_2, ..., \sum_{i=1}^{k-1} \vec{x}_i$, and $\sum_{i=1}^{k} \vec{x}_i = \vec{x}$.

$$P_p[0 \longleftarrow^{h_{r_0}}\!\!\longrightarrow \vec{x} \text{ ; reg pts: } \vec{x}_1, \vec{x}_1 + \vec{x}_2, ..., \textstyle\sum_{i=1}^{k-1} \vec{x}_i]$$

$$= P[0 \longleftarrow^{f_{r_0}}\!\!\longrightarrow \vec{x}_1] \times$$

$$P[\vec{x}_1 \longleftarrow^{f_{r_0}}\!\!\longrightarrow \vec{x}_1 + \vec{x}_2] ... P[\textstyle\sum_{i=1}^{k-1} \vec{x}_i \longleftarrow^{f_{r_0}}\!\!\longrightarrow \vec{x}]$$

$$= e^{-\vec{r}\cdot\vec{x}} Q_{r_0}^{r}(\vec{x}_1) ... Q_{r_0}^{r}(\vec{x}_k).$$

Shrinking: The hitting area of the orthogonal hyper-planes shrinks

$\Rightarrow$ for n large enough, all the points of the scaled cluster are within an ε-neighborhood of the points in the "skeleton".

Observed: If $\vec{\mathbf{r}} = \nabla \xi_p(\vec{\mathbf{r}}_0)$ then $Q^r_{r_0}$ is a probability measure.

Used the properties of $Q^r_{r_0}$ and convex analysis to bound the probability of

$$\{\max_i |x_i - x_{i-1}| > n^{1/3}\}$$

with x_i's being all the regeneration points, conditioned on $0 \overset{h}{\longleftrightarrow} n\vec{\mathbf{a}}$.