Problem 1. Consider a one-dimensional Brownian motion $B_t$, $t \in [0, \infty)$.

a. Prove that $B^2_t - t$ is a martingale.

b. Prove $e^{\theta B_t - \frac{\theta^2 t}{2}}$ is a martingale.

Problem 2. Consider a one-dimensional standard Brownian motion $B(t)$, $t \in [0, \infty)$.

a. Take $a \neq 0$. Show that $X(t) = \frac{1}{a} B(a^2 t)$ is also a standard Brownian motion.

b. Show that $M(t) = \begin{cases} 0 & \text{for } t = 0, \\ t B \left( \frac{1}{t} \right) & \text{for } t > 0 \end{cases}$ is also a standard Brownian motion.

Problem 3. Consider a one-dimensional standard Brownian motion $B(t)$. The process $V(t) = B(t) - tB(1)$ for $t \in [0, 1]$ is called a Brownian Bridge.

a. Show that for $0 \leq s \leq t \leq 1$, the covariance $Cov (V(s), V(t)) = s(1-t)$.

b. Show that $W(t) = (1 + t) V \left( \frac{t}{t+1} \right)$ is a standard Brownian motion.