Problem 1. Suppose $X$ and $Y$ are two random variable defined on the probability space $(\Omega, P, \mathcal{F})$ such that
\[ X \leq Y \text{ a.s.} \]
For $\mathcal{G} \subseteq \mathcal{F}$ prove that
\[ E[X | \mathcal{G}] \leq E[Y | \mathcal{G}] \text{ a.s.} \]

Problem 2 (Markov Inequality). Suppose $X$ is a nonnegative random variable defined on the probability space $(\Omega, P, \mathcal{F})$. For $\mathcal{G} \subseteq \mathcal{F}$ and $a > 0$, prove that
\[ P(X \geq a | \mathcal{G}) = E[1_{X \geq a} | \mathcal{G}] \leq \frac{E[X | \mathcal{G}]}{a} \text{ a.s.} \]

Problem 3 (Random Walk). Suppose $X_1, X_2, \ldots$ is a sequence of i.i.d. Bernoulli random variables with
\[ P(X_j = 1) = P(X_j = -1) = \frac{1}{2} \]
Then $S_n = X_1 + \ldots + X_n$ is a simple random walk. Prove that $M_n = S_n^2 - n$ is a martingale.

Problem 4 (Gambling system). For a given gambling game, let $X_0$ be your initial capital and $X_n (n \geq 1)$ be the net amount of money you would have won at time $n$ if you have bet one dollar each time. That is, $X_j - X_{j-1}$ is the profit resulted from a stake of one unit on the $j$-th betting. We can assume the independence of the margins $X_j - X_{j-1}$. It is natural to assume that the game is fair, i.e., assume $X_n$ is a martingale.

In reality, you stake $H_j \geq 0$ dollars on the $j$-th betting, where the choice of $H_j$ is based on the history of the game up to time $j - 1$. So, $H_j$ is a predictable process, i.e., $H_j$ is $\mathcal{F}_{j-1}$ measurable for each $j \geq 1$. Therefore, your capital at time $n$ can be expressed as
\[ (H \cdot X)_n = X_0 + \sum_{j=1}^{n} H_j (X_j - X_{j-1}) \]
Prove that $(H \cdot X)_n$ is a martingale.

Problem 5 (Nearest neighbor walk). Consider a state space $S = \{0, 1, 2, \ldots\}$ and a Markov chain $\{X_t\}_{t=0,1,\ldots}$ on $S$ with transition probabilities
\[ p(i, i+1) = p_i, \quad p(i, i-1) = q_i, \quad \text{and} \quad p(i, i) = r_i \]
satisfying $q_0 = 0$ and $q_i + r_i + p_i = 1$ for all $i \in S$. Suppose the walk begins at $X_0 = x$.

Find a (nonconstant) probability harmonic function. Next, for $0 < x < M$, find the probability that the walker reaches 0 before reaching $M$. 