Math 664
Homework #1: Solutions

1. An urn contains n green and m black balls. The balls are withdrawn one at a time until only those of the same color are left. Show that with probability $\frac{n}{n+m}$ they are all green.

   SOLUTION: The outcome of the experiment will not change if you pull all but one last marble from the urn, and then check its color. Now, removing $n+m-1$ marbles from the urn, and checking the color of the last marble is no different from selecting one marble and checking its color. Both are equivalent to separating marbles into two groups, one of size one, and the other of size $n+m-1$. Thus the marble is green with probability $\frac{n}{n+m}$.

2. For a nonnegative integer-valued random variable X, show that

$$E[X] = \sum_{j=1}^{\infty} \text{Prob}(X \geq j)$$

   Hint: Write $\sum_{j=1}^{\infty} \text{Prob}(X \geq j) = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \text{Prob}(X = k)$, interchange the order of summation.

   SOLUTION:

$$\sum_{j=1}^{\infty} P(X \geq j) = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} P(X = k) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} P(X = k) = \sum_{k=1}^{\infty} kP(X = k) = E[X]$$

   as the double sum was the sum over all different integer pairs (j, k) such that

$$1 \leq j \leq k < \infty$$

3. Recall that a *geometric* random variable with parameter $p \in (0,1)$ is defined by its geometric probability mass function

$$p(j) = p(1-p)^{j-1} \text{ for } j = 1, 2, 3, ...$$

Suppose X is such random variable. Find $E[X]$ and $Var(X)$.

SOLUTION:

$$P(X > j) = \sum_{i=j+1}^{\infty} p(1-p)^{i-1} = p(1-p)^j \sum_{k=0}^{\infty}(1-p)^k = p(1-p)^j \cdot \frac{1}{1-(1-p)} = (1-p)^j$$

Now, by the result in the previous problem,

$$E[W] = \sum_{i=1}^{\infty} P(X \geq i) = \sum_{j=0}^{+\infty} P(X > j) = \sum_{j=0}^{+\infty}(1-p)^j = \frac{1}{1-(1-p)} = \frac{1}{p}$$

Alternatively, here $\sum\limits_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$ as

$$\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty}(x^k)' = \left(\sum_{k=0}^{\infty} x^k\right)' = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

and therefore

$$E[X] = p \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = p \cdot \frac{1}{(1-(1-p))^2} = \frac{1}{p}$$

Let $\mu = E[X] = \frac{1}{p}$, then

$$Var(X) = E[X^2] - \mu^2 = \sum_{k=1}^{\infty} k^2 \cdot p \cdot (1-p)^{k-1} - \mu^2$$

$$= \sum_{k=1}^{\infty} k(k-1) \cdot p \cdot (1-p)^{k-1} + \sum_{k=1}^{\infty} k \cdot p \cdot (1-p)^{k-1} - \mu^2$$

$$= p \cdot (1-p) \cdot \sum_{k=0}^{\infty} k(k-1) \cdot (1-p)^{k-2} + \mu - \mu^2$$

Now, for $|x| < 1$,

$$\sum_{k=0}^{\infty} k(k-1) \cdot x^{k-2} = \sum_{k=0}^{\infty} \left(x^k\right)'' = \frac{d^2}{dx^2}\left(\sum_{k=0}^{\infty} x^k\right) = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{2}{(1-x)^3}$$

Hence,

$$Var(X) = p \cdot (1-p) \cdot \frac{2}{p^3} + \mu - \mu^2 = 2 \cdot \frac{1-p}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

4. Let $f(t)$ be the probability density function, and $F(t) = \int\limits_{-\infty}^{t} f(x)dx$ be the corresponding cumulative distribution function. Define the *hazard function* $h(t) = \frac{f(t)}{1-F(t)}$. Show that if X is an exponential random variable with parameter $\lambda > 0$, then its hazard function will be a constant

$$h(t) = \lambda$$

for all $t > 0$. Think of how this relates to the memorylessness property of exponential random variables.

SOLUTION: Here for $t > 0$, $f(t) = \lambda e^{-\lambda t}$ and $F(t) = 1 - e^{-\lambda t}$. Thus

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{\lambda e^{-\lambda t}}{1-(1-e^{-\lambda t})} = \lambda$$

In other words, because of the memorylessness property, the hazard rate is constant at all times.

5. The *gamma function* $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$$

for all $\alpha > 0$. Use integration by parts to prove that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Compute $\Gamma(1)$ and show that $\Gamma(k) = (k-1)!$ for all positive integer k.

SOLUTION:

$$\Gamma(\alpha+1) = \int_0^\infty e^{-y} y^{\alpha} dy = \int_0^\infty \left(-e^{-y}\right)' y^{\alpha} dy = \left(-e^{-y} y^{\alpha}\right)_0^\infty - \int_0^\infty \left(-e^{-y}\right) \left(y^{\alpha}\right)' dy$$

$$= 0 + \int_0^\infty e^{-y} \alpha y^{\alpha-1} dy = \alpha\Gamma(\alpha)$$

Now,

$$\Gamma(1) = \int_0^\infty e^{-y} dy = 1 = 0!$$

Thus $\Gamma(2) = 1 \cdot \Gamma(1) = 1!$, $\Gamma(3) = 2 \cdot \Gamma(2) = 2!$ and by induction, $\Gamma(k) = (k-1)!$ for all positive integer k.

6. A gamma distributed random variable with parameters (α, λ) is defined by its probability density function

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\lambda e^{-\lambda x}(\lambda x)^{\alpha-1} \text{ when } x \geq 0 \\ 0 \quad \text{otherwise,} \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$. Suppose X is a gamma distributed random variable with parameters (α, λ), where $\alpha > 0$ and $\lambda > 0$. Compute $E[e^{-X}]$.

SOLUTION:

$$E[e^{-X}] = \frac{1}{\Gamma(\alpha)}\int_0^\infty \lambda e^{-x}e^{-\lambda x}(\lambda x)^{\alpha-1}dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-(\lambda+1)x}x^{\alpha-1}dx$$

Let $y = (\lambda+1)x$, then

$$E[e^{-X}] = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-y}\frac{y^{\alpha-1}}{(\lambda+1)^{\alpha-1}}\frac{dy}{\lambda+1} = \left(\frac{\lambda}{\lambda+1}\right)^\alpha \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-y}y^{\alpha-1}dy = \left(\frac{\lambda}{\lambda+1}\right)^\alpha$$

7. The standard normal random variable Z is characterized by its density function

$$f_Z(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}.$$

Compute $E[e^Z]$.

SOLUTION:

$$E[e^Z] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^x e^{\frac{-x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{\frac{-(x^2-2x)}{2}}dx$$

Now, $x^2 - 2x = (x-1)^2 - 1$, and

$$E[e^Z] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{\frac{-(x-1)^2+1}{2}}dx = e^{\frac{1}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{\frac{-(x-1)^2}{2}}dx$$

Let $y = x - 1$, then

$$E[e^Z] = e^{\frac{1}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{\frac{-y^2}{2}}dy = e^{\frac{1}{2}} = \sqrt{e}$$

8. An experiment consists of $1,210$ independent Bernoulli trials with probability $p = \frac{1}{11}$ of success. Use the Central Limit Theorem for Bernoulli Trials (the DeMoivre-Laplace theorem) and the table of values for the standard normal distribution to estimate the probability of the event that

$$\{98 \leq \text{ the number of successes } \leq 116\}.$$

Remember: it is best to consider $P\{97.5 \leq \text{ the number of successes } \leq 116.5\}$.

SOLUTION: Denote by S the number of successes. Then S is binomial with parameters $(n = 1210,\ p = \frac{1}{11})$ and

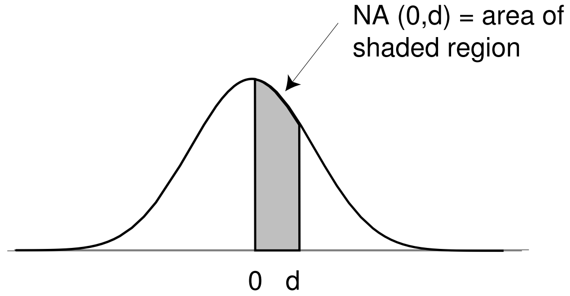
$$E[S] = np = 110, \quad \sigma(S) = \sqrt{np(1-p)} = 10$$

Now, by the DeMoivre-Laplace limit theorem,

$$P(97.5 \leq S \leq 116.5) = P(-12.5 \leq S - np \leq 6.5) = P\left(-1.25 \leq \frac{S - np}{\sqrt{np(1-p)}} \leq 0.65\right)$$

$$\approx P(-1.25 \leq Z \leq 0.65) \approx 0.2422 + 0.3944 = 0.6366$$

Normal distribution table



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000