MTH 664

Lectures 24 - 27

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Topics:

- Martingales.
- Filtration. Stopping times.
- Probability harmonic functions.
- Optional Stopping Theorem.
- Martingale Convergence Theorem.

Conditional expectation.

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $X \in \mathcal{F}$.

Let $\mathcal{G} \subseteq \mathcal{F}$ be a smaller σ-algebra.

Definition. Conditional expectation $E[X|\mathcal{G}]$ is a unique function from Ω to $\mathbb{R}$ satisfying:

1. $E[X|\mathcal{G}]$ is $\mathcal{G}$-measurable
2. $\int\limits_A E[X|\mathcal{G}]\; dP(\omega) = \int\limits_A X\; dP(\omega)$ for all $A \in \mathcal{G}$

The existence and uniqueness of $E[X|\mathcal{G}]$ comes from the Radon-Nikodym theorem.

Lemma. If $X \in \mathcal{G}$, $Y(\omega) \in L^1(\Omega, P)$, and $X(\omega) \cdot Y(\omega) \in L^1(\Omega, P)$, then

$$E[X \cdot Y|\mathcal{G}] = X \cdot E[Y|\mathcal{G}]$$

Conditional expectation.

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $X \in \mathcal{F}$.

Lemma. If $\mathcal{G} \subseteq \mathcal{F}$, then $E\Big[E[X|\mathcal{G}]\Big] = E[X]$

Let $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$ be smaller sub-σ-algebras.

Lemma.

$$E\Big[E[X|\mathcal{G}_2] \mid \mathcal{G}_1\Big] = E[X|\mathcal{G}_1]$$

Proof. For any $A \in \mathcal{G}_1 \subseteq \mathcal{G}_2$,

$$\int_A E\Big[E[X|\mathcal{G}_2] \mid \mathcal{G}_1\Big](\omega)\ dP(\omega) = \int_A E[X|\mathcal{G}_2](\omega)\ dP(\omega)$$

$$= \int_A X(\omega)\, dP(\omega) = \int_A E[X|\mathcal{G}_1](\omega)\, dP(\omega)$$

$\square$

Filtration.

Definition. Consider an arbitrary linear ordered set T: A sequence of sub-σ-algebras $\{\mathcal{F}_t\}_{t\in T}$ of $\mathcal{F}$ is said to be a filtration if

$$\mathcal{F}_s \subseteq \mathcal{F}_t \quad a.s. \quad \forall s < t \ \in T$$

Example. Consider a sequence of random variables $X_1, X_2, \ldots$ on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ is the smallest σ-algebra such that $X_1, X_2, \ldots, X_n$ are $\mathcal{F}_n$-measurable. Then $\mathcal{F}_n$ is the smallest filtration that X_n is adapted to, i.e. $X_n \in \mathcal{F}_n$.

Important: When filtration $\mathcal{F}_n$ is not mentioned in defining the martingale, submartingale, or supermartingale,

$$\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$$

Definition. Consider a filtration $\{\mathcal{F}_n\}$. A sequence of random variables $X_1, X_2, \ldots \in L^1(\Omega, P)$ adapted to $\mathcal{F}_n$ (i.e. $X_n \in \mathcal{F}_n$) is said to be a martingale with respect to $\{\mathcal{F}_n\}$ if

$$E[X_{n+1} \mid \mathcal{F}_n] = X_n \quad a.s. \quad \forall n > 1$$

Martingales.

Definition. Consider a filtration $\{\mathcal{F}_n\}$. A sequence of random variables $X_1, X_2, \ldots \in L^1(\Omega, P)$ adapted to $\mathcal{F}_n$ (i.e. $X_n \in \mathcal{F}_n$) is said to be a martingale with respect to $\{\mathcal{F}_n\}$ if

$$E[X_{n+1} \mid \mathcal{F}_n] = X_n \quad a.s. \quad \forall n > 1$$

Example. Let $\xi_1, \xi_2, \ldots$ be independent $L^1(\Omega, P)$ random variables such that

$$E[\xi_j] = 0 \quad \forall j \in \mathbb{N}$$

Now, let $X_n = \xi_1 + \ldots + \xi_n$. Then

$$E[X_{n+1} \mid \mathcal{F}_n] = E[X_n + \xi_{n+1} \mid \mathcal{F}_n] = X_n + E[\xi_{n+1} \mid \mathcal{F}_n] = X_n + E[\xi_{n+1}] = X_n$$

as ξ_{n+1} is independent of $\mathcal{F}_n$. Specifically, $\forall m \in \mathbb{N}$ s.t. $1 \le m \le n$, and any Borel $A \in \mathcal{B}$,

$$\int_{X_m^{-1}(A)} E[\xi_{n+1} \mid \mathcal{F}_n](\omega)\, dP(\omega) = \int_{X_m^{-1}(A)} \xi_{n+1}(\omega)\, dP(\omega) = E[\xi_{n+1} \cdot \mathbf{1}_{X_m \in A}]$$

$$= E[\xi_{n+1}] \cdot E[\mathbf{1}_{X_m \in A}] = \int_{X_m^{-1}(A)} E[\xi_{n+1}]\, dP(\omega)$$

Martingales.

Definition. Consider a filtration $\{\mathcal{F}_n\}$. A sequence of random variables $X_1, X_2, \ldots \in L^1(\Omega, P)$ adapted to $\mathcal{F}_n$ (i.e. $X_n \in \mathcal{F}_n$) is said to be a martingale with respect to $\{\mathcal{F}_n\}$ if

$$E[X_{n+1} \mid \mathcal{F}_n] = X_n \quad a.s. \quad \forall n > 1$$

Definition. Consider a filtration $\{\mathcal{F}_n\}$. A sequence of random variables $X_1, X_2, \ldots \in L^1(\Omega, P)$ adapted to $\mathcal{F}_n$ is said to be a supermartingale with respect to $\{\mathcal{F}_n\}$ if

$$E[X_{n+1} \mid \mathcal{F}_n] \leq X_n \quad a.s. \quad \forall n > 1$$

Definition. Consider a filtration $\{\mathcal{F}_n\}$. A sequence of random variables $X_1, X_2, \ldots \in L^1(\Omega, P)$ adapted to $\mathcal{F}_n$ is said to be a submartingale with respect to $\{\mathcal{F}_n\}$ if

$$E[X_{n+1} \mid \mathcal{F}_n] \geq X_n \quad a.s. \quad \forall n > 1$$

All these definitions can be extended to an arbitrary linear ordered set T: Consider a filtration $\{\mathcal{F}_t\}_{t \in T}$. A sequence of random variables $\{X_t\}_{t \in T}$ adapted to $\mathcal{F}_t$ is said to be a martingale if

$$E[X_t \mid \mathcal{F}_s] = X_s \quad a.s. \quad \forall s < t \in T$$

Probability harmonic functions.

Consider a sequence of random variables $X_1, X_2, \ldots$ with associated σ-algebras $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.

Definition. A function $h(x)$ is said to be a probability harmonic function if $M_t = h(X_t)$ is a martingale sequence.

Example. Random walk on $\mathbb{Z}$. Take $p \in (0,1)$, and let $\xi_1, \xi_2, \ldots$ be i.i.d. Bernoulli random variables such that

$$\xi_j = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } q = 1-p \end{cases}$$

If $p = \frac{1}{2}$, the random walk $X_n = X_0 + \xi_1 + \ldots + \xi_n$ is a martingale.

Suppose $p \neq \frac{1}{2}$, then $X_n = X_0 + \xi_1 + \ldots + \xi_n$ is not a martingale. We need a **probability harmonic function** $h(x)$ such that $M_n = h(X_n)$ is a martingale. For this, we solve

$$p \cdot h(X_n + 1) + q \cdot h(X_n - 1) = E[h(X_{n+1}) \mid \mathcal{F}_n] = h(X_n)$$

arriving at $h(x) = A \cdot \left(\frac{q}{p}\right)^x + B$ for any choice of constants A and B.

Filtration. Stopping time.

Definition. Consider an arbitrary linear ordered set T: A sequence of sub-σ-algebras $\{\mathcal{F}_t\}_{t\in T}$ of $\mathcal{F}$ is said to be a filtration if

$$\mathcal{F}_s \subseteq \mathcal{F}_t \quad a.s. \quad \forall s < t \ \in T$$

Example. Consider a sequence of random variables $X_1, X_2, \ldots$ on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$. Then $\mathcal{F}_n$ is a filtration.

Definition. Consider an arbitrary linear ordered set T, and a filtration $\{\mathcal{F}_t\}_{t\in T}$. A random variable τ is a stopping time if for any $t \geq 0$,

$$\{\tau \leq t\} \in \mathcal{F}_t$$

In other words knowing the trajectory of the process up to time m is sufficient to determine whether $\{\tau \leq t\}$ occurred.

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In other words knowing the trajectory of the process up to time m is sufficient to determine whether $\{\tau \leq t\}$ occurred.

For every stopping time τ we associate a stopped σ-algebra $\mathcal{F}_\tau \subseteq \mathcal{F}$ defined as

$$\mathcal{F}_\tau = \Big\{ A \in \mathcal{F} \ : \ A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \Big\}$$

Observe that if $\{\mathcal{F}_n\}$ is a filtration, and if $X_1, X_2, \ldots$ is a sequence of random variables adapted to $\mathcal{F}_n$, and τ is a stopping time w.r.t. $\{\mathcal{F}_n\}$, then

$$X_\tau = \sum_{j=1}^{\infty} X_j \cdot \mathbf{1}_{\tau=j} \ \in \mathcal{F}_\tau$$

Filtration. Stopping time.

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$$\mathcal{F}_\tau = \Big\{ A \in \mathcal{F} \ : \ A \cap \{\tau \le t\} \in \mathcal{F}_t \quad \forall t \Big\}$$

Lemma. Suppose τ_1 and τ_2 are two stopping times w.r.t. $\mathcal{F}_n$ such that $P(\tau_1 \le \tau_2) = 1$, then

$$\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$$

Proof. Take $A \in \mathcal{F}_{\tau_1}$, then $\forall t$,

$$A \cap \{\tau_2 \le t\} = A \cap \{\tau_1 \le t\} \cap \{\tau_2 \le t\} \qquad P - a.s.$$

and therefore $A \cap \{\tau_2 \le t\} \in \mathcal{F}_t$ as both $A \cap \{\tau_1 \le t\}$ and $\{\tau_2 \le t\}$ are in $\mathcal{F}_t$.

$\square$

Filtration. Stopping time.

For every stopping time τ we associate a stopped σ-algebra $\mathcal{F}_\tau \subseteq \mathcal{F}$ defined as

$$\mathcal{F}_\tau = \Big\{ A \in \mathcal{F} \ : \ A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \Big\}$$

Lemma. Suppose τ is a stopping time w.r.t. $\mathcal{F}_n$ such that $\tau \leq K \quad a.s.$ for some integer $K > 0$. Then, if the sequence $\{X_t\}$ is a martingale,

$$E[X_K \mid \mathcal{F}_\tau] = X_\tau$$

Proof. Take $A \in \mathcal{F}_\tau$, then

$$\int_A X_K(\omega)\, dP(\omega) = \sum_{j=0}^{K} \int_{A \cap \{\tau = j\}} X_K(\omega)\, dP(\omega) = \sum_{j=0}^{K} \int_{A \cap \{\tau = j\}} E[X_K | \mathcal{F}_j](\omega)\, dP(\omega)$$

$$= \sum_{j=0}^{K} \int_{A \cap \{\tau = j\}} X_j(\omega)\, dP(\omega) = \sum_{j=0}^{K} \int_{A \cap \{\tau = j\}} X_\tau(\omega)\, dP(\omega) = \int_A X_\tau(\omega)\, dP(\omega)$$

$\square$

Optional Stopping Theorem.

Doob's Optional Stopping Theorem. Consider a sequence of random variables $X_1, X_2, \ldots$ on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$. Suppose τ_1 and τ_2 are two stopping times w.r.t. $\mathcal{F}_n$ such that **either** of the following conditions is satisfied:

(a) $P(\tau_1 \leq \tau_2 \leq K) = 1$ for some $K > 0$

(b) $P(\tau_1 \leq \tau_2 < \infty) = 1$ and $S = \sup\limits_{0 \leq k \leq \tau_2} \left| X_k \right| \in L^1(\Omega, P)$

Then, if the sequence $\{X_t\}$ is a martingale,

$$E[X_{\tau_2} \mid \mathcal{F}_{\tau_1}] = X_{\tau_1}$$

Similarly, if the sequence $\{X_t\}$ is a supermartingale, $E[X_{\tau_2} \mid \mathcal{F}_{\tau_1}] \leq X_{\tau_1}$, and if the sequence $\{X_t\}$ is a submartingale, $E[X_{\tau_2} \mid \mathcal{F}_{\tau_1}] \geq X_{\tau_1}$.

Proof. (part **(a)**) Suppose $P(\tau_1 \leq \tau_2 \leq K) = 1$ for some integer $K > 0$. Then

$$\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2} \subseteq \mathcal{F}_K$$

Optional Stopping Theorem.

Proof. (part **(a)**) Suppose $P(\tau_1 \leq \tau_2 \leq K) = 1$ for some integer $K > 0$. Then

$$\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2} \subseteq \mathcal{F}_K$$

and

$$E[X_{\tau_2}|\mathcal{F}_{\tau_1}] = E\Big[E[X_K|\mathcal{F}_{\tau_2}] \mid \mathcal{F}_{\tau_1}\Big] = E[X_K|\mathcal{F}_{\tau_1}] = X_{\tau_1}$$

□

Proof. (part **(b)**) Suppose $\{X_t\}$ is a martingale. For $K > 0$, consider a stopped process $Y_t = X_{t \wedge K}$. Then, $\tau_1^* = \tau_1 \wedge K$ and $\tau_2^* = \tau_2 \wedge K$ are both bounded stopping times, as in part **(a)**, and

$$E[Y_{\tau_2^*} \mid \mathcal{F}_{\tau_1^*}] = Y_{\tau_1^*} \quad \Leftrightarrow \quad E[Y_{\tau_2} \mid \mathcal{F}_{\tau_1}] = Y_{\tau_1}$$

Therefore

$$E[X_{\tau_2} \mid \mathcal{F}_{\tau_1}] + E[(X_K - X_{\tau_2}) \cdot \mathbf{1}_{\tau_2 > K} \mid \mathcal{F}_{\tau_1}] = X_{\tau_1} \cdot \mathbf{1}_{\tau_1 \leq K} + X_K \cdot \mathbf{1}_{\tau_1 > K}$$

Optional Stopping Theorem.

Proof. (part **(b)**) Suppose $\{X_t\}$ is a martingale. For $K > 0$, consider a stopped process $Y_t = X_{t\wedge K}$. Then, $\tau_1^* = \tau_1 \wedge K$ and $\tau_2^* = \tau_2 \wedge K$ are both bounded stopping times, as in part **(a)**, and

$$E[Y_{\tau_2^*} \mid \mathcal{F}_{\tau_1^*}] = Y_{\tau_1^*} \quad \Leftrightarrow \quad E[Y_{\tau_2} \mid \mathcal{F}_{\tau_1}] = Y_{\tau_1}$$

Therefore

$$E[X_{\tau_2} \mid \mathcal{F}_{\tau_1}] + E[(X_K - X_{\tau_2}) \cdot \mathbf{1}_{\tau_2 > K} \mid \mathcal{F}_{\tau_1}] = X_{\tau_1} \cdot \mathbf{1}_{\tau_1 \le K} + X_K \cdot \mathbf{1}_{\tau_1 > K},$$

where $\forall A \in \mathcal{F}_{\tau_1}$,

$$\int_A E[(X_K - X_{\tau_2}) \cdot \mathbf{1}_{\tau_2 > K} \mid \mathcal{F}_{\tau_1}](\omega)\, dP(\omega) = \int_A (X_K(\omega) - X_{\tau_2}(\omega)) \cdot \mathbf{1}_{\tau_2 > K}(\omega)\, dP(\omega) \to 0$$

uniformly (in A) as $K \to \infty$ by the DCT as

$$\frac{1}{2} \cdot |X_K(\omega) - X_{\tau_2}(\omega)| \;\le\; S(\omega) = \sup_{0 \le k \le \tau_2} \Big| X_k(\omega) \Big| \;\in L^1(\Omega, P)$$

Finally,

$$X_{\tau_1} \cdot \mathbf{1}_{\tau_1 \le K} + X_K \cdot \mathbf{1}_{\tau_1 > K} \;\to\; X_{\tau_1} \quad \text{in} \quad L^1(\Omega, P)$$

as

$$E[|X_K| \cdot \mathbf{1}_{\tau_1 > K}] \;\le\; E[S \cdot \mathbf{1}_{\tau_1 > K}] \;\to\; 0$$

□

Optional Stopping Theorem.

Example. Random walk on $\mathbb{Z}$. Take $p \in (0,1)$, and let $\xi_1, \xi_2, \ldots$ be i.i.d. Bernoulli random variables such that

$$\xi_j = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } q = 1-p \end{cases}$$

Consider integers $0 < x_0 < M$. Let $X_0 = x_0$ and $X_n = X_0 + \xi_1 + \ldots + \xi_n$. Then, the first hitting time

$$\tau = \min\{t > 0 \ : \ X_t = 0 \ \text{ or } \ X_t = M\}$$

is a stopping time w.r.t. filtration $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.

We want to find $P(X_\tau = M)$.

If $p = \frac{1}{2}$, the random walk $X_n = X_0 + \xi_1 + \ldots + \xi_n$ is a martingale, and by part **(b)** of the Optional Stopping Theorem,

$$P(X_\tau = M) = \frac{x_0}{M}$$

Optional Stopping Theorem.

Example. Random walk on $\mathbb{Z}$. Take $p \in (0,1)$, and let $\xi_1, \xi_2, \ldots$ be i.i.d. Bernoulli random variables such that

$$\xi_j = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } q = 1-p \end{cases}$$

Consider integers $0 < x_0 < M$. Let $X_0 = x_0$ and $X_n = X_0 + \xi_1 + \ldots + \xi_n$. Then, the first hitting time

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is a stopping time w.r.t. filtration $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.

We want to find $P(X_\tau = M)$.

If $p \neq \frac{1}{2}$, then $X_n = X_0 + \xi_1 + \ldots + \xi_n$ is not a martingale, but $M_n = h(X_n)$ is a martingale when $h(x) = A \cdot \left(\frac{q}{p}\right)^x + B$ for any choice of constants A and B. Then, taking $A \neq 0$, by part **(b)** of the Optional Stopping Theorem,

$$P(X_\tau = M) = \frac{h(x_0) - h(0)}{h(M) - h(0)} = \frac{1 - \left(\frac{q}{p}\right)^{x_0}}{1 - \left(\frac{q}{p}\right)^{M}}$$

Martingale Convergence Theorem.

Jensen's inequality: If φ is a convex function, then

$$E[\varphi(X)|\mathcal{G}] \geq \varphi\Big(E[X|\mathcal{G}]\Big) \qquad a.s.$$

Proposition. If X_n is a submartingale w.r.t. $\mathcal{F}_n$ and φ is an non-decreasing convex function with $E[|\varphi(X_n)| < \infty$ for all n, then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$.

Proof. By Jensen's inequality,

$$E[\varphi(X_{n+1})|\mathcal{F}_n] \geq \varphi\Big(E[X_{n+1}|\mathcal{F}_n]\Big) \geq \varphi(X_n) \qquad a.s.$$

□

Martingale Convergence Theorem.

Suppose X_n is a submartingale:

$$E[X_{n+1} \mid \mathcal{F}_n] \geq X_n \quad a.s. \quad \forall n > 1$$

Let $a < b$ and let $N_0 = -1$,

$$N_{2k+1} = \inf\{n > N_{2k} \;:\; X_n \leq a\} \qquad k = 0, 1, \ldots,$$

$$N_{2k} = \inf\{n > N_{2k-1} \;:\; X_n \geq b\} \qquad k = 1, 2, \ldots$$

Then N_j are stopping times,

$$\{N_{2k-1} < n \leq N_{2k}\} = \{N_{2k-1} \leq n-1\} \cap \{N_{2k} \leq n-1\}^c \;\in \mathcal{F}_{n-1}$$

and

$$H_n = \begin{cases} 1 & \text{if } N_{2k-1} < n \leq N_{2k} \text{ for some } k \geq 1 \\ 0 & \text{otherwise} \end{cases} \qquad \in \mathcal{F}_{n-1}$$

Such time intervals $\left[N_{2k-1}, N_{2k}\right]$ are called upcrossings.

Let $U_n = \sup\{k \;:\; N_{2k} \leq n\}$ denote the number of upcrossings by time n.

Martingale Convergence Theorem.

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Such time intervals $\big[N_{2k-1}, N_{2k}\big]$ are called upcrossings.

Let $U_n = \sup\{k \ : \ N_{2k} \le n\}$ denote the number of upcrossings by time n.

The Upcrossing Inequality. If $\{X_n\}_{n=0,1,\ldots}$ is a submartingale, then

$$(b-a)\cdot E[U_n] \le E\Big[(X_n-a)^+\Big] - E\Big[(X_0-a)^+\Big]$$

Proof. Observe that $Y_n = a+(X_n-a)^+$ is also a submartingale, and it upcrosses $[a,b]$ the same number of times as X_n does, and therefore

$$(b-a)\cdot U_n \le (H\cdot Y)_n = \sum_{m=1}^{n} H_m \cdot (Y_m - Y_{m-1})$$

as $(H\cdot Y)_n$ adds up the upcrossings $Y(N_{2k}) - Y(N_{2k-1}) \ge b-a$ of Y.

Finally, $(b-a)\cdot E[U_n] \le E[(H\cdot Y)_n] \le E[Y_n - Y_0] = E\Big[(X_n-a)^+\Big] - E\Big[(X_0-a)^+\Big]$

Martingale Convergence Theorem.

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Finally, $(b-a)\cdot E[U_n] \le E[(H\cdot Y)_n] \le E[Y_n - Y_0] = E\Big[(X_n-a)^+\Big] - E\Big[(X_0-a)^+\Big]$

as $H_n \in \mathcal{F}_{n-1}$ and

$$E[Y_n - Y_0] - E[(H\cdot Y)_n] = E\left[\sum_{m=1}^{n}(1-H_m)\cdot(Y_m - Y_{m-1})\right]$$

$$= E\left[\sum_{m=1}^{n}(1-H_m)\cdot E[Y_m - Y_{m-1}|\mathcal{F}_{m-1}]\right] \ge 0$$

□

Martingale Convergence Theorem.

The Martingale Convergence Theorem. Suppose X_n is a submartingale such that

$$\sup_n E[X_n^+] < \infty$$

Then, as $n \to \infty$,

$$X_n \to X \qquad a.s.$$

where $X \in L^1(\Omega, P)$.

Proof. From the Upcrossing Inequality, $\forall a < b$,

$$(b-a) \cdot E[U_n] \leq E\left[(X_n - a)^+\right] - E\left[(X_0 - a)^+\right]$$

and, as $(x-a)^+ \leq x^+ + |a|$,

$$E[U_n] \leq \frac{E\left[X_n^+\right] + |a|}{b-a}$$

Thus, since $\sup_n E[X_n^+] < \infty$, and since U_n is an increasing sequence,

$$U_n \uparrow U, \quad \text{where } E[U] < \infty \text{ and } U < \infty \ \ a.s.$$

Martingale Convergence Theorem.

Proof. From the Upcrossing Inequality, $\forall a < b$,

$$(b-a)\cdot E[U_n] \le E\Big[(X_n - a)^+\Big] - E\Big[(X_0 - a)^+\Big]$$

and, as $(x-a)^+ \le x^+ + |a|$,

$$E[U_n] \le \frac{E\Big[X_n^+\Big] + |a|}{b-a}$$

Thus, since $\sup_n E[X_n^+] < \infty$, and since U_n is an increasing sequence,

$$U_n \uparrow U, \quad \text{where } E[U] < \infty \ \text{ and } \ U < \infty \quad a.s.$$

Thus

$$P\Bigg(\bigcup_{a,b\in\mathbb{Q}} \Big\{ \liminf_{n\to\infty} X_n < a < b < \limsup_{n\to\infty} X_n \Big\} \Bigg) = 0$$

and therefore

$$\liminf_{n\to\infty} X_n = \limsup_{n\to\infty} X_n \quad a.s.$$

Martingale Convergence Theorem.

Proof. (continued)

$$\liminf_{n\to\infty} X_n = \limsup_{n\to\infty} X_n \quad a.s.$$

Finally, we need to show that $X = \lim_{n\to\infty} X_n$ is in $L^1(\Omega, P)$.

By Fatou's Lemma,

$$E[X^+] \;\le\; \liminf_{n\to\infty} E[X_n^+] \;<\infty$$

Now, since X_n is a submartingale,

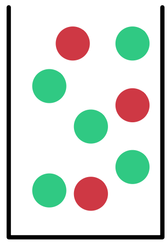
$$E[X_n^-] \;=\; E[X_n^+] - E[X_n] \;\le\; E[X_n^+] - E[X_0],$$

and by Fatou's Lemma,

$$E[X^-] \;\le\; \liminf_{n\to\infty} E[X_n^-] \;\le\; \sup_n E[X_n^+] - E[X_0] \;<\infty$$

□

Polya's Urn.



Polya's Urn

We begin with R_0 red marbles and G_0 green marbles in the urn, at time $t = 0$. At each iteration, a marble is selected from the urn, uniformly at random. Then the marble is returned to the urn, and D marbles of the same color as the selected marble are added into the urn.

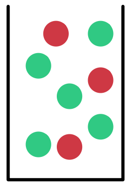
Let R_n and G_n denote respectively the number of red and green marbles after n iterations. Then the fraction of the red marbles at time n,

$$\rho_n = \frac{R_n}{R_n + G_n}$$

is a **martingale**:

$$E[\rho_{n+1} \mid \mathcal{F}_n] = \frac{R_n + D}{R_n + G_n + D} \cdot \frac{R_n}{R_n + G_n} + \frac{R_n}{R_n + G_n + D} \cdot \frac{G_n}{R_n + G_n} = \frac{R_n}{R_n + G_n} = \rho_n$$

Polya's Urn.



The fraction of the red marbles at time n,

$$\rho_n = \frac{R_n}{R_n + G_n}$$

is a **martingale**:

$$E[\rho_{n+1} \mid \mathcal{F}_n] = \frac{R_n + D}{R_n + G_n + D} \cdot \frac{R_n}{R_n + G_n} + \frac{R_n}{R_n + G_n + D} \cdot \frac{G_n}{R_n + G_n} = \frac{R_n}{R_n + G_n} = \rho_n$$

Thus, by the Martingale Convergence Theorem,

$$\rho_n \longrightarrow \rho_\infty \qquad a.s.$$

Here one can show that ρ_∞ is a beta random variable with parameters $(R_0 + D, G_0 + D)$ and density function

$$f(x) = \frac{1}{\mathbf{B}\big(R_0 + D, G_0 + D\big)} \cdot x^{R_0 + D - 1}(1 - x)^{G_0 + D - 1} \qquad 0 \le x \le 1$$