Math 664
Homework #3: Solutions

1. Recall that events $A_1, \ldots, A_n$ in a probability space $(\Omega, \mathcal{F}, P)$ are independent if
   
   $$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$$

   for any subcollection of distinct indices $i_1 < i_2 < \ldots < i_k$.

   Also recall that $X_1, \ldots, X_n$ are independent random variables if
   
   $$P(X_1 \in B_1, \ldots, X_n \in B_n) = \prod_{j=1}^n P(X_j \in B_j)$$

   for all Borel sets $B_1, \ldots, B_n \in \mathcal{B}$.

   (a). Show that if $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are independent events, then so are $A_1^c, A_2^c, \ldots, A_n^c$.

   (b). Show that if $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are independent events, then $1_{A_1}, 1_{A_2}, \ldots, 1_{A_n}$ are independent random variables.

   (c). Show that if $X_1, \ldots, X_n$ are independent random variables then the distribution of $X = (X_1, \ldots, X_n)$ is a product measure

   $$\nu_n = \mu_1 \times \ldots \times \mu_n$$

   and $d\nu(x_1, \ldots, x_n) = d\mu_1(x_1) \cdot \ldots \cdot d\mu_n(x_n)$.

   That is, show that $P\left((X_1, \ldots, X_n) \in B_1 \times \ldots \times B_n\right) = \mu_1(B_1) \cdot \ldots \cdot \mu_n(B_n)$ for all Borel sets $B_1, \ldots, B_n \in \mathcal{B}$.

   Conclude using the Fubini-Tonelli theorem that

   $$E[\phi_1(X_1)\ldots\phi_n(X_n)] = \int_{\mathbb{R}^n} \phi_1(x_1)\ldots\phi_n(x_n) \, d\nu(x_1, \ldots, x_n) = E[\phi_1(X_1)] \cdot \ldots \cdot E[\phi_n(X_n)]$$

   for all Borel measurable $\{\phi_j\}$ such that $E[|\phi_j(X_j)|] < \infty$.

   **SOLUTION:** (a). Let $i_1 = 1$. We need to show that

   $$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$$

   for any subcollection of distinct indices $i_1 = 1 < i_2 < \ldots < i_k$.

   Here

   $$P(A_1^c \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_1^c \cap A_{i_2} \cap \ldots \cap A_{i_k}) - P(A_1 \cap A_{i_2} \cap \ldots \cap A_{i_k})$$

   $$= P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k}) - P(A_1) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$$

   $$= (1 - P(A_1)) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$$

   $$= P(A_1^c) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$$
Hence $A_1^c, A_2, \ldots, A_n$ are independent events.

(b). Suppose $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are independent events. Consider a collection of Borel sets $B_1, \ldots, B_n \in \mathcal{B}$. Then for any $j \in \{1, \ldots, n\}$, the event

$$\{1_{A_j} \in B_j\} = \begin{cases} 
\emptyset & \text{if } 0, 1 \not\in B_j \\
A_j & \text{if } 1 \in B_j \text{ and } 0 \not\in B_j \\
A_j^c & \text{if } 0 \in B_j \text{ and } 1 \not\in B_j \\
\Omega & \text{if } 0, 1 \in B_j
\end{cases}$$

Thus, by part (a) of this exercise, for any choice of $B_1, \ldots, B_n \in \mathcal{B}$, the events

$$\{1_{A_1} \in B_1\}, \ldots, \{1_{A_n} \in B_n\}$$

are independent, and therefore

$$P(1_{A_1} \in B_1, \ldots, 1_{A_n} \in B_n) = P(1_{A_1} \in B_1) \cdot \ldots \cdot P(1_{A_n} \in B_n)$$

(c). Consider a collection of Borel sets $B_1, \ldots, B_n \in \mathcal{B}$. Then

$$P\left((X_1, \ldots, X_n) \in B_1 \times \ldots \times B_n\right) = P(X_1 \in B_1, \ldots, X_n \in B_n)$$

$$= \prod_{j=1}^{n} P(X_j \in B_j)$$

$$= \mu_1(B_1) \cdot \ldots \cdot \mu_n(B_n)$$

and therefore $d\nu(x_1, \ldots, x_n) = d\mu_1(x_1) \cdot \ldots \cdot d\mu_n(x_n)$ is the distribution of $(X_1, \ldots, X_n)$.

Hence, by the Fubini-Tonelli theorem,

$$E[\phi_1(X_1) \cdot \ldots \cdot \phi_n(X_n)] = \int_{\mathbb{R}^n} \phi_1(x_1) \cdot \ldots \cdot \phi_n(x_n) \, d\nu(x_1, \ldots, x_n)$$

$$= \int_{\mathbb{R}^n} \phi_1(x_1) \cdot \ldots \cdot \phi_n(x_n) \, d\mu_1(x_1) \cdot \ldots \cdot d\mu_n(x_n)$$

$$= \int_{\mathbb{R}} \phi_1(x_1) \, d\mu_1(x_1) \cdot \ldots \cdot \int_{\mathbb{R}} \phi_n(x_n) \, d\mu_n(x_n)$$

$$= E[\phi_1(X_1)] \cdot \ldots \cdot E[\phi_n(X_n)]$$

for all Borel measurable $\{\phi_j\}$ such that $E[|\phi_j(X_j)|] < \infty$. 
2. Given \( p \in [0, 1] \). Let \( X_1, X_2, \ldots \) be i.i.d. random variables such that for each \( j \),

\[
X_j = \begin{cases} 
1 & \text{with probability } p \\
-1 & \text{with probability } 1 - p 
\end{cases}
\]

Variables \( X_1, X_2, \ldots \) will represent the steps of a random walk on \( \mathbb{Z} \). Let \( S_0 = 0 \) and \( S_k = X_1 + \ldots + X_k \) be the location of the walker at time \( k \).

Let \( A_n = \{S_{2n} = 0\} \) be the event that the walker returns to the origin after \( 2n \) steps. Use the Stirling’s formula to prove that

\[
\sum_{n=1}^{\infty} P(A_n) = \begin{cases} 
< \infty & \text{if } p \neq \frac{1}{2} \\
\infty & \text{if } p = \frac{1}{2} 
\end{cases}
\]

Use the Borel-Cantelli Lemma to conclude that the random walk is transient whenever \( p \neq \frac{1}{2} \). A random walk that begins at the origin is said to be recurrent if the walker comes back to the origin infinitely often a.s., and is said to be transient if the walker returns to the origin at most finitely many times a.s.

**SOLUTION:** Recall the Stirling’s formula:

\[
n! = \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} \cdot (1 + o(1))
\]

Now,

\[
P(A_n) = \binom{2n}{n} \cdot (p(1-p))^n \\
= \frac{\sqrt{2\pi}(2n)^{2n+\frac{1}{2}}e^{-2n}}{2\pi n^{2n+1}e^{-2n}} \cdot (p(1-p))^n \cdot (1 + o(1)) \\
= (4p(1-p))^n \cdot \frac{1 + o(1)}{\sqrt{\pi \cdot n}},
\]

where

\[
4p(1-p) = \begin{cases} 
< 1 & \text{if } p \neq \frac{1}{2} \\
1 & \text{if } p = \frac{1}{2} 
\end{cases}
\]

and therefore

\[
\sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{\sqrt{n}} = \begin{cases} 
< \infty & \text{if } p \neq \frac{1}{2} \\
\infty & \text{if } p = \frac{1}{2} 
\end{cases}
\]

Hence,

\[
\sum_{n=1}^{\infty} P(A_n) = \begin{cases} 
< \infty & \text{if } p \neq \frac{1}{2} \\
\infty & \text{if } p = \frac{1}{2} 
\end{cases}
\]
By the Borel-Cantelli Lemma, the random walk is \textit{transient} whenever \( p \neq \frac{1}{2} \) as in that case, almost surely, no more than finitely many return events \( A_n = \{S_{2n} = 0\} \) occur.

3. (Bhattacharya and Waymire, p.57 # 4) Let \( X_1, X_2, \ldots \) be an i.i.d. sequence of positive random variables such that \( E[|\ln X_j|] < \infty \). Calculate the a.s. limiting geometric mean \( \lim_{n \to \infty} (X_1 \cdot X_2 \cdot \ldots \cdot X_n)^{\frac{1}{n}} \). Determine the numerical value of this limit in the case of uniformly distributed random variables on \( (0, 1) \).

**SOLUTION:** Since \( E[|\ln X_j|] < \infty \), by the Strong Law of Large Numbers,

\[
\lim_{n \to \infty} \frac{\ln X_1 + \ln X_2 + \ldots + \ln X_n}{n} = E[\ln X_1] \quad \text{a.s.}
\]

and therefore,

\[
\lim_{n \to \infty} (X_1 \cdot X_2 \cdot \ldots \cdot X_n)^{\frac{1}{n}} = \lim_{n \to \infty} e^{\frac{\ln X_1 + \ln X_2 + \ldots + \ln X_n}{n}} = e^{E[\ln X_1]} \quad \text{a.s.}
\]

Suppose \( X_1, X_2, \ldots \) is a sequence of i.i.d. uniformly distributed random variables on \( (0, 1) \). Then

\[
E[\ln X_1] = \int_0^1 \ln x \, dx = [x \ln x - x]_0^1 = -1
\]

by L'Hôpital’s rule. Therefore,

\[
\lim_{n \to \infty} (X_1 \cdot X_2 \cdot \ldots \cdot X_n)^{\frac{1}{n}} = \frac{1}{e} \quad \text{a.s.}
\]

4. (From “Probability” by D. Khoshnevisan) Prove the One-Series Theorem of Kolmogorov (1930): If \( X_1, X_2, \ldots \) are independent mean-zero random variables, and if \( \sum_{j=1}^{\infty} E[X_j^2] < \infty \), then \( \sum_{j=1}^{\infty} X_j \) converges almost surely.

Hint: Show that the sequence of partial sums \( S_n = X_1 + \ldots + X_n \) is almost surely a Cauchy sequence. Here is one approach. Let for each \( i \in \mathbb{N} \), a number \( N_i \) be such that \( \sum_{j=N_i}^{\infty} E[X_j^2] \leq \frac{1}{2^i} \). Use Kolmogorov’s Maximal Inequality to bound the probability of the event

\[
A_i = \{ \exists n > N_i : |S_n - S_{N_i}| > \epsilon/2 \}
\]

Next, use the Borel-Cantelli Lemma.
**SOLUTION:** We want to show that the sequence of partial sums \( S_n = X_1 + \ldots + X_n \) is almost surely a Cauchy sequence. Let for each \( i \in \mathbb{N} \), a number \( N_i \) be such that \( \sum_{j=N_i}^{\infty} \mathbb{E}[X_j^2] \leq \frac{1}{2^i} \). For any \( \epsilon > 0 \) consider event

\[ A_i = \{ \exists \, n > N_i : |S_n - S_{N_i}| > \epsilon/2 \} \]

Then, by Kolmogorov’s Maximal Inequality,

\[
P(A_i) = \lim_{m \to \infty} P \left( \max_{n : N_i \leq n \leq N_i + m} |S_n - S_{N_i}| > \epsilon/2 \right) \leq \lim_{m \to \infty} \frac{\text{Var}(S_{N_i+m} - S_{N_i})}{(\epsilon/2)^2} = \lim_{m \to \infty} \frac{\sum_{j=N_i+1}^{N_i+m} \mathbb{E}[X_j^2]}{(\epsilon/2)^2} = \frac{\sum_{j=N_i+1}^{\infty} \mathbb{E}[X_j^2]}{(\epsilon/2)^2} \leq \frac{1}{\epsilon^2 \cdot 2^{i-2}}
\]

as the probability measure \( P \) is continuous from below.

Therefore \( \sum_{i=1}^{\infty} P(A_i) < \infty \), and the Borel-Cantelli Lemma implies no more than finitely many events \( A_i \) occur at the same time, almost surely. That is, almost surely,

\[ \exists \, i \text{ such that } |S_n - S_{N_i}| \leq \epsilon/2 \quad \forall n > N_i \]

and therefore \( \forall n, m > N_i \),

\[ |S_n - S_m| \leq |S_n - S_{N_i}| + |S_m - S_{N_i}| \leq \epsilon \]

Thus the sequence \( S_n \) is almost surely Cauchy. Hence it converges.