1. Let $\Omega = \mathbb{R}$, $\mathcal{F} = \{ A \subseteq \mathbb{R} : \text{either } A \text{ or } A^c \text{ is countable} \}$, and
   
   
   $$P(A) = \begin{cases} 
   0 & \text{if } A \text{ is countable} \\
   1 & \text{if } A \text{ is uncountable.}
   \end{cases}$$

   Show that $(\Omega, \mathcal{F}, P)$ is a probability measure space.

   **SOLUTION:** First we show that $\mathcal{F}$ is a $\sigma$-algebra:

   (a) $\Omega \in \mathcal{F}$ since $\Omega^c = \emptyset$ countable.

   (b) $A \in \mathcal{F}$ if and only if $A^c \in \mathcal{F}$ if and only if $A$ or $A^c$ is countable.

   (c) If $A_1, A_2, A_3, \ldots \in \mathcal{F}$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$ because if all $A_j$ are countable then so is $\bigcup_{j=1}^{\infty} A_j$, and if at least one of $A_1, A_2, A_3, \ldots \in \mathcal{F}$ is uncountable then $\left( \bigcup_{j=1}^{\infty} A_j \right)^c = \bigcap_{j=1}^{\infty} A_j$ is countable.

   Next, we show that $P$ is a probability measure: $P(\Omega) = P(\mathbb{R}) = 1$, $P(\emptyset) = 0$, and for any sequence of disjoint (mutually non-intersecting) events $A_1, A_2, \cdots \in \mathcal{F}$, either all of them are countable and therefore $\bigcup_{j=1}^{\infty} A_j$ is countable, implying

   $$P \left( \bigcup_{j=1}^{\infty} A_j \right) = 0 = \sum_{j=1}^{\infty} P(A_j),$$

   or exactly one of $A_1, A_2, \cdots \in \mathcal{F}$ is uncountable and

   $$P \left( \bigcup_{j=1}^{\infty} A_j \right) = 1 = \sum_{j=1}^{\infty} P(A_j)$$

   This is because two disjoint events $A$ and $B$ in

   $$\mathcal{F} = \{ A \subseteq \mathbb{R} : \text{either } A \text{ or } A^c \text{ is countable} \}$$

   cannot be both uncountable as then $A$ and $A^c \supseteq B$ are both uncountable, contradicting $A \in \mathcal{F}$.

2. Suppose $X \geq 0$ is a nonnegative random variable over a probability measure space $(\Omega, \mathcal{F}, P)$, and suppose

   $$0 < \rho = E[X] < \infty.$$
Define for every $A \in \mathcal{F}$,
\[ Q(A) = \frac{1}{\rho} \int_A X(\omega) \, dP(\omega) \]
Show that $(\Omega, \mathcal{F}, Q)$ is another probability measure space, and $Q \ll P$, i.e. $Q$ is absolutely continuous with respect to $P$.

**SOLUTION:** First, we check that $Q$ is a probability measure:
\[ Q(\emptyset) = \frac{1}{\rho} \int_\emptyset X(\omega) \, dP(\omega) = 0. \]
For any sequence of disjoint (mutually non-intersecting) events $A_1, A_2, \cdots \in \mathcal{F}$,
\[ Q \left( \bigcup_{j=1}^\infty A_j \right) = \frac{1}{\rho} \int_{\bigcup_{j=1}^\infty A_j} X(\omega) \, dP(\omega) = \sum_{j=1}^\infty \frac{1}{\rho} \int_{A_j} X(\omega) \, dP(\omega) = \sum_{j=1}^\infty Q(A_j) \]
by the Monotone Convergence Theorem. Finally,
\[ Q(\Omega) = \frac{1}{\rho} \int_{\Omega} X(\omega) \, dP(\omega) = \frac{\rho}{\rho} = 1. \]
Next, we show that $Q \ll P$:
\[ \text{if } P(A) = 0 \text{ then } Q(A) = \frac{1}{\rho} \int_A X(\omega) \, dP(\omega) = 0. \]

3. Suppose, for $p > 0$, a sequence of random variables $X_n \rightarrow 0$ in $L^p$. That is
\[ E[|X_n|^p] = \int |X_n(\omega)|^p \, dP(\omega) \rightarrow 0 \quad \text{as} \ n \rightarrow \infty. \]
Show that $X_n \rightarrow 0$ in probability: for any $\epsilon > 0$,
\[ P(|X_n| \geq \epsilon) \rightarrow 0 \quad \text{as} \ n \rightarrow \infty. \]

**SOLUTION:** By Markov inequality,
\[ P(|X_n| \geq \epsilon) = P(|X_n|^p \geq \epsilon^p) \leq \frac{E[|X_n|^p]}{\epsilon^p} \rightarrow 0 \quad \text{as} \ n \rightarrow \infty. \]
4. Use Jensen’s inequality and size-biasing to show that if $X \geq 0$ is a random variable such that

$$0 < \rho = E[X] < \infty, \quad \sigma^2 = Var(X) < \infty, \quad \text{and } \frac{\sigma}{\rho} \leq M$$

for a given constant $M > 0$, then

$$\rho^{3/2} \leq E[X^{3/2}] \leq C \cdot \rho^{3/2}$$

for some constant $C > 1$, e.g. $C = \sqrt{1 + M^2}$.

**SOLUTION:** Since $(x^{3/2})'' = \frac{3}{4\sqrt{x}} > 0$ for $x > 0$, function $x^{3/2}$ is convex on $[0, \infty)$ and by Jensen’s inequality,

$$\rho^{3/2} \leq E[X^{3/2}]$$

For the upper bound, we will use size-biasing. Let

$$\nu(A) = \int_A \frac{x}{\rho} \, d\mu(x) \quad \forall A \in \mathcal{B}$$

Then $\nu$ is a probability measure over $(\mathbb{R}, \mathcal{B})$, and $\nu \ll \mu$.

Next, we apply Jensen’s inequality for the concave function $\sqrt{x}$ and obtain

$$E_\mu[X^{3/2}] = E_\mu[X \cdot \sqrt{X}] = \rho \cdot E_\nu[\sqrt{X}] \leq \rho \cdot \sqrt{E_\nu[X]} = \rho \cdot \sqrt{\frac{E_\mu[X^2]}{\rho}} = \rho \cdot \sqrt{\frac{\sigma^2 + \rho^2}{\rho}}$$

Therefore

$$E_\mu[X^{3/2}] \leq \rho^{3/2} \cdot \sqrt{\frac{\sigma^2 + \rho^2}{\rho^2}} \leq \rho^{3/2} \cdot \sqrt{1 + M^2}$$