1. (Easy!) Recall that events $A_1, \ldots, A_n$ in a probability space $(\Omega, \mathcal{F}, P)$ are independent if

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$$

for any subcollection of distinct indices $i_1 < i_2 < \ldots < i_k$.

Also recall that $X_1, \ldots, X_n$ are independent random variables if

$$P(X_1 \in B_1, \ldots, X_n \in B_n) = \prod_{j=1}^{n} P(X_j \in B_j)$$

for all Borel sets $B_1, \ldots, B_n \in \mathcal{B}$.

(a). Show that if $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are independent events, then so are $A_{i_1}^c, A_{i_2}^c, \ldots, A_{i_k}^c$.

(b). Show that if $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are independent events, then $1_{A_1}, 1_{A_2}, \ldots, 1_{A_n}$ are independent random variables.

(c). Show that if $X_1, \ldots, X_n$ are independent random variables then the distribution of $X = (X_1, \ldots, X_n)$ is a product measure

$$\nu_n = \mu_1 \times \ldots \times \mu_n$$

and $d\nu(x_1, \ldots, x_n) = d\mu_1(x_1) \cdot \ldots \cdot d\mu_n(x_n)$.

That is, show that $P\left( (X_1, \ldots, X_n) \in B_1 \times \ldots \times B_n \right) = \mu_1(B_1) \cdot \ldots \cdot \mu_n(B_n)$ for all Borel sets $B_1, \ldots, B_n \in \mathcal{B}$.

Conclude using the Fubini-Tonelli theorem that

$$E[\phi_1(X_1) \cdot \ldots \cdot \phi_n(X_n)] = \int_{\mathbb{R}^n} \phi_1(x_1) \cdot \ldots \cdot \phi_n(x_n) \, d\nu(x_1, \ldots, x_n) = \int_{\mathbb{R}^n} \phi_1(x_1) \cdot \ldots \cdot \phi_n(x_n) \, d\nu_n(x_1, \ldots, x_n) = E[\phi_1(X_1)] \cdot \ldots \cdot E[\phi_n(X_n)]$$

for all Borel measurable $\{\phi_j\}$ such that $E[|\phi_j(X_j)|] < \infty$. 
2. Given $p \in [0, 1]$. Let $X_1, X_2, \ldots$ be i.i.d. random variables such that for each $j$,
\[ X_j = \begin{cases} 
1 & \text{with probability } p \\
-1 & \text{with probability } 1 - p 
\end{cases} \]
Variables $X_1, X_2, \ldots$ will represent the steps of a random walk on $\mathbb{Z}$. Let $S_0 = 0$ and $S_k = X_1 + \ldots + X_k$ be the location of the walker at time $k$.

Let $A_n = \{S_{2n} = 0\}$ be the event that the walker returns to the origin after $2n$ steps. Use the Stirling’s formula to prove that
\[ \sum_{n=1}^{\infty} P(A_n) = \begin{cases} 
\infty & \text{if } p \neq \frac{1}{2} \\
< \infty & \text{if } p = \frac{1}{2} 
\end{cases} \]
Use the Borel-Cantelli Lemma to conclude that the random walk is transient whenever $p \neq \frac{1}{2}$. A random walk that begins at the origin is said to be recurrent if the walker comes back to the origin infinitely often a.s., and is said to be transient if the walker returns to the origin at most finitely many times a.s.

3. (Bhattacharya and Waymire, p.57 # 4) Let $X_1, X_2, \ldots$ be an i.i.d. sequence of positive random variables such that $E[\ln X_j] < \infty$. Calculate the a.s. limiting geometric mean $\lim_{n \to \infty} \left( X_1 \cdot X_2 \cdot \ldots \cdot X_n \right)^{\frac{1}{n}}$. Determine the numerical value of this limit in the case of uniformly distributed random variables on $(0, 1)$.

4. (From “Probability” by D. Khoshnevisan) Prove the One-Series Theorem of Kolmogorov (1930): If $X_1, X_2, \ldots$ are independent mean-zero random variables, and if $\sum_{j=1}^{\infty} E[X_j^2] < \infty$, then $\sum_{j=1}^{\infty} X_j$ converges almost surely.

Hint: Show that the sequence of partial sums $S_n = X_1 + \ldots + X_n$ is almost surely a Cauchy sequence. Here is one approach. Let for each $i \in \mathbb{N}$, a number $N_i$ be such that $\sum_{j=N_i}^{\infty} E[X_j^2] \leq \frac{1}{2^i}$. Use Kolmogorov’s Maximal Inequality to bound the probability of the event
\[ A_i = \{ \exists n > N_i : |S_n - S_{N_i}| > \epsilon/2 \} \]
Next, use the Borel-Cantelli Lemma.