1. Let $\Omega = \mathbb{R}$, 

$$
\mathcal{F} = \{ A \subseteq \mathbb{R} : \text{either } A \text{ or } A^c \text{ is countable} \},
$$

and 

$$
P(A) = \begin{cases} 
0 & \text{if } A \text{ is countable} \\
1 & \text{if } A \text{ is uncountable}.
\end{cases}
$$

Show that $(\Omega, \mathcal{F}, P)$ is a probability measure space.

2. Suppose $X \geq 0$ is a nonnegative random variable over a probability measure space $(\Omega, \mathcal{F}, P)$, and suppose 

$$0 < \rho = E[X] < \infty.$$

Define for every $A \in \mathcal{F}$, 

$$Q(A) = \frac{1}{\rho} \int_A X(\omega) \, dP(\omega)$$

Show that $(\Omega, \mathcal{F}, Q)$ is another probability measure space, and 

$$Q \ll P,$$

i.e. $Q$ is absolutely continuous with respect to $P$.

3. Suppose, for $p > 0$, a sequence of random variables $X_n \to 0$ in $L^p$. That is 

$$E[|X_n|^p] = \int |X_n(\omega)|^p \, dP(\omega) \to 0 \quad \text{as } n \to \infty.$$

Show that $X_n \to 0$ in probability: for any $\epsilon > 0$, 

$$P(|X_n| \geq \epsilon) \to 0 \quad \text{as } n \to \infty.$$

4. Use Jensen’s inequality and size-biasing to show that if $X \geq 0$ is a random variable such that 

$$0 < \rho = E[X] < \infty, \quad \sigma^2 = Var(X) < \infty, \quad \text{and } \frac{\sigma}{\rho} \leq M$$

for a given constant $M > 0$, then 

$$\rho^{3/2} \leq E[X^{3/2}] \leq C \cdot \rho^{3/2}$$

for some constant $C > 1$, e.g. $C = \sqrt{1 + M^2}$. 
