MTH 467/567

Lecture 24-27

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- The adjustment coefficient.
- Lundberg inequality.
- Solving the differential equations.

The Cramér-Lundberg Process.

$$C_t = u + ct - \sum_{j=1}^{N_t} Y_j,$$

where

- C_t is the insurance portfolio at time $t \geq 0$
- u is the initial capital
- c is the premium rate
- N_t is the Poisson process with rate $\lambda > 0$

Premium calculation: finding the right c

The Cramér-Lundberg Process.

$$C_t = u + ct - \sum_{j=1}^{N_t} Y_j$$

- **Integral equation:**

$$\delta(u) = e^{-\lambda h}\delta(u+ch) + \int_0^h \int_0^{u+ct} \delta(u+ct-y) f_Y(y)\, dy \lambda e^{-\lambda t}\, dt$$

- **Differential equation:**

$$c\delta'(u) = \lambda \left[\delta(u) - \int_0^u \delta(u-y) f_Y(y)\, dy \right]$$

Initial conditions: if $c > \lambda\mu$,

$$\lim_{u\to\infty} \delta(u) = 1 \qquad \text{and} \qquad \delta(0) = 1 - \frac{\lambda\mu}{c}.$$

The (Lundberg) adjustment coefficient.

Recall **the compound Poisson model**:

There, $N \sim Poisson(\lambda)$. Then $E[S] = \lambda\mu$,

$Var(S) = \lambda\mu_2$, and $M_S(s) = \exp\{\lambda(M_Y(s) - 1)\}$.

For $0 \le t' \le t$, consider the difference

$$C_{t'} - C_t = -c(t - t') + \sum_{j=N_{t'}+1}^{N_t} Y_j,$$

where $N_t - N_{t'} \sim Poisson\big(\lambda(t - t')\big)$. Thus,

$$E\left[\exp\left\{s \cdot \sum_{j=N_{t'}+1}^{N_t} Y_j\right\}\right] = \exp\left\{\lambda(t - t')(M_Y(s) - 1)\right\}.$$

The (Lundberg) adjustment coefficient.

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and therefore, the moment generating function

$$E\left[\exp\left\{s\big(C_{t'} - C_t\big)\right\}\right] = \exp\Big\{-(t-t')\big(cs - \lambda M_Y(s) + \lambda\big)\Big\}.$$

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The value of $s^* > 0$ (if exists) such that

$$cs^* - \lambda M_Y(s^*) + \lambda = 0$$

is called **the adjustment coefficient** (aka the **Lundberg exponent**).

The (Lundberg) adjustment coefficient.

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$$cs^* - \lambda M_Y(s^*) + \lambda = 0$$

is called **the adjustment coefficient**.

- **Example:** If $Y_j \sim Exponential(\alpha)$, then

$$cs^* - \lambda \frac{\alpha}{\alpha - s^*} + \lambda = 0$$

implies $cs^* - \frac{\lambda s^*}{\alpha - s^*} = 0$ and therefore

$$s^* = \alpha - \frac{\lambda}{c}.$$

Recall: $\frac{\lambda}{\alpha c} < 1$ and

$$\psi(u) = \frac{\lambda}{\alpha c} \exp\left\{ -\left(\alpha - \frac{\lambda}{c} \right) u \right\} < e^{-s^* u}.$$

Lundberg inequality.

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- **Lundberg inequality:**

$$\psi(u) < e^{-s^* u}$$

Lundberg inequality.
Recall

$$c\big(\delta(u)-\delta(0)\big) = \lambda \int_0^u \delta(x) \int_{u-x}^{\infty} f_Y(y)\,dy\,dx = \lambda \int_0^u \delta(u-x) \int_x^{\infty} f_Y(y)\,dy\,dx,$$

where $\delta(0) = 1 - \frac{\lambda\mu}{c}$. Plugging $\psi(u) = 1-\delta(u)$, obtain

$$c\psi(u) = \lambda\mu - \lambda \int_0^u \big(1-\psi(u-x)\big) \int_x^{\infty} f_Y(y)\,dy\,dx.$$

Next, since $\mu = \int_0^{\infty}\int_x^{\infty} f_Y(y)\,dy\,dx$,

$$c\psi(u) = \lambda\left(\int_u^{\infty}\int_x^{\infty} f_Y(y)\,dy\,dx + \int_0^u \psi(u-x)\int_x^{\infty} f_Y(y)\,dy\,dx\right).$$

Lundberg inequality.

$$\psi(u) < e^{-s^* u}$$

- **Proof:**

$$c\psi(u) = \lambda \left(\int_u^\infty \int_x^\infty f_Y(y)\,dy\,dx + \int_0^u \psi(u-x) \int_x^\infty f_Y(y)\,dy\,dx \right).$$

Here, ψ is continuous and $\psi(0) < 1 = e^{-s^* \cdot 0}$. Suppose there is smallest $u_0 > 0$ such that $\psi(u_0) = e^{-s^* u_0}$, then

$$c\psi(u_0) = \lambda \left(\int_{u_0}^\infty \int_x^\infty f_Y(y)\,dy\,dx + \int_0^{u_0} \psi(u_0-x) \int_x^\infty f_Y(y)\,dy\,dx \right)$$

$$< \lambda \left(\int_{u_0}^\infty \int_x^\infty f_Y(y)\,dy\,dx + \int_0^{u_0} e^{-s^*(u_0-x)} \int_x^\infty f_Y(y)\,dy\,dx \right)$$

Lundberg inequality: $\psi(u) < e^{-s^* u}$

- **Proof (cont.):**

$$c\psi(u_0) < \lambda\left(\int_{u_0}^{\infty}\int_{x}^{\infty} f_Y(y)\,dy\,dx + \int_0^{u_0} e^{-s^*(u_0-x)}\int_x^{\infty} f_Y(y)\,dy\,dx\right)$$

$$\le \lambda\int_0^{\infty} e^{-s^*(u_0-x)}\int_x^{\infty} f_Y(y)\,dy\,dx = \lambda e^{-s^* u_0}\int_0^{\infty} e^{s^* x}\int_x^{\infty} f_Y(y)\,dy\,dx$$

$$= \lambda e^{-s^* u_0}\int_0^{\infty}\int_0^{y} e^{s^* x} f_Y(y)\,dx\,dy = \lambda e^{-s^* u_0}\int_0^{\infty}\frac{1}{s^*}(e^{s^* y}-1)f_Y(y)\,dy$$

$$= \lambda e^{-s^* u_0}\frac{M_Y(s^*)-1}{s^*} = ce^{-s^* u_0}$$ - CONTRADICTION.

Thus, there is no $u_0 > 0$ such that $\psi(u_0) = e^{-s^* u_0}$.

Existence of the adjustment coefficient.

The value of $s^* > 0$ (if exists) such that

$$cs^* - \lambda M_Y(s^*) + \lambda = 0$$

is called **the adjustment coefficient**.

• **Claim:** The adjustment coefficient $s^* > 0$ exists if and only if

$$c > \lambda\mu.$$

The (Lundberg) adjustment coefficient.

• **Theorem.** Suppose the adjustment coefficient coefficient exists ($c > \lambda\mu$), and

$$M_Y'(s^*) = \int_0^\infty x e^{s^* x} f_Y(x)\, dx \ < \infty.$$

Then

$$\lim_{u\to\infty} e^{s^* u}\psi(u) = \frac{c - \lambda\mu}{\lambda M_Y'(s^*) - c}.$$

Solution to the differential equation.

We substitute $\delta(x) = 1 - \psi(x)$ into

$$c\delta'(x) = \lambda \left[\delta(x) - \int_0^x \delta(x-y) f_Y(y)\, dy \right]$$

and obtain

$$\psi'(x) = \frac{\lambda}{c} \left[\psi(x) - \int_x^\infty f_Y(y)\, dy - \int_0^x \psi(x-y) f_Y(y)\, dy \right].$$

Since by Lundberg inequality $\psi(u) < e^{-s^* u}$, the following **integral transform**

$$\widehat{\psi}(s) = \int_0^\infty e^{sx} \psi(x)\, dx \qquad \text{is well defined for } \; s < s^*.$$

Solution to the differential equation.

$$\psi'(x) = \frac{\lambda}{c}\left[\psi(x) - \int_x^\infty f_Y(y)\,dy - \int_0^x \psi(x-y) f_Y(y)\,dy\right].$$

Since by Lundberg inequality $\psi(u) < e^{-s^* u}$, the following **integral transform**

$$\widehat{\psi}(s) = \int_0^\infty e^{sx}\psi(x)\,dx \qquad \text{is well defined for} \quad s < s^*.$$

Here, for $s < s^*$, we have

$$\int_0^\infty e^{sx}\psi'(x)\,dx = \frac{\lambda}{c}\left[\int_0^\infty e^{sx}\psi(x)\,dx - \int_0^\infty e^{sx}\int_x^\infty f_Y(y)\,dy\,dx - \int_0^\infty e^{sx}\int_0^x \psi(x-y) f_Y(y)\,dy\,dx\right].$$

Solution to the differential equation.

$$\int_0^\infty e^{sx}\psi'(x)\,dx = \frac{\lambda}{c}\left[\int_0^\infty e^{sx}\psi(x)\,dx - \int_0^\infty e^{sx}\int_x^\infty f_Y(y)\,dy\,dx - \int_0^\infty e^{sx}\int_0^x \psi(x-y)f_Y(y)\,dy\,dx\right],$$

where

$$\int_0^\infty e^{sx}\psi'(x)\,dx = e^{sx}\psi(x)\Big|_0^\infty - s\int_0^\infty e^{sx}\psi(x)\,dx = -\frac{\lambda\mu}{c} - s\widehat{\psi}(s),$$

$$\int_0^\infty e^{sx}\int_x^\infty f_Y(y)\,dy\,dx = \int_0^\infty\int_0^y e^{sx}f_Y(y)\,dx\,dy = \int_0^\infty \frac{1}{s}(e^{sy}-1)f_Y(y)\,dy = \frac{M_Y(s)-1}{s},$$

and $$\int_0^\infty e^{sx}\int_0^x \psi(x-y)f_Y(y)\,dy\,dx = \int_0^\infty\int_0^x e^{s(x-y)}\psi(x-y)e^{sy}f_Y(y)\,dy\,dx = \widehat{\psi}(s)M_Y(s).$$

Solution to the differential equation.

$$\int_0^\infty e^{sx}\psi'(x)\,dx = \frac{\lambda}{c}\left[\int_0^\infty e^{sx}\psi(x)\,dx - \int_0^\infty e^{sx}\int_x^\infty f_Y(y)\,dy\,dx - \int_0^\infty e^{sx}\int_0^x \psi(x-y)f_Y(y)\,dy\,dx\right],$$

where

$$\int_0^\infty e^{sx}\psi'(x)\,dx = -\frac{\lambda\mu}{c} - s\widehat{\psi}(s), \qquad \int_0^\infty e^{sx}\int_x^\infty f_Y(y)\,dy\,dx = \frac{M_Y(s)-1}{s},$$

and $$\int_0^\infty e^{sx}\int_0^x \psi(x-y)f_Y(y)\,dy\,dx = \widehat{\psi}(s)M_Y(s).$$

Thus,

$$-\frac{\lambda\mu}{c} - s\widehat{\psi}(s) = \frac{\lambda}{c}\left[\widehat{\psi}(s) - \frac{M_Y(s)-1}{s} - \widehat{\psi}(s)M_Y(s)\right].$$

Solution to the differential equation.

For $s < s^*$, the **integral transform**

$$\widehat{\psi}(s) = \int_0^\infty e^{sx}\psi(x)\,dx \qquad \text{is well defined}$$

and

$$-\frac{\lambda\mu}{c} - s\widehat{\psi}(s) = \frac{\lambda}{c}\left[\widehat{\psi}(s) - \frac{M_Y(s)-1}{s} - \widehat{\psi}(s)M_Y(s)\right].$$

Thus, we have the following solution:

$$\widehat{\psi}(s) = \frac{c-\lambda\mu}{cs-\lambda(M_Y(s)-1)} - \frac{1}{s} \qquad (s < s^*).$$

Solution to the differential equation.

For $s < s^*$, the **integral transform**

$$\hat{\psi}(s) = \int_0^\infty e^{sx}\psi(x)\,dx = \frac{c - \lambda\mu}{cs - \lambda(M_Y(s) - 1)} - \frac{1}{s}.$$

For any choice of $r > 0$, let $s = s^* - r$, then

$$\int_0^\infty e^{-rx}e^{s^*x}\psi(x)\,dx = \frac{c - \lambda\mu}{c(s^* - r) - \lambda(M_Y(s^* - r) - 1)} - \frac{1}{s^* - r},$$

where for r small enough,

$$M_Y(s^* - r) = M_Y(s^*) - M_Y'(s^*)r + O(r^2).$$

Solution to the differential equation.

For any choice of $r > 0$,

$$\int_0^\infty e^{-rx} e^{s^* x} \psi(x)\, dx = \frac{c - \lambda\mu}{c(s^* - r) - \lambda(M_Y(s^* - r) - 1)} - \frac{1}{s^* - r},$$

where for r small enough,

$$M_Y(s^* - r) = M_Y(s^*) - M_Y'(s^*) r + O(r^2).$$

Thus,

$$\int_0^\infty e^{-rx} e^{s^* x} \psi(x)\, dx = \frac{c - \lambda\mu}{\lambda M_Y'(s^*) - c + O(r)} \cdot \frac{1}{r} - \frac{1}{s^* - r}$$

as

$$cs^* - \lambda M_Y(s^*) + \lambda = 0.$$

Hardy-Littlewood Tauberian Theorem.

• **Hardy-Littlewood Tauberian Theorem (Feller, Vol. II):** Suppose $f(x)\begin{cases} \geq 0 & \text{if } x \geq 0 \\ = 0 & \text{if } x < 0 \end{cases}$.

Consider the Laplace transform of $f(x)$,

$$\omega(r) = \int_0^\infty e^{-rx} f(x)\,dx \quad (r \geq 0).$$

If

$$\lim_{r \to 0+} r^\rho \omega(r) = C$$

then

$$\lim_{x \to \infty} x^{1-\rho} f(x) = \frac{C}{\Gamma(\rho)}.$$

The (Lundberg) adjustment coefficient.

- **Theorem.** Suppose the adjustment coefficient coefficient exists ($c > \lambda\mu$), and

$$M_Y'(s^*) = \int_0^\infty x e^{s^* x} f_Y(x)\, dx \ < \infty.$$

Then

$$\lim_{u\to\infty} e^{s^* u}\psi(u) = \frac{c-\lambda\mu}{\lambda M_Y'(s^*) - c}.$$

- **Proof:** Recall

$$\int_0^\infty e^{-rx} e^{s^* x}\psi(x)\, dx = \frac{c-\lambda\mu}{\lambda M_Y'(s^*) - c + O(r)} \cdot \frac{1}{r} - \frac{1}{s^* - r}$$

The (Lundberg) adjustment coefficient.

- **Proof (cont.):** Recall

$$\int_0^\infty e^{-rx} e^{s^* x} \psi(x)\, dx = \frac{c - \lambda\mu}{\lambda M_Y'(s^*) - c + O(r)} \cdot \frac{1}{r} - \frac{1}{s^* - r}$$

- **Hardy-Littlewood Tauberian Theorem:**

If $\lim\limits_{r \to 0+} r^\rho \int_0^\infty e^{-rx} f(x)\, dx = C$ then $\lim\limits_{x \to \infty} x^{1-\rho} f(x) = \frac{C}{\Gamma(\rho)}$.

Thus, since $\lim\limits_{r \to 0+} r \int_0^\infty e^{-rx} e^{s^* x} \psi(x)\, dx = \frac{c - \lambda\mu}{\lambda M_Y'(s^*) - c}$,

$$\lim_{x \to \infty} e^{s^* x} \psi(x) = \frac{c - \lambda\mu}{\lambda M_Y'(s^*) - c} \qquad \text{(for } \rho = 1\text{).}$$

Connection to martingales.

For $0 \leq t' \leq t$, the moment generating function

$$E\left[\exp\left\{s\big(C_{t'} - C_t\big)\right\}\right] = \exp\Big\{-(t-t')\big(cs - \lambda M_Y(s) + \lambda\big)\Big\}.$$

The value of $s^* > 0$ (if exists) such that

$$cs^* - \lambda M_Y(s^*) + \lambda = 0$$

is called **the adjustment coefficient** (aka the **Lundberg exponent**).

Let $h(u) = e^{-s^* u}$. Then, $M_t = h(C_t)$ is a **martingale**: for $0 \leq t' \leq t$,

$$E[M_t \mid C_{t'}] = M_{t'}.$$

Such $h(u)$ is said to be the probability harmonic function, and we know from the Lundberg inequality that

$$\psi(u) < h(u).$$