# Math 465/565
## Homework #3 - Due Friday, June 4th

1. (15 pts) **Poisson process.** Let $\{N(t)\}_{t\geq 0}$ be the Poisson process with intensity $\lambda > 0$. For $t > 0$, the transition probability $p_t(i,j) = P\big(N(t) = j - i\big)$.

   **i.** (5 pts) Find the expectation $E[\,N(t)\,N(t+s)\,]$ for $t, s \geq 0$.

   Hint: Recall the independence of increments property. That is

   $$N(t_1) - N(t_0),\ N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$$

   are independent for any sequence $0 \leq t_1 < t_2 < \ldots < t_n$.

   **ii.** (10 pts) Recall that the matrix $P_t$ of transition probabilities satisfies Kolmogorov's backward equation:

   $$\frac{d}{dt} P_t = G P_t,$$

   where $G$ denotes the generator. That is

   $$\frac{d}{dt} p_t(i,j) = \sum_{k:k\neq i} g(i,k) p_t(k,j) - \Big(\sum_{k:k\neq i} g(i,k)\Big) p_t(i,j).$$

   Write down the above Kolmogorov's backward equation for the Poisson process $N(t)$. Show that $p_t(i,j) = e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!}$ solves the equation for all $0 \leq i \leq j$.

2. (20 pts) **M/M/1 queue with balking.** Imagine a bank with **one** teller that serves customers who queue in a single line. Customers arrive at times of a Poisson process with rate $\lambda > 0$ but only join the queue with probability $\frac{1}{n+1}$ if there are $n$ customers in line (not counting the customer that is being served by the teller). Each customer requires an independent amount of service time that has an exponential distribution with rate $\mu > 0$. Do the following parts.

   **i.** (5 pts) Write down the transition rates $g(\cdot,\cdot)$ for the number of customers in the system.

   **ii.** (5 pts) Write down the system of equations that comes out of the detailed balance condition.

   **iii.** (10 pts) Find the stationary distribution $\pi = (\pi(0), \pi(1), \ldots)$. Show your work.

3. (10 pts) Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively. Find the probability that the first Poisson process reaches $n$ before the second reaches $m$.

Hint: If $T$ and $S$ are independent continuous random variables with respective density functions $f_T(x)$ and $f_S(x)$, then

$$P(T < S) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_T(x) f_S(y)\, dx\, dy = \int_{-\infty}^{\infty} f_T(x)\big(1 - F_S(x)\big)\, dx,$$

where $F_S(x)$ is the cumulative distribution function of $S$. Also, by integration by parts, the cumulative distribution function of a gamma random variable with parameters $(\alpha, \lambda)$ equals

$$F(x) = 1 - \sum_{j=0}^{\alpha-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \quad \text{whenever } \alpha \text{ is a positive integer.}$$