# Math 465/565
## Homework #2 - Due Friday, May 14th

1. (35 pts) Consider a birth-and-death chain $X_t$ on $S = \{0, 1, 2, \ldots\}$ with the following transition probabilities: $p(0, 1) = 1$, and

$$p(j, j + 1) = p \quad \text{and} \quad p(j, j - 1) = q$$

for $j = 1, 2, \ldots$, where $p \in \left(0, \frac{1}{2}\right)$ and $q = 1 - p$. Do the following parts.

**i.** (5 pts) Show the Markov chain is recurrent.

**ii.** (10 pts) Let $T = \min\{t \geq 0 : X_t = 0\}$, and $\varphi(j) = E[T \,|\, X_0 = j]$ for all $j \geq 0$. Assuming it is known that the expectation $\varphi(1)$ is finite[1], find $\varphi(1)$.

Hint: Let $N$ be the number of excursions to the right of state 1 before the first visit to state 0. Find the probability mass function of $N$ and $E[N]$. Next, show that $\varphi(1) = 1 + E[N]\big(\varphi(1) + 1\big)$.

**iii.** (10 pts) Find $\varphi(j) = E[T \,|\, X_0 = j]$ for all $j \geq 0$.

Hint: Convert the recursion into a difference equation with known $\varphi(0)$ and $\varphi(1)$.

**iv.** (5 pts) For $x \in S$, let $T_x = \min\{t > 0 : X_t = x\}$. Find $E_0[T_0]$.

Hint: $E_0[T_0] = 1 + \varphi(1)$.

**v.** (5 pts) Conclude that the Markov chain is positive recurrent. That is, all of its states $x \in S$ satisfy $E_x[T_x] < \infty$.

Hint: You can use the fact that any irreducible Markov chain over a **finite** state space is positive recurrent.

Show your work.

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[1] Showing finiteness of $\varphi(1)$ requires a different argument. One option is to use Stirling's formula for establishing

$$\varphi(1) = E[T \,|\, X_0 = 1] = \sum_{n=0}^{\infty} (2n + 1) P(T = 2n + 1 \,|\, X_0 = 1) = \sum_{n=0}^{\infty} (2n + 1) C_n p^n q^{n+1} < \infty,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is known as the *Catalan number*. Indeed, by Stirling's formula, $C_n \sim \frac{4^n}{\sqrt{\pi} n^{3/2}}$ and $(2n + 1) C_n p^n q^{n+1} \sim 2q \frac{(4pq)^n}{\sqrt{\pi n}}$. Since, $p < \frac{1}{2}$, we have $4pq = 4p(1 - p) < 1$ and $\sum_{n=0}^{\infty} \frac{(4pq)^n}{\sqrt{\pi n}} < \infty$.

2. (15 pts) Let $\xi_1, \xi_2, ...$ be independent identically distributed Bernoulli random variables with $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$ for all $i = 1, 2, \ldots$. Let $S_0 = s_0$, where $s_0$ is a fixed constant, and $S_n = s_0 + \xi_1 + ... + \xi_n$ for all $n \geq 1$. Prove that $M_n = S_n^2 - n$ is a martingale with respect to $S_n$.

3. (15 pts) For $n \in \mathbb{N}$, let $X_t$ be a time homogeneous Markov chain on $S = \{0, 1, 2, \ldots, n\}$ with transition probabilities

$$p(i,j) = \binom{n}{j} \frac{i^j (n-i)^{n-j}}{n^n} \qquad \forall i, j \in S,$$

where we let $0 \cdot \ln 0 = 0$, and therefore, $0^0 = 1$. Notice that 0 and $n$ are the trap states. Show that $X_t$ is a martingale. Next, use Optional Stopping Theorem to find $P(X_T = n \,|\, X_0 = k)$, where $T = \min\{t \geq 0 \,:\, X_t = 0 \text{ or } n\}$.