Math 465/565
Homework #1 - Due Monday, April 26th

1. (25 pts) In a video game the player travels between the three universes, Universe **1**, Universe **2**, and Universe **3**. Universe **1** has three planets, Universe **2** has one planet, and Universe **3** has five planets. The video game starts in Universe **1** ($X_0 = 1$). Every epoch, the player is relocated to one of the two universes, other than the one the player is in. The relocation happens at random, with the next universe selected according to the probabilities proportional to the number of planets in a universe: if one universe has i planets and the other has j planets, then the probabilities for relocating to the respective universes are $\frac{i}{i+j}$ and $\frac{j}{i+j}$. Do the following parts.

   i. (5 pts) Let X_t represent the universe that the player is in during the epoch t. Write down the transition matrix P for the Markov chain $X_0, X_1, X_2, \ldots$.

   ii. (5 pts) List all the recurrent states. You don't have to explain your answer.

   iii. (5 pts) Is the Markov chain X_t aperiodic? Explain your answer.

   iv. (5 pts) We know from the statement of the problem that $P(X_0 = 1) = 1$. Find the distribution

   $$\mu_2 = \Big(P(X_2 = 1),\ P(X_2 = 2),\ P(X_2 = 3) \Big).$$

   Show your work.

   v. (5 pts) In the long run (the game never ends), what is the fraction of time that the player spends in each universe? Explain your answer.

2. (15 pts) **Reflecting random walk.** Let $0 < a < \frac{1}{2}$ be given. Consider a random walk on $\{0, 1, 2, ...\}$ with the following transition probabilities: $p(0,0) = 1 - a$,

   $$p(i, i+1) = a \text{ when } i \geq 0 \qquad \text{and} \qquad p(i, i-1) = 1 - a \text{ when } i \geq 1.$$

   Show that $\pi = (\pi(0), \pi(1), \pi(2), \ldots)$ with

   $$\pi(i) = (1 - 2a)\frac{a^i}{(1-a)^{i+1}} \qquad i = 0, 1, 2, \ldots$$

   is a stationary distribution.

3. (10 pts) Suppose

   $$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

   is the transition matrix. Which ones of the five states (1, 2, 3, 4, and 5) are recurrent?

4. (10 pts) Consider a Markov chain on $S = \{0, 1, \ldots, n\}$ with transition probabilities

$$p(j, j+1) = \frac{n-j}{n} \quad \text{and} \quad p(j, j-1) = \frac{j}{n}.$$

Find its stationary distribution.