MTH 465/565

Lectures 20 - 28

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Topics:

- Continuous-time Markov chains.
- Stationary distributions. Ergodicity.
- Poisson process.
- Thinning and superposition.
- Reversibility.
- Continuous-time birth-and-death chain.
- Queueing processes.

Continuous-time Markov chains.

Consider a family of random variables (process) $\{X(t)\}$ with $t \in [0, \infty)$, taking values in a discrete state space S. W.l.o.g. we enumerate the state space by letting

$$S = \{0, 1, 2, \ldots\}.$$

Process $X(t)$ is a continuous-time Markov chain if it satisfies the following continuous Markov property:

$$P\Big(X(t) = j \mid X(t_n) = i_n, X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1\Big) \;=\; P(X(t) = j \mid X(t_n) = i_n)$$

for any sequence

$$0 \le t_1 < t_2 < \ldots < t_n < t.$$

Continuous-time Markov chain $X(t)$ is said to be time homogeneous if there are probabilities $p_t(i, j)$ with $t \in [0, \infty)$ and $i, j \in S$ such that

$$P(X(t) = j \mid X(s) = i) = p_{t-s}(i, j) \qquad \text{for all } 0 \le s \le t, \quad i, j \in S.$$

Continuous-time Markov chains.

Consider a time homogeneous continuous-time Markov chain $X(t)$.

$$P(X(t) = j \mid X(s) = i) = p_{t-s}(i,j) \qquad \text{for all } 0 \le s < t, \quad i,j \in S.$$

For $t \in [0, \infty)$, let

$$P_t = \Big(p_t(i,j)\Big)_{i,j \in S}$$

Observe that $P_0 = I$, and P_t is stochastic, i.e., non-negative with rows adding to one.

Chapman-Kolmogorov equation.

$$P_{s+t} = P_s P_t, \qquad \forall s,t \in [0,\infty)$$

i.e., $\{P_t\}_{t \ge 0}$ form a semigroup.

Continuous-time Markov chains. $P_0 = I$, and

$$P_{s+t} = P_s\, P_t, \qquad \forall s, t \in [0, \infty)$$

i.e., $\{P_t\}_{t \geq 0}$ form a semigroup.

Generator:

$$G = \lim_{h \to 0+} \frac{P_h - I}{h}$$

Forward equations:

$$\frac{d}{dt} P_t = P_t G, \qquad P_0 = I$$

Backward equations:

$$\frac{d}{dt} P_t = G P_t, \qquad P_0 = I$$

Representation:

$$P_t = \exp\{tG\} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k G^k, \qquad t \in [0, \infty)$$

Continuous-time Markov chains: generator.

Properties of generators $G = \lim_{h \to 0+} \dfrac{P_h - I}{h}$

- Rows of G add up to one;
- Diagonal elements of G are nonpositive;
- Off-diagonal elements of G are nonnegative.

Stationary distribution:

$$\pi G = 0 \quad \Leftrightarrow \quad \pi P_t = \pi \quad \forall t \in [0, \infty)$$

Ergodicity: Under irreducibility and positive recurrence conditions, there exists a unique stationary distribution π, and

$$\lim_{t \to \infty} p_t(i, j) = \pi(j) \qquad \forall i, j \in S.$$

Continuous-time Markov chains: generator.
Stationary distribution:

$$\pi G = 0 \quad \Leftrightarrow \quad \pi P_t = \pi \quad \forall t \in [0, \infty)$$

Proof: If $\pi G = 0$, then

$$\pi P_t = \pi \exp\{tG\} = \pi \left(I + \sum_{k=1}^{\infty} \frac{1}{k!} t^k G^k \right) = \pi + \sum_{k=1}^{\infty} \frac{1}{k!} t^k \pi G^k = \pi.$$

Conversely, if $\pi P_t = \pi$ for all $t \geq 0$, then

$$\pi G = \pi \left(\lim_{h \to 0+} \frac{P_h - I}{h} \right) = \lim_{h \to 0+} \frac{\pi P_h - \pi}{h} = 0.$$

Ergodicity: Under irreducibility ($\forall i, j \; p_t(i,j) > 0$ for $t > 0$) and positive recurrence conditions, there exists a unique stationary distribution π, and

$$\lim_{t \to \infty} p_t(i,j) = \pi(j) \qquad \forall i, j \in S.$$

Continuous-time Markov chains.

Example. Consider $S = \{0, 1\}$, and for $\lambda, \mu > 0$, let

$$G = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

Interpretation: Process X_t jumps from state 0 to state 1 after waiting for an interarrival time, distributed as an exponential random variable with parameter $\lambda > 0$. Process X_t jumps from state 1 to state 0 after waiting for an interarrival time, distributed as an exponential random variable with parameter $\mu > 0$. The interarrival times are independent random variables.

We want to find P_t. We use backward equations:

$$\frac{d}{dt} P_t = G P_t, \qquad P_0 = I$$

Continuous-time Markov chains.

Example (continued). Consider $S = \{0, 1\}$, and for $\lambda, \mu > 0$, let

$$G = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

We want to find P_t. We use backward equations:

$$\frac{d}{dt} P_t = G P_t, \qquad P_0 = I.$$

Denote $\varphi(t) = p_t(0, 1)$ and $\psi(t) = p_t(1, 0)$, then

$$P_t = \begin{pmatrix} 1 - \varphi(t) & \varphi(t) \\ \psi(t) & 1 - \psi(t) \end{pmatrix} \qquad \text{and}$$

$$\begin{pmatrix} -\varphi'(t) & \varphi'(t) \\ \psi'(t) & -\psi'(t) \end{pmatrix} = \frac{d}{dt} P_t = G P_t = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} 1 - \varphi(t) & \varphi(t) \\ \psi(t) & 1 - \psi(t) \end{pmatrix}.$$

Continuous-time Markov chains.

Example (continued).
Denote $\varphi(t) = p_t(0,1)$ and $\psi(t) = p_t(1,0)$, then

$$\begin{pmatrix} -\varphi'(t) & \varphi'(t) \\ \psi'(t) & -\psi'(t) \end{pmatrix} = \frac{d}{dt}P_t = GP_t = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} 1-\varphi(t) & \varphi(t) \\ \psi(t) & 1-\psi(t) \end{pmatrix}.$$

Hence,

$$\varphi'(t) = \lambda\big(1-\varphi(t)-\psi(t)\big) \qquad \text{and} \qquad \psi'(t) = \mu\big(1-\varphi(t)-\psi(t)\big)$$

with the initial conditions $\varphi(0) = \psi(0) = 0$ as $P_0 = I$.
First,

$$\frac{d}{dt}\big(\mu\varphi(t) - \lambda\psi(t)\big) = 0,$$

implying $\psi(t) = \frac{\mu}{\lambda}\varphi(t)$. Substituting, we have

$$\varphi'(t) = \lambda - (\lambda+\mu)\varphi(t), \qquad \varphi(0) = 0.$$

Continuous-time Markov chains.

Example (continued).
Denote $\varphi(t) = p_t(0,1)$ and $\psi(t) = p_t(1,0)$, then

$$\varphi'(t) = \lambda\big(1 - \varphi(t) - \psi(t)\big) \qquad \text{and} \qquad \psi'(t) = \mu\big(1 - \varphi(t) - \psi(t)\big)$$

with the initial conditions $\varphi(0) = \psi(0) = 0$ as $P_0 = I$.

We deduce $\psi(t) = \frac{\mu}{\lambda}\varphi(t)$. Substituting, we have

$$\varphi'(t) = \lambda - (\lambda + \mu)\varphi(t), \qquad \varphi(0) = 0.$$

Solving it, obtain

$$p_t(0,1) = \varphi(t) = \frac{\lambda}{\lambda + \mu}\left(1 - e^{-(\lambda+\mu)t}\right)$$

and $$p_t(1,0) = \psi(t) = \frac{\mu}{\lambda}\varphi(t) = \frac{\mu}{\lambda + \mu}\left(1 - e^{-(\lambda+\mu)t}\right).$$

Continuous-time Markov chains.

Example (continued). Consider $S = \{0, 1\}$, and for $\lambda, \mu > 0$, let

$$G = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

We find

$$P_t = \begin{pmatrix} \frac{\mu}{\lambda+\mu}\left(1 + \frac{\lambda}{\mu} e^{-(\lambda+\mu)t}\right) & \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right) \\ \frac{\mu}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right) & \frac{\lambda}{\lambda+\mu}\left(1 + \frac{\mu}{\lambda} e^{-(\lambda+\mu)t}\right) \end{pmatrix}.$$

Notice that

$$\pi = \left(\frac{\mu}{\lambda+\mu}, \quad \frac{\lambda}{\lambda+\mu} \right)$$

is the stationary distribution for the Markov chain. Observe ergodicity:

$$\lim_{t\to\infty} p_t(i, j) = \pi(j) \qquad \forall i, j \in S.$$

Interarrival times.

Let $Y_1, Y_2, \ldots, Y_n$ be independent exponential random variables with respective parameters

$$\lambda_1 > 0,\ \lambda_2 > 0, \ldots, \lambda_n > 0.$$

Then, $Y = \min\{Y_1, Y_2, \ldots, Y_n\}$ is also an exponential random variable with the parameter

$$\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

Indeed, for $a \geq 0$,

$$P(Y > a) = P\big(Y_1 > a,\ Y_2 > a, \ldots, Y_n > a\big)$$

$$= P(Y_1 > a)P(Y_2 > a) \ldots P(Y_n > a) = e^{-\lambda_1 a} e^{-\lambda_2 a} \ldots e^{-\lambda_n a}$$

$$= e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_n)a} = e^{-\lambda a}.$$

Hence, $F_Y(a) = P(Y \leq a) = 1 - P(Y > a) = 1 - e^{-\lambda a}$.

Interarrival times.

Let $Y_1, Y_2, \ldots, Y_n$ be independent exponential random variables with respective parameters

$$\lambda_1 > 0,\ \lambda_2 > 0, \ldots, \lambda_n > 0.$$

Then, for each $j = 1, 2, \ldots, n$,

$$P\Big(\min\{Y_1, Y_2, \ldots, Y_n\} = Y_j\Big) = \frac{\lambda_j}{\lambda},$$

where $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. Indeed, for $j = 1$,

$$P\Big(\min\{Y_1, Y_2, \ldots, Y_n\}\Big) = \int_0^\infty P\big(Y_2 > x, \ldots, Y_n > x\big)\lambda_1 e^{\lambda_1 x}\, dx$$

$$= \int_0^\infty P(Y_2 > x) \ldots P(Y_n > x)\lambda_1 e^{\lambda_1 x} dx = \int_0^\infty \lambda_1 e^{-\lambda_1 x} e^{-\lambda_2 x} \ldots e^{-\lambda_n x} dx$$

$$= \int_0^\infty \lambda_1 e^{-\lambda x}\, dx = \frac{\lambda_1}{\lambda}.$$

Continuous-time Markov chains.

Example. Let $S = \{1, 2, \ldots, n\}$ and the generator

$$G = \begin{pmatrix} -\sum\limits_{j:j\neq 1} g_{1,j} & g_{1,2} & g_{1,3} & \cdots & g_{1,n} \\ g_{2,1} & -\sum\limits_{j:j\neq 2} g_{2,j} & g_{2,3} & \cdots & g_{2,n} \\ g_{3,1} & g_{3,2} & -\sum\limits_{j:j\neq 3} g_{3,j} & \ddots & g_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ g_{n,1} & g_{n,2} & g_{n,3} & \cdots & -\sum\limits_{j:j\neq n} g_{n,j} \end{pmatrix},$$

where $g_{i,j} \geq 0$ are the transition rates.

• Once at state i, the duration of stay at state i is an exponential random variable with parameter $g_i = \sum\limits_{j:j\neq i} g_{i,j}$.

• Then it relocates to one of the states $j \neq i$, selected with probability $g_{i,j}/g_i$.

Poisson process $N(t)$ with rate $\lambda > 0$.

Let $X_1, X_2, \ldots$ be independent exponential random variables with parameter $\lambda > 0$.

- **Arrival times:** Let $T_0 = 0$ and $T_n = \sum_{k=1}^{n} X_k$ for $n = 1, 2, \ldots$.
- **Interarrival times:** $X_n = T_n - T_{n-1}$
- Poisson process with intensity λ is defined as
$$N(t) = \max\{n \geq 0 : T_n \leq t\} \quad (t \geq 0).$$

Here, $N(t)$ counts the number of arrivals between 0 and t.

- The increment $N(t_0+L)-N(t_0)$ counts the number of arrivals between t_0 and $t_0 + L$.
- Because of memorylessness property of exponential random variables X_j, the increment $N(t_0+L)-N(t_0)$ is distributed as $N(L)$.
- $N(t_0+L)-N(t_0)$ is a Poisson random variable with parameter λL:
$$P\Big(N(t_0 + L) - N(t_0) = k\Big) = e^{-\lambda L}\frac{(\lambda L)^k}{k!} \quad (k = 0, 1, \ldots).$$

Poisson process.

$$P\Big(N(t_0+L)-N(t_0)=k\Big) = e^{-\lambda L}\frac{(\lambda L)^k}{k!} \quad (k=0,1,\ldots).$$

Proof: Recall $N(t_0+L)-N(t_0)$ is distributed as $N(L)$.

Now, since

$$\{T_k \le L\} = \{T_k \le L < T_{k+1}\} \cup \{T_{k+1} \le L\},$$

$$P(N(L)=k) = P(T_k \le L < T_{k+1}) = P(T_k \le L) - P(T_{k+1} \le L)$$

Since $T_k = X_1 + \ldots + X_k$ is a gamma random variable with parameters (k,λ),

$$P(N(L)=k) = \int_0^L \frac{\lambda^k x^{k-1}}{(k-1)!}e^{-\lambda x}dx - \int_0^L \frac{\lambda^{k+1}x^k}{k!}e^{-\lambda x}dx$$

Integration by parts:

$$\int_0^L \frac{x^{k-1}}{(k-1)!}e^{-\lambda x}dx = e^{-\lambda x}\frac{x^k}{k!}\Big|_{x=0}^{L} + \int_0^L \frac{x^k}{k!}\lambda e^{-\lambda x}dx$$

Plugging in: $\quad P(N(L)=k) = e^{-\lambda L}\frac{(\lambda L)^k}{k!}$

Poisson process.

$$P\Big(N(t_0+L)-N(t_0)=k\Big)=e^{-\lambda L}\frac{(\lambda L)^k}{k!} \quad (k=0,1,\ldots).$$

Thus, $p_t(i,j)=0$ if $j<i$, and

$$p_t(i,j)=e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}, \qquad 0\le i\le j<\infty.$$

So, as $h\to 0+$, we have

- $p_h(i,i)=e^{-\lambda h}=1-\lambda h+o(h)$;
- $p_h(i,i+1)=e^{-\lambda h}\lambda h=\lambda h+o(h)$;
- $\sum_{k=2}^{\infty} p_h(i,i+k)=o(h)$.

Poisson process.

- $p_h(i,i) = e^{-\lambda h} = 1 - \lambda h + o(h)$;
- $p_h(i,i+1) = e^{-\lambda h}\lambda h = \lambda h + o(h)$;
- $p_h(i,i+k) = o(h)$ for all $k \geq 2$.

Generator:

$$G = \lim_{h \to 0+} \frac{P_h - I}{h} = \begin{pmatrix} -\lambda & \lambda & 0 & \dots \\ 0 & -\lambda & \lambda & \dots \\ 0 & 0 & -\lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

$G = \left(g_{i,j}\right)_{i,j \geq 0}$ satisfies $g_{i,j} = \begin{cases} -\lambda & \text{if } j = i, \\ \lambda & \text{if } j = i+1, \\ 0 & \text{otherwise}. \end{cases}$

Poisson process.

Independent increments: the increments

$$N(t_1) - N(t_0),\ N(t_2) - N(t_1),\ \ldots,\ N(t_n) - N(t_{n-1})$$

are independent for any $0 \le t_0 < t_1 < t_2 < \ldots < t_n$.

Alternative definition: $N(t)$ is a Poisson process with rate λ if

- $N(0) = 0$;
- $N(t)$ has independent increments;
- For any $s, t \ge 0$, $N(s+t) - N(s)$ is a Poisson random variable with parameter λt.

Poisson process.

Another alternative definition: $N(t)$ is a Poisson process with rate λ if

- $N(0) = 0$;
- $N(t)$ has independent increments;
- for $t \geq 0$, $P\big(N(t+h) - N(t) = 0\big) = 1 - \lambda h + o(h)$ as $h \to 0+$;
- for $t \geq 0$, $P\big(N(t+h) - N(t) = 1\big) = \lambda h + o(h)$ as $h \to 0+$.

Poisson process: thinning.

Consider a Poisson process $N(t)$ with rate $\lambda > 0$.

For $0 < p < 1$, let $B_1, B_2, \ldots$ be i.i.d. Bernoulli random variable with parameter p.

Theorem. $S(t) = \sum_{i=1}^{N(t)} B_i$ and $R(t) = \sum_{i=1}^{N(t)} (1 - B_i)$ are independent Poisson processes with respective rates λp and $(1-\lambda)p$.

In other words, $S(t)$ counts an arrival with probability p and $R(t) = N(t) - S(t)$ counts the rest of the arrivals.

$$\sum_{k=1}^{\infty} f_{T_k}(x)\,(1-p)^{k-1}p = \sum_{k=1}^{\infty} \frac{1}{(k-1)!}\lambda^k x^{k-1} e^{-\lambda x}(1-p)^{k-1}p$$

$$= \lambda p e^{-\lambda x} \sum_{k=1}^{\infty} \frac{\big(\lambda(1-p)x\big)^{k-1}}{(k-1)!} = \lambda p e^{-\lambda x} e^{\lambda(1-p)x} = \lambda p e^{-\lambda p x}.$$

Poisson process: superposition.

Theorem. Suppose $N_1(t)$ and $N_2(t)$ are independent Poisson processes with respective rates λ_1 and λ_2. Then $N(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$. Moreover, $N_1(t)$ and $N_2(t)$ can be represented as thinning of $N(t)$ with probabilities

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} \quad \text{and} \quad \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Recall that if X and X' are independent exponential random variables with respective parameters λ_1 and λ_2, then, $Y = \min\{X, X'\}$ is also an exponential random variable with the parameter $\lambda_1 + \lambda_2$.

Continuous-time Markov chains: generator.

Stationary distribution:

$$\pi G = 0 \quad \Leftrightarrow \quad \pi P_t = \pi \quad \forall t \in [0, \infty)$$

Ergodicity: Under irreducibility ($\forall i, j \ p_t(i,j) > 0$ for $t > 0$) and positive recurrence conditions, there exists a unique stationary distribution π, and

$$\lim_{t \to \infty} p_t(i,j) = \pi(j) \qquad \forall i, j \in S.$$

Reversibility: for continuous time, the detailed balance conditions (d.b.c.) are

$$\pi(i) g_{i,j} = \pi(j) g_{j,i}$$

If π satisfies d.b.c., then for a given $j \in S$,

$$\sum_{i:i \neq j} \pi(i) g_{i,j} - \pi(j) \sum_{i:i \neq j} g_{j,i} = 0$$

as the rows of G add up to zero. Thus, $\pi G = 0$.

Continuous-time birth-and-death chain.

Consider $S = \{0, 1, 2, \ldots\}$ and generator

$$G = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \ldots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \ldots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \ddots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\lambda_j, \mu_j > 0$.

The process is called continuous-time birth-and-death chain or birth-and-death process.

Similarly to discrete time Markov chains, if positive recurrent, the birth-and-death process satisfies d.b.c.:

$$\pi(i)\lambda_i = \pi(i+1)\mu_{i+1}$$

Continuous-time birth-and-death chain.

In a birth-and-death process, $\pi(i+1) = \frac{\lambda_i}{\mu_{i+1}}\pi(i)$ yields

$$\pi(i) = \frac{\lambda_0 \dots \lambda_{i-1}}{\mu_1 \dots \mu_i}\pi(0) \qquad \text{for } i = 1, 2, \dots. \qquad \text{Next,}$$

$$1 = \sum_{i=0}^{\infty} \pi(i) = \pi(0) + \sum_{i=1}^{\infty} \frac{\lambda_0 \dots \lambda_{i-1}}{\mu_1 \dots \mu_i}\pi(0) = \pi(0)\left(1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \dots \lambda_{i-1}}{\mu_1 \dots \mu_i}\right).$$

Thus, the stationary distribution exists if and only if

$$\sum_{i=1}^{\infty} \frac{\lambda_0 \dots \lambda_{i-1}}{\mu_1 \dots \mu_i} < \infty. \qquad \text{Then,}$$

$$\pi(0) = \frac{1}{1 + \sum\limits_{i=1}^{\infty} \frac{\lambda_0 \dots \lambda_{i-1}}{\mu_1 \dots \mu_i}} \quad \text{and} \quad \pi(i) = \frac{\frac{\lambda_0 \dots \lambda_{i-1}}{\mu_1 \dots \mu_i}}{1 + \sum\limits_{j=1}^{\infty} \frac{\lambda_0 \dots \lambda_{j-1}}{\mu_1 \dots \mu_j}} \quad \text{for } i = 1, 2, \dots.$$

Queueing processes.

$M/M/1$ **queue.** Fix $\lambda > 0$ and $\mu > 0$. Consider a system with one single server that serves customers who queue in a single line. Customers arrive at times of a Poisson process with rate λ. Each customer requires an independent service time, which is an exponential random variable with parameter μ.

$$\xrightarrow{\lambda} \boxed{\text{server}} \xrightarrow{\mu}$$

Let $X(t)$ denote the number of customers in a queue (including the ones being served). Suppose $X(0) = 0$.

The $M(\lambda)/M(\mu)/1$ **queueing process** $X(t)$ is a birth-and-death process with the generator

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & \dots \\ \mu & -(\lambda+\mu) & \lambda & \dots \\ 0 & \mu & -(\lambda+\mu) & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Queueing processes.

The $M(\lambda)/M(\mu)/1$ **queueing process** $X(t)$ is a birth-and-death process with the generator

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & \cdots \\ 0 & \mu & -(\lambda+\mu) & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

For $j = 1, 2, \ldots$, if $X_t = j$, then the probability of jumping to $j+1$ is $\frac{\lambda}{\lambda+\mu}$, and the probability of jumping to $j-1$ is $\frac{\mu}{\lambda+\mu}$. Hence,

- If $\mu > \lambda$, the process is positive recurrent.
- If $\mu = \lambda$, the process is null recurrent.
- If $\mu < \lambda$, the process is transient.

Queueing processes.

The $M(\lambda)/M(\mu)/1$ **queueing process** $X(t)$ is a birth-and-death process with the generator

$$\lambda_j = \lambda \quad \text{for} \quad j = 0, 1, \ldots \qquad \text{and} \qquad \mu_j = \mu \quad \text{for} \quad j = 1, 2, \ldots .$$

Here,

$$\sum_{i=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i} = \sum_{i=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^i < \infty \quad \text{if and only if} \quad \lambda < \mu.$$

If $\lambda < \mu$, $\quad 1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i} = \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i = 1\Big/\left(1 - \frac{\lambda}{\mu}\right) = \frac{\mu}{\mu - \lambda}$

and

$$\pi(i) = \frac{\frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i}}{1 + \sum_{j=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j}} = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^i \quad i = 0, 1, 2, \ldots .$$

Queueing processes.

$M/M/r$ **queue.** Fix $\lambda > 0$ and $\mu > 0$. Consider a system with r servers that serve customers who queue in a single line. Customers arrive at times of a Poisson process with rate λ. Each customer requires an independent service time, which is an exponential random variable with parameter μ.

$$\xrightarrow{\lambda} \left\{ \begin{array}{ll} \boxed{\text{server 1}} & \xrightarrow{\mu} \\ \boxed{\text{server 2}} & \xrightarrow{\mu} \\ \quad\vdots & \vdots \\ \boxed{\text{server r}} & \xrightarrow{\mu} \end{array} \right.$$

Let $X(t)$ denote the number of customers in a queue (including the ones being served). Suppose $X(0) = 0$.

Queueing processes.

The $M(\lambda)/M(\mu)/r$ **queueing process** $X(t)$ is a birth-and-death process with the generator

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \ddots & \ddots & \ddots & \dots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \ddots & \ddots & \dots \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & r\mu & -(\lambda+r\mu) & \lambda & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & r\mu & -(\lambda+r\mu) & \lambda & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & r\mu & -(\lambda+r\mu) & \lambda & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

with the transition rates

$$g_{i,i+1} = \lambda \quad (i = 0, 1, \dots) \qquad \text{and} \qquad g_{i,i-1} = \begin{cases} i\mu & i = 1, \dots, r \\ r\mu & i = r+1,\ r+2, \dots \end{cases}$$

Queueing processes.

$$\lambda_i = \lambda \quad (i = 0, 1, \ldots) \qquad \text{and} \qquad \mu_i = \begin{cases} i\mu & i = 1, \ldots r \\ r\mu & i = r+1,\, r+2, \ldots \end{cases}$$

$$\sum_{i=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i} = \sum_{i=1}^{r} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \frac{r^r}{r!}\sum_{i=r+1}^{\infty}\left(\frac{\lambda}{r\mu}\right)^i < \infty \text{ if and only if } \lambda < r\mu.$$

If $\lambda < r\mu$, $1+\sum_{i=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i} = \mathsf{T}_r(\lambda/\mu) + \frac{r^r}{r!}\left(\frac{\lambda}{r\mu}\right)^r \frac{\lambda}{r\mu-\lambda}$, where $\mathsf{T}_r(x) = \sum_{k=0}^{r} \frac{1}{k!}x^k$ is the r^{th} degree Taylor polynomial for $f(x) = e^x$.

$$\pi(i) = \frac{\frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i}}{1 + \sum_{j=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j}} = \begin{cases} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i \left(\mathsf{T}_r(\lambda/\mu) + \frac{r^r}{r!}\left(\frac{\lambda}{r\mu}\right)^r \frac{\lambda}{r\mu-\lambda}\right)^{-1} & i \le r \\ \frac{r^r}{r!}\left(\frac{\lambda}{r\mu}\right)^i \left(\mathsf{T}_r(\lambda/\mu) + \frac{r^r}{r!}\left(\frac{\lambda}{r\mu}\right)^r \frac{\lambda}{r\mu-\lambda}\right)^{-1} & i > r \end{cases}$$

Queueing processes.

$M/M/\infty$ **queue.** Fix $\lambda > 0$ and $\mu > 0$. Consider a system with **infinitely many** servers that serve customers who queue in a single line. Customers arrive at times of a Poisson process with rate λ. Each customer requires an independent service time, which is an exponential random variable with parameter μ.

$$\xrightarrow{\lambda} \left\{ \begin{array}{cc} \boxed{\text{server 1}} & \xrightarrow{\mu} \\ \boxed{\text{server 2}} & \xrightarrow{\mu} \\ \boxed{\text{server 3}} & \xrightarrow{\mu} \\ \vdots & \vdots \end{array} \right.$$

Let $X(t)$ denote the number of customers in a queue (including the ones being served). Suppose $X(0) = 0$.

Queueing processes.

The $M(\lambda)/M(\mu)/\infty$ **queueing process** $X(t)$ is a birth-and-death process with the transition rates

$$\lambda_i = \lambda \quad (i = 0, 1, \ldots) \qquad \text{and} \qquad \mu_i = i\mu \quad (i = 1, 2, \ldots)$$

$$\sum_{i=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i} = \sum_{i=1}^{\infty} \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i < \infty \quad \text{for all } \lambda \text{ and } \mu.$$

Thus, for all λ and μ, the process is positive recurrent, and $1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i} = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i = e^{\lambda/\mu}$ and

$$\pi(i) = \frac{\frac{\lambda_0 \ldots \lambda_{i-1}}{\mu_1 \ldots \mu_i}}{1 + \sum_{j=1}^{\infty} \frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j}} = \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i e^{-\lambda/\mu} \quad i = 0, 1, 2, \ldots.$$

Brownian motion.

R. Brown (1827) observed erratic movement of grains of pollen suspended in water.

L. J. B. A. Bachelier (1900) used ideas of modern day Brownian motion to study market fluctuations. His work was forgotten for a number of decades.

A. Einstein (1905) studied diffusion of particles. Coined the name *Brownian motion*. Assumed Brownian motion existed.

N. Wiener (1923) proved Brownian motion existed.

Brownian motion.

Definition. A real-valued stochastic process $\{B(t)\}_{t\in[0,\infty)}$ is called a one-dimensional Brownian motion that originates at $x \in \mathbb{R}$ if the following hold:

- $B(0) = x$
- Independent increments:

  $$B(t_n)-B(t_{n-1}),\ B(t_{n-1})-B(t_{n-2}), \ldots,\ B(t_2)-B(t_1)$$

  are independent for any finite increasing sequence $0 \le t_1 < t_2 < \ldots < t_n$.
- Gaussian increments: For $s, t \ge 0$, $B(s+t)-B(s)$ is a normal random variable with mean 0 and variance t, i.e., $\mathcal{N}(0, t)$.
- The trajectory $B(t)$ is continuous with probability one.

Brownian motion.

A one-dimensional Brownian motion that originates at $x = 0$ is called standard Brownian motion.

A one-dimensional Brownian motion $B(t)$ that originates at $x \in \mathbb{R}$ can be expressed as

$$B(t) = x + W(t),$$

where $W(t)$ is a standard Brownian motion.

Theorem (N. Wiener, 1923). Standard Brownian motion exists.

We will give a constructive proof by P. Lévy (1937).

Brownian motion.

Properties:

$$(\mathbf{1}). \qquad Cov\big(B(s), B(t)\big) = s \wedge t \qquad \forall s, t \geq 0$$

Indeed, assume $0 \leq s \leq t$, then

$$Cov\big(B(s), B(t)\big) = Cov\big(B(s), B(t) - B(s)\big) + Cov\big(B(s), B(s)\big)$$

$$= Var\big(B(s)\big) = s$$

as $B(s) - B(0)$ and $B(t) - B(s)$ are independent increments.

$(\mathbf{2})$. If $B(t)$ is standard Brownian motion, then for a given $a > 0$, $\frac{1}{\sqrt{a}} B(at)$ is also standard Brownian motion.

$(\mathbf{3})$. If $B(t)$ is standard Brownian motion. Let $Y(0) = 0$ and $Y(t) = tB(1/t)$ for $t > 0$, then $Y(t)$ is also standard Brownian motion.