Math 464/564
Homework #3 - Due Monday, February 22nd

1. (10 pts) Let X be a gamma random variable with the probability density function

$$f_{\mathsf{x}}(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\,\lambda e^{-\lambda x}(\lambda x)^{\alpha-1} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

and Y be a discrete random variable with the probability mass function p_{y}. Suppose that for each $x > 0$, the conditional probability mass function $p_{Y|X}(n|x)$ is Poisson with parameter x:

$$p_{Y|X}(n|x) = e^{-x}\,\frac{x^n}{n!} \qquad n = 0, 1, 2, \ldots$$

i. (5 pts) Find the probability mass function p_{y}.

ii. (5 pts) Find the conditional probability density function $f_{X|Y}(x|n)$.

Hint: $f_{X|Y}(x|n)p_{\mathsf{y}}(y) = p_{Y|X}(n|x)f_{\mathsf{x}}(x)$.

2. (5 pts) Suppose U is a uniform random variable over $[0,1]$. Let X be a discrete random variable with the probability mass function p_{x}. Suppose there is a positive integer n such that the conditional probability mass function $p_{X|U}(k|u)$ is binomial with parameters (n, u), i.e.,

$$p_{X|U}(k|u) = \binom{n}{k} u^k (1-u)^{n-k} \qquad k = 0, 1, \ldots, n$$

Find the probability mass function p_{x}.

3. (10 pts) In an online video game, there are N players fighting a fleet of enemy spaceships. When the fleet of 10 enemy spaceships arrives, each player chooses one of the 10 spaceships uniformly at random and fires one missile. The players select targets independently of each other, and each missile hits the target with probability $p = 3/5$. Note that the same spaceship may be selected by more than one player. We assume that the number N of players is a Poisson random variables with parameter $\lambda = 8$. In order to compute the expected number of enemy spaceships hit (and destroyed) in the attack, one needs to complete the following two steps:

i. (5 pts) Let H_i be the event that the i-th spaceship is hit, and let X_i be the indicator variable of H_i. Compute $E\big[X_i \,\big|N = n\big]$ for $n = 1, 2, \ldots$.
Hint: notice that $E\big[X_i \,\big|N = n\big] = P\big(H_i \,\big|N = n\big)$.

ii. (5 pts) Use conditioning to compute $E\left[\sum_{i=1}^{10} X_i\right]$.

4. (5 pts) Let X be a Poisson random variable with parameter $\lambda > 0$. Use the moment generating function of X to compute $E[X^3]$.

5. (5 pts) X and Y are independent random variables with density functions given by

$$f_X(x) = \begin{cases} 3e^{-3x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_Y(x) = \begin{cases} \frac{3e^{-3x}(3x)^4}{4!} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

respectively. Compute the moment generating function $M_{X+Y}(t)$ of $X+Y$. Use $M_{X+Y}(t)$ to identify the density function $f_{X+Y}(x)$ of $X+Y$.

6. (10 pts) Suppose $X_1, X_2, \ldots$ is an infinite sequence of independent identically distributed random variables with mean $E[X_i] = 6$ and variance $Var(X_i) = 9$, and suppose N is a discrete random variable, independent from $X_1, X_2, \ldots$, with the moment generating function given by $M_N(t) = e^{5(e^t - 1)}$.

   Let $X = \sum_{i=1}^{N} X_i$ be the sum of first N members of the sequence. Compute $Var(X)$.