Math 464/564
Homework #2 - Due Monday, February 8th

1. (15 pts) Consider continuous random variables X and Y with a joint probability density function

$$f(x,y) = \begin{cases} 2e^{-x-y} & \text{if } 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Do the following parts.

i. (4 pts) Find the marginal probability density functions f_{x} and f_{y}. Are X and Y independent?

ii. (3 pts) Compute $E[X]$ and $E[Y]$.

iii. (4 pts) Is the covariance $Cov(X,Y)$ well defined? If yes, find $Cov(X,Y)$. If no, explain why.

iv. (4 pts) Is the correlation $\operatorname{corr}(X,Y)$ well defined? If yes, find $\operatorname{corr}(X,Y)$. If no, explain why.

Hint: To simplify the computations, recall that $\int_0^\infty y^n e^{-y}dy = \Gamma(n+1) = n!$ for all integer $n \geq 0$.

2. (10 pts) Consider discrete random variables X and Y such that

$$p_{\mathsf{y}}(m) = P(Y = m) = e^{-2}\frac{2^m}{m!}, \qquad m = 0, 1, 2, \ldots,$$

and for each integer $m \geq 0$,

$$p_{X|Y}(k|m) = P(X = k \,|\, Y = m) = e^{-(m+1)}\frac{(m+1)^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

Do the following parts.

i. (2 pts) Find the joint probability mass function $p(k,m)$.

ii. (4 pts) Find the conditional expectation $E[X|Y]$.

iii. (4 pts) Compute $E\big[(E[X|Y]-1)^2\big]$.

Hint: Recall that if Z is a Poisson random variable with parameter $\lambda > 0$, then $E[Z] = Var(Z) = \lambda$. Also notice that if $g(m) = E[X \,|\, Y = m]$, then, for any function $\phi : \mathbb{R} \to \mathbb{R}$,

$$E[\phi(g(Y))] = \sum_m \phi(g(m))P(Y = m)$$

3. (5 pts) Consider continuous random variables X and Y with a joint probability density function

$$f(x,y) = \begin{cases} ye^{-(x+1)y} & \text{if } x > 0,\ y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Do the following parts.

i. (3 pts) For a given $y > 0$, find the conditional probability density function $f_{X|Y}(x|y)$.

ii. (2 pts) Find the conditional expectation $E[X|Y]$.