1. (15 pts) Consider continuous random variables $X$ and $Y$ with a joint probability density function

$$f(x, y) = \begin{cases} 
2e^{-x-y} & \text{if } 0 < x < y < \infty, \\
0 & \text{otherwise}.
\end{cases}$$

Do the following parts.

i. (4 pts) Find the marginal probability density functions $f_X$ and $f_Y$. Are $X$ and $Y$ independent?

ii. (3 pts) Compute $E[X]$ and $E[Y]$.

iii. (4 pts) Is the covariance $\text{Cov}(X, Y)$ well defined? If yes, find $\text{Cov}(X, Y)$. If no, explain why.

iv. (4 pts) Is the correlation $\text{corr}(X, Y)$ well defined? If yes, find $\text{corr}(X, Y)$. If no, explain why.

Hint: To simplify the computations, recall that $\int_0^\infty y^n e^{-y} dy = \Gamma(n+1) = n!$ for all integer $n \geq 0$.

2. (10 pts) Consider discrete random variables $X$ and $Y$ such that

$$p_Y(m) = P(Y = m) = e^{-2} \frac{2^m}{m!}, \quad m = 0, 1, 2, \ldots,$$

and for each integer $m \geq 0$,

$$p_{X|Y}(k|m) = P(X = k \mid Y = m) = e^{-(m+1)} \frac{(m+1)^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

Do the following parts.

i. (2 pts) Find the joint probability mass function $p(k, m)$.

ii. (4 pts) Find the conditional expectation $E[X|Y]$.


Hint: Recall that if $Z$ is a Poisson random variable with parameter $\lambda > 0$, then $E[Z] = \text{Var}(Z) = \lambda$. Also notice that if $g(m) = E[X \mid Y = m]$, then, for any function $\phi : \mathbb{R} \to \mathbb{R}$,

$$E[\phi(g(Y))] = \sum_m \phi(g(m)) P(Y = m)$$
3. (5 pts) Consider continuous random variables $X$ and $Y$ with a joint probability density function

$$f(x, y) = \begin{cases} \frac{ye^{-(x+1)y}}{y} & \text{if } x > 0, \ y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Do the following parts.

i. (3 pts) For a given $y > 0$, find the conditional probability density function $f_{X|Y}(x|y)$.

ii. (2 pts) Find the conditional expectation $E[X|Y]$. 